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Randomised observation, control and stabilization of waves

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The problems of observing, controlling and stabilizing wave processes arise in many different contexts ranging from structural mechanics to seismic waves. In a suitable functional setting, they are closely interconnected and sometimes completely equivalent. In a series of previous articles we have addressed the problem of the optimal design of sensors for purely conservative wave models. We analyzed a relaxed version of the optimal observation problem, considering the expectation of solutions under a randomisation procedure, rather than that where all possible solutions are considered in a purely deterministic setting. From an analytical point of view, this randomisation procedure had the advantage of leading to a spectral diagonalisation of the observations. In this way, using fine asymptotic spectral properties of the Laplacian, we disclosed the links between the geometric properties of the domain where waves propagate and the existence of optimal locations for the sensors or, by the contrary, the emergence of relaxation phenomena. Here we show that spectral randomised observability is equivalent to the property of spectral controllability by means of a discrete set of lumped controls acting everywhere on the domain, and distributed according to the shape of the eigenfunctions. Our results on optimal observation then find natural equivalents on the problem of optimal spectral control. We also give an interpretation of these results in terms of a feedback stabilization property, ensuring the exponential decay of the energy of solutions as time tends to infinity.

1 Introduction

In this paper we analyse the control and stabilization counterparts of our previous works [8–10, 12] on the randomised observability for wave processes.

The property of observability of a system, that refers to the possibility of recovering the full energy of the solutions out of partial measurements done on the solutions of the model under consideration, plays a key role in Control and Inverse problems theory. In our previous works we analyzed the optimal design and location of sensors, so as to ensure the optimal observation of the energy of all possible solutions, an issue that plays a key role in applications (vibrations of structures, acoustic and seismic waves, etc.).

One of our main contributions in that series of articles was to underline the relevance of considering a randomised version of the problem to optimally observe the expectation of solutions, rather than all possible solutions in a deterministic manner. This randomisation procedure, implemented on the Fourier coefficients of the initial data, leads to a spectral diagonalisation of the observation criterium that can then be handled by using fine properties of the spectrum of the Laplacian.

Here we are mainly interested in describing the counterparts of those results in the context of active control and stabilization. Roughly, in this article we prove that the spectral randomised observability inequality is equivalent to the properties of optimal spectral control and stabilization.

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According to the property of spectral controllability all wave perturbations can be driven to rest within arbitrarily small time by means of lumped controls, acting on each eigenfunction of the system, everywhere in the domain. Similarly, the spectral stabilization property ensures the uniform exponential decay of the energy by means of a feedback control of the same nature. In both cases the localisation properties of controls are lost and they turn out to be distributed everywhere in the domain where waves propagate, acting separately in each spectral component.

Our arguments can be easily extended to an abstract setting of purely conservative semigroups as in [13] and, actually, they apply to Schrödinger and plate like equations. However, in order to simplify the exposition, throughout the paper we restrict ourselves to the wave equation.

In [11] we developed the parabolic counterpart of the problem of optimal randomised observability which leads to weighted spectral observabillity inequalities. As in the present context of purely conservative systems, these results lead also to spectral controllability properties. This issue will be analysed in a forthcoming article.

The rest of this paper is organised as follows. In Sect. 2 we present the main consequences of spectral observability in the context of spectral controllability and stabilization. In Sect. 3 we give the proof of the result of spectral controllability. Section 4 is devoted to prove the spectral stabilization results. Section 5 is devoted to discuss some other possible extensions of the results of this paper.

2 Problem formulation and main results

2.1 Preliminaries on observability and spectral observability

Given a bounded and smooth domain Ω of $\mathbb{R}^n$, $n \geq 1$, and a measurable subset ω of Ω, we consider the following controlled wave equation, where χ_ω stands for the characteristic function of the set ω where the controls are supported:

$$\begin{aligned} &\partial_{tt} y = \triangle y + f\chi_\omega \quad \text{in } \Omega \times (0,T),\\ &y(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega,\\ &y(0,x) = y^0(x), \quad \partial_t y(0,x) = y^1(x) \quad \text{in } \Omega. \end{aligned} \tag{1}$$

Here, (y^0, y^1) stand for the initial data to be controlled, say in the energy space $H_0^1(\Omega)\times L^2(\Omega)$, and $f = f(x,t) \in L^2(\omega\times(0,T))$ for the control.

This system is well known to be exactly controllable when the support of the controls ω is an open set, and ω and the time horizon T are large enough so that the so-called Geometric Control Condition (GCC) is satisfied (see [1, 2]). This condition requires that all rays of Geometric Optics enter the control set ω within time T.

When GCC holds, the system is exactly controllable meaning that, for all initial data (y^0, y^1) in $H_0^1(\Omega)\times L^2(\Omega)$, there exists a control $f = f(x,t) \in L^2(\omega\times(0,T))$ such that the solution reaches the equilibrium at time $t = T$, i.e.

$$y(T,x) = \partial_t y(T,x) = 0 \quad \text{in } \Omega. \tag{2}$$

Furthermore, this is done by means of controls f with support in ω.

We emphasize however that, as a consequence of the hyperbolic nature of the model under consideration, for this property to hold, the control time $T > 0$ has to be large enough, and the control set ω to be so that all generalized rays enter into it within time T.

This property of exact controllability is well known to be equivalent to an observability inequality for the adjoint system:

$$\begin{aligned} &\partial_{tt} p = \triangle p \quad \text{in } \Omega\times(0,T),\\ &p(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega,\\ &p(0,x) = p^0(x),\ \partial_t p(0,x) = p^1(x) \quad \text{in } \Omega. \end{aligned} \tag{3}$$

More precisely, exact controllability holds if and only if there exists $C > 0$ such that all solutions $p = p(x,t)$ of this adjoint system satisfy

$$C\|(p^0,p^1)\|^2_{L^2\times H^{-1}} \leq \int_0^T\int_\omega |p(t,x)|^2\,dx\,dt. \tag{4}$$

In practice, the constant C ensuring the observability inequality (4) is also a measure of the quality of the observation and depends on the location ω of the observation and the length of time interval T. In fact, the larger $C = C(T,\omega)$ is, the more energy of solutions we observe from ω and, accordingly, the better that location of the observation is.

In previous papers we have analyzed the problem of optimizing the choice of the shape and location of the observation set ω among, say, the class of all measurable subdomains of Ω of a given measure or volume fraction. This problem seems currently to be out of reach in its full version, when considering all possible solutions of the wave equation. The problem is even challenging from a computational point of view, as explained e.g. in [8, 9], due to the emergence of numerical instabilities and the necessity of taking into consideration high frequency phenomena. On the other hand, when considering all possible solutions of the wave equation, one faces, in fact, the worst possible cases. But in practice, when having access to a large number of measures, it is natural to search for a sensor design performing optimally in an average sense. This is why, rather than considering the full observability inequality, we addressed the same issue for a randomised version in which the key inequality (4) is replaced by its expectation with respect to randomised initial data. Furthermore, the randomised observability problem makes sense in the context of applications in which, often times, observations and measures are frequently submitted to small perturbations of the relevant parameter values.

Let us be more precise. Given a fixed horizon of time $T > 0$, the problem of optimal observability would a priori consist on maximizing the functional $\chi_\omega \mapsto C(T,\omega)$ over the set

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of Lebesgue measure } |\omega| = L|\Omega|\}.$$

Here, the notation χ_ω stands for the characteristic function of the set ω.

A spectral expansion of the solutions shows the emergence of crossed terms in the functional to be minimized, that are difficult to treat.

To see this, in what follows we fix a Hilbert basis $(\phi_j)_{j\geq 1}$ of $L^2(\Omega)$ consisting of (real-valued) eigenfunctions of the Dirichlet-Laplacian operator on Ω, associated with the negative eigenvalues $(-\lambda_j^2)_{j\geq 1}$. Then any solution p of (3) can be expanded as

$$p(t,x) = \sum_{j=1}^{+\infty} p_j(t)\phi_j(x) = \sum_{j=1}^{+\infty} \left(a_j \,\mathrm{e}^{i\lambda_j t} + b_j \,\mathrm{e}^{-i\lambda_j t}\right)\phi_j(x), \tag{5}$$

where the coefficients a_j and b_j account for initial data. It follows that

$$C(T,\omega) = \frac{1}{2} \inf_{\substack{(a_j),(b_j)\in\ell^2(\mathbb{C}) \\ \sum_{j=1}^{+\infty}(|a_j|^2+|b_j|^2)=1}} \int_0^T \int_\omega \left| \sum_{j=1}^{+\infty} \left(a_j \,\mathrm{e}^{i\lambda_j t} + b_j \,\mathrm{e}^{-i\lambda_j t}\right)\phi_j(x)\right|^2 dx\, dt, \tag{6}$$

and maximizing this functional over $\mathcal{U}_L$ appears to be very difficult, due to the crossed terms $\int_\omega \phi_j\phi_k\, dx$ measuring the interaction over ω between distinct eigenfunctions.

The observability constant defined by (6) is deterministic and provides an account for the worst possible case. But, as mentioned above, in practical applications, one realizes a large number of measures, and it is therefore natural to consider an averaged version of the observability inequality over random initial data.

We then define the *randomised observability constant* by

$$C_{\mathrm{rand}}(T,\omega) = \frac{1}{2} \inf_{\substack{(a_j),(b_j)\in\ell^2(\mathbb{C}) \\ \sum_{j=1}^{+\infty}(|a_j|^2+|b_j|^2)=1}} \mathbb{E}\left(\int_0^T \int_\omega \left| \sum_{j=1}^{+\infty} \left(\beta_{1,j}^\nu a_j \,\mathrm{e}^{i\lambda_j t} + \beta_{2,j}^\nu b_j \,\mathrm{e}^{-i\lambda_j t}\right)\phi_j(x)\right|^2 dx\, dt \right),$$

where $(\beta_{1,j}^\nu)_{j\geq 1}$ and $(\beta_{2,j}^\nu)_{j\geq 1}$ are two sequences of (for example) i.i.d. Bernoulli random laws on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$, and $\mathbb{E}$ is the expectation over $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$. It corresponds to an averaged version of the observability inequality over random initial data.

The following characterization of the randomised observability constant was proved in [12]:

For every measurable subset ω of Ω, we have

$$C_{\mathrm{rand}}(T,\omega) = \frac{T}{2} \inf_{j\geq 1} \int_\omega \phi_j(x)^2\, dx.$$

Thus, we obtain a purely spectral formulation of this randomised optimal observability problem, i.e., that of maximizing the functional

$$J(\chi_\omega) = \inf_{j\geq 1} \int_\omega \phi_j(x)^2\, dx$$

over the set $\mathcal{U}_L$.

The functional J can be interpreted as a criterion giving an account for the concentration properties of eigenfunctions.

Definition 1. *Given the orthonormal basis $(\phi_j)_{j\geq 1}$ of eigenfunctions under consideration, given a measurable subset ω of Ω, we say that the spectral observability inequality holds true if there exists $C > 0$ such that*

$$\int_\omega \phi_j(x)^2\,dx \geq C, \tag{7}$$

for every $j \geq 1$.

Obviously, the actual value of the constant $C > 0$ in (7) is a measure of the quality of ω for spectral observability.

The one-dimensional case is the simplest situation in which (7) holds true since, in view of the explicit structure of the eigenfunctions (sinusoidal functions), the inequality is obviously satisfied for any measurable set ω of positive measure. In this particular case, it is notable that the spectral observability inequality (7) is equivalent to the standard observability inequality (4) provided that the observability time T be large enough. But this fails to be true in the multi-dimensional case as we shall see below.

It is interesting to note that we always have $C(T,\omega) \leq C_{\text{rand}}(T,\omega)$, and that the strict inequality may hold true. Accordingly, spectral observability does not guarantee observability in the sense of (4). In fact, as mentioned above, the latter is equivalent to GCC on (ω, T), while the spectral observability inequality is independent on T, and requires weaker conditions on ω.

The main reason for the possible gap between $C(T,\omega)$ and $C_{\text{rand}}(T,\omega)$ is as follows. The necessity of the GCC for the observability inequality to hold in the classical deterministic sense is that, whenever a ray of Geometric Optics escapes the observation set, one can build gaussian beam solutions that constitute an impediment for the observability inequality to hold. But in order for these wave packets to be spectrally localized, stronger stability conditions on the billiard dynamics generated by the rays are required. For this reason, in practice, there are situations where spectral observability holds but the stronger version of the observability inequality fails.[1]

It is also interesting to observe that the quantity $J(\chi_\omega)$ can also be recovered by considering a time-asymptotic version of the observability inequality (4) as $T \to \infty$. In fact $J(\chi_\omega)$ is the largest possible constant such that

$$C\|(p^0, p^1)\|^2_{L^2\times H^{-1}} \leq \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_\omega |p(t,x)|^2\,dx\,dt, \tag{8}$$

for all $(p^0, p^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ (see [12]).

The problem of the existence and characterization of the optimal set ω for this randomised observability problem is complex. The answer actually depends in a very sensitive manner on the fine properties of the spectrum of the Laplacian that turns out to be linked to the geometry of the domain of Ω and, more precisely, to the dynamical systems properties of the billiard it generates. In some cases it was proved that the optimal set does not exist and that, in fact, through an homogenization process, minimizing sequences tend to cover the whole domain Ω, as solutions of the relaxed version that we introduce now.

To formulate the convexified or relaxed version of the problem, we consider the convex closure of the set $\mathcal{U}_L$ for the L^∞ weak star topology, that is

$$\overline{\mathcal{U}}_L = \{a \in L^\infty(\Omega, [0,1]) \mid \int_\Omega a(x)\,dx = L|\Omega|\}. \tag{9}$$

The convexified problem then consists of maximizing the functional (still denoted J)

$$J(a) = \inf_{j\geq 1} \int_\Omega a(x)\phi_j(x)^2\,dx \tag{10}$$

over $\overline{\mathcal{U}}_L$. It is easy to see that, for this relaxed problem, a maximizer exists. Nevertheless since the functional J is not lower semi-continuous it is not clear whether or not there may be a gap between the original spectral problem and its convexified version. The analysis of this question turned out to be very interesting and revealed deep connections with the theory of quantum chaos and, more precisely, with quantum ergodicity properties of Ω (see [12]).

With these preliminaries on the problem of spectral observability and the corresponding optimal location/design problem, we are now in a position to address the controllability analogs.

2.2 Spectral controllability

Our first main result concerns the consequences of spectral observability at the level of the controllability of the system.

[1] An example of such a situation for the wave equation is the following: take $\Omega = (0,\pi)^2$ with Dirichlet boundary conditions and $\omega = \{(x,y) \in \Omega \mid x < \pi/2\}$. Clearly, such a domain does not satisfy GCC, and one has $C(T,\omega) = 0$, whereas $C_{\text{rand}}(T,\omega) = 1/4$. We refer to [12, Remark 4] for additional examples and comments.

Theorem 1. *The spectral observability inequality (7) holds true in Ω from the subset ω if and only if the wave equation enjoys the following property of exact spectral controllability in any time $T > 0$: for all $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a sequence of controls $(f_j(\cdot))_{j\geq 1}$ in $L^2(0, T; \ell^2(\mathbb{R}))$ such that the solution y of*

$$\begin{aligned} &\partial_{tt} y = \triangle y + \sum_{j\geq 1} f_j(t) \left(\int_\omega \phi_j(x)^2 \, dx \right) \phi_j(x) \quad \textit{in}\,(0,T)\times\Omega, \\ &y(t,x) = 0 \quad \textit{on}\,(0,T)\times\partial\Omega, \\ &y(x,0) = y^0(x),\ \partial_t y(x,0) = y^1(x) \quad \textit{in}\,\Omega, \end{aligned} \tag{11}$$

satisfies the null controllability condition at time T,

$$y(T,x) = \partial_t y(T,x) = 0 \quad \textit{in}\,\Omega.$$

In that case, furthermore, there exists a constant $C'(T, \omega) > 0$ such that

$$\|(f_j(\cdot))_{j\geq 1}\|^2_{L^2(0,T;\ell^2(\mathbb{R}))} \leq C'(T,\omega) \|(y^0, y^1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}, \tag{12}$$

for all $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Several remarks are in order.

- We observe that the control consists of a sequence $(f_j(t))_{j\geq 1}$ of time-dependent controls, acting diagonally on the eigenfunctions of the system.
- As expected, this spectral controllability property is independent of the control time T. This is so since the control is distributed everywhere in the domain Ω, acting on the system through the profiles of the eigenfunctions ϕ_j.
- The amplitude of each control f_j is weighted by the term $\int_\omega \phi_j(x)^2 \, dx$. Accordingly, for the uniform estimate (12) to hold true, the spectral observability inequality (7) is required.
- Exact spectral controls f_j, realizing exact spectral controllability in time T for the system (11), can be built according to the usual Hilbert Uniqueness Method (HUM, see [7]), by minimizing some appropriate quadratic functional over the set of adjoint solutions. This functional is defined in Sect. 3, where the expression of the resulting controls is also given.
- The constant $C(T, \omega)$ on the cost of spectral control in (12) is inversely proportional to the spectral observability constant. Thus, the better ω is for spectral observability, the better it is also as location for the controllers.
- This spectral controllability result can also be extended to the context of the relaxed spectral observability inequality in which the observation is distributed everywhere in the domain according to the density function $a(x)$ rather than localized in ω. In that case the corresponding controlled system reads:

  $$\partial_{tt} y = \triangle y + \sum_{j\geq 1} f_j(t) \left(\int_\Omega a(x)\phi_j(x)^2 \, dx \right) \phi_j(x) \quad \text{in}\,(0,T)\times\Omega. \tag{13}$$

  The control being distributed everywhere in the domain, under (7), the fact that the system is controllable is straightforward. However, as we shall see, spectral observability has important consequences on the amplitudes of the spectral controls f_j.

Optimal design. The equivalence between spectral observability and controllability allows also to transfer the main results on the optimal design and location of sensors (obtained in [8, 12]) to the context of controllability. This goes as follows.

Defining the operator

$$\Lambda_{T,\omega} : H_0^1(\Omega) \times L^2(\Omega) \ni (y^0, y^1) \mapsto u \in L^2(0, T, L^2(\Omega)),$$

where u is the control function defined by

$$u(t,x) = \sum_{j\geq 1} f_j(t) \left(\int_\omega \phi_j(x)^2 \, dx \right) \phi_j(x),$$

it is easy to see (using the arguments of the proof of Theorem 1) that

$$\|\Lambda_{T,\omega}\| = 1/C_{\text{rand}}(T,\omega).$$

Therefore, for a given $L \in (0, 1)$, minimizing the operator norm $\|\Lambda_{T,\omega}\|$ over $\mathcal{U}_L$ is equivalent to maximizing $C_{\text{rand}}(T, \omega)$ over $\mathcal{U}_L$.

We have proved in [12] that, under appropriate spectral assumptions on the functions $(\phi_j)_{j\geq 1}$, namely the uniform boundedness of this sequence in $L^p(\Omega)$ for a given $p > 1$ and the convergence of the sequence $(\phi_j^2\,dx)_{j\geq 1}$ of probability measures to the uniform measure, there is no gap between the problem of maximizing J over $\mathcal{U}_L$ and the convexified problem of maximizing J over $\overline{\mathcal{U}}_L$. Moreover, the function $a = L$ is a solution of the convexified problem, and the maximal possible value of $C_{T,\text{rand}}(\chi_\omega)$ over $\mathcal{U}_L$ is equal to $TL/2$.

The above spectral assumptions are satisfied in 1D, and in multi-D are related to deep quantum ergodicity properties, and quantum chaos (see [12] for a discussion on such issues). They are sufficient but not sharp: indeed we can prove that the no-gap result is still valid if Ω is a 2D disk (with the usual eigenfunctions parametrized by Bessel functions), although the eigenfunctions do not equidistribute as the eigenfrequencies increase, as illustrated by the well-known whispering gallery effect.

Our results eventually show intimate connections between domain optimization and fine spectral properties (quantum ergodicity properties) of Ω.

The maximum of the convexified functional J over $\overline{\mathcal{U}}_L$ is always reached, and in general, it is reached in an infinite number of ways. The question of the reachability of the supremum of J over $\mathcal{U}_L$, that is, the existence of an optimal classical set, is a difficult one. In particular cases it can however be addressed using harmonic analysis. For instance in 1D we have proved in [8] that the supremum is reached if and only if $L = 1/2$ (and that, in that particular case, there is an infinite number of optimal sets). In multi-D the question is open, and we conjecture that, for generic domains Ω and generic values of L, the supremum is not reached and hence there does not exist any optimal set. It can however be noted that, in the 2D square, if we restrict the search of optimal sets to Cartesian products of 1D subsets, then the supremum is reached if and only if $L \in \{1/4, 1/2, 3/4\}$.

In view of that, it is then natural to study a finite-dimensional spectral approximation of the problem, namely:

$$\textit{Maximize the functional } J_N(\chi_\omega) = \min_{1\leq j\leq N} \int_\omega \phi_j(x)^2\,dx \textit{ over } \mathcal{U}_L.$$

For this problem, given a fixed value of N, the existence and uniqueness of an optimal set ω^N is easy to obtain, as well as a Γ-convergence property of J_N towards J for the weak star topology of L^∞. Moreover, the set ω^N is semi-analytic and thus it has a finite number of connected components, expected to increase as N grows (as confirmed by numerical simulations). The increasing complexity as N grows is in accordance with the conjecture of the nonexistence of an optimal set maximizing J. It can be noted that, in 1D and for L sufficiently small, loosely speaking, the optimal domain ω^N for N modes is the worst possible one when considering the truncated problem with $N + 1$ modes (spillover phenomenon: see [5, 8]).

2.3 Spectral stabilization

Our second main result concerns the consequences of spectral observability for the problem of feedback stabilization.

As proved by A. Haraux in [4], in the context of the classical deterministic observability inequality (4), the observability of the adjoint wave equation is equivalent to the property of stabilization of the following dissipative one:

$$\begin{aligned} &\partial_{tt} z = \triangle z + \chi_\omega z_t \quad \text{in } (0, T) \times \Omega, \\ &z(t, x) = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ &z(0, x) = z^0(x),\ \partial_t z(0, x) = z^1(x) \quad \text{in } \Omega. \end{aligned} \tag{14}$$

More precisely, the property of observability (4) is equivalent to the property of exponential decay of the energy of solutions of (14), i.e.,

$$E(t) \leq CE(0)\exp(-\alpha t) \quad \forall t \geq 0, \tag{15}$$

for suitable constants $C > 0$ and $\alpha > 0$ (not depending on the solution) and for all solutions of (14), where

$$E(t) = \frac{1}{2}\int_\Omega \left(|\partial_t z(x, t)|^2 + |\nabla z(x, t)|^2\right) dx.$$

The weaker spectral observability inequality (7) is equivalent to a weaker stabilization property as well, in the sense that the damping term that is required to ensure the exponential stabilization is of a spectral nature, and is distributed everywhere in the domain.

To be more precise, as a consequence of (7), and arguing as in [4], the following stabilization result can be proved.

Theorem 2. *The spectral observability inequality (7) holds true if and only if the wave equation enjoys the property of spectral stabilization in the sense that (15) holds for the solutions of the spectrally damped wave equation*

$$\begin{aligned} &\partial_{tt} z = \triangle z + \sum_{j\geq 1} \int_\omega \phi_j(x)^2\,dx \int_\Omega \partial_t z(t,x)\phi_j(x)\,dx\,\phi_j(x) \quad \text{in}\,(0,T)\times\Omega, \\ &z(t,x) = 0 \quad \text{on}\,(0,T)\times\partial\Omega, \\ &z(0,x) = z^0(x),\; z_t(0,x) = z^1(x) \quad \text{in}\,\Omega. \end{aligned} \tag{16}$$

For the system (16), the energy dissipation law is given by

$$E'(t) = -\sum_{j\geq 1} \int_\omega \phi_j(x)^2\,dx \left| \int_\Omega \partial_t z(t,x)\phi_j(x)\,dx \right|^2, \tag{17}$$

while for (14) it was given by

$$E'(t) = -\int_\omega |\partial_t z(t,x)|^2\,dx. \tag{18}$$

The nature of the feedback mechanism in (16), distributed everywhere in the domain, and acting on each spectral component of the system separately, mimics the actual meaning of the spectral observability inequality.

Note however that one cannot guarantee that the optimal choice of the observation set for spectral observability ensures the optimality at the level of spectral stabilization. This is so since, as it is well known, there is no one-to-one correspondence between the observability constant and the exponential decay rate. Actually, these two quantities are only clearly related in the limit as the amplitude of the feedback tends to zero ([3, 5]).

Thus, even if, for a given specific subdomain ω, spectral observability and the spectral stabilization result above are equivalent properties, this equivalence cannot be employed to transfer to the stabilization frame our previous results on the optimal location and design of sensors for spectral observability.

3 Proof of the spectral controllability result

First of all, let us prove that the spectral observability inequality (7) is equivalent to the fact that there exists $C_1 > 0$ such that

$$C_1\|(p^0,p^1)\|^2_{L^2\times H^{-1}} \leq \sum_{j\geq 1} \int_0^T |p_j(t)|^2\,dt \int_\omega \phi_j(x)^2\,dx, \tag{19}$$

for all $(p^0,p^1) \in L^2(\Omega)\times H^{-1}(\Omega)$, where p, given by (5), solves (3).

Indeed, if the spectral observability inequality is satisfied, then the left-hand side inequality of (19) is equivalent to the fact that

$$\|(p^0,p^1)\|^2_{L^2\times H^{-1}} \leq C \sum_{j\geq 1} \int_0^T |p_j(t)|^2\,dt, \tag{20}$$

which is obvious by using the Fourier expansion of the solutions and the expression of the Fourier components $p_j(t)$ (given by (5)).

Conversely, if (19) holds true, then, by considering solutions that only involve one Fourier mode, we infer that the spectral observability inequality (7) is satisfied.

Note that, actually, the converse inequality of (19) is always true, so that we have

$$C_1\|(p^0,p^1)\|^2_{L^2\times H^{-1}} \leq \sum_{j\geq 1} \int_0^T |p_j(t)|^2\,dt \int_\omega \phi_j(x)^2\,dx \leq C_2\|(p^0,p^1)\|^2_{L^2\times H^{-1}}, \tag{21}$$

for all solutions of (3), where $C_2 > 0$ is some positive constant. Indeed, the upper bound in (21) is a direct consequence of the fact that, for $(p^0,p^1) \in L^2(\Omega)\times H^{-1}(\Omega)$, the solution p lies in $C([0,T];L^2(\Omega))\cap C^1([0,T];H^{-1}(\Omega))$, with linear continuous dependence on the initial data (p^0,p^1). Furthermore, $(\phi_j)_{j\geq 1}$ being an orthonormal basis of $L^2(\Omega)$, we have $\int_\omega \phi_j(x)^2\,dx \leq 1$ for every $j \geq 1$.

Let us now show that this spectral observability inequality (20) is equivalent to spectral controllability as stated in Theorem 1.

First we observe that, as it is classical, if the spectral controllability property holds with the continuity estimate (12) on the controls, then spectral observability inequality (20) also holds true. To see this it is sufficient to multiply by the adjoint state p in the controlled equation (11) and to integrate by parts.

The fact that spectral observability implies spectral controllability can be shown to hold by a standard duality argument, as in the usual HUM method (see [7]). Let us briefly describe it.

Setting $z = \begin{pmatrix} y \\ \partial_t y \end{pmatrix}$, the wave equation (11) can be written in the more abstract form

$$\partial_t z = Az + Bf, \tag{22}$$

with the unbounded operator A defined by

$$A = \begin{pmatrix} 0 & \mathrm{id} \\ \triangle & 0 \end{pmatrix} \qquad \text{on} \quad D(A) = H_0^1(\Omega) \times L^2(\Omega),$$

and the (bounded) control operator B defined by

$$B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \qquad B_2 f = \sum_{j\geq 1} f_j b_j, \qquad b_j = \left(\int_\omega \phi_j(x)^2\, dx \right) \phi_j,$$

where $f = (f_j)_{j\geq 1}$.

Denoting by $S(t)$ the C_0 semi-group generated by A, it is well known (see, e.g., [13]) that the system (22) is exactly null controllable in time T if and only if one has the observability inequality

$$\int_0^T \|B^* S(T-t)^* \psi\|^2\, dt \geq C \|S(T)^* \psi\|^2,$$

for some constant $C > 0$. We do not provide all details, that are standard. Using the orthonormal basis $(\phi_j)_{j\geq 1}$, the operator B_2 is represented by an infinite-dimensional matrix, which is diagonal and whose j^{th} diagonal term is $\int_\omega \phi_j(x)^2\, dx$. Thanks to that framework, it is now easy to see that the exact null controllability property for (11) is equivalent to the observability inequality (19).

Therefore, at this step, we have proved that the exact null controllability property for (11) is equivalent to the spectral observability inequality (7).

It remains to build the spectral controls and to derive the estimate (12). We consider the following quadratic functional defined over the class of solutions p of (3):

$$K(p^0, p^1) = \frac{1}{2} \sum_{j\geq 1} \int_\omega \phi_j(x)^2\, dx \int_0^T \left| \int_\Omega p(t,x)\phi_j(x)\, dx \right|^2 dt + \int_\Omega p^0(x) y^1(x)\, dx - \langle p^1, y^0 \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}.$$

Under (21), the functional K is continuous, strictly convex (it is actually quadratic) and coercive in the Hilbert space $L^2(\Omega) \times H^{-1}(\Omega)$. Hence it has a minimum at a unique point $(\bar{p}^0, \bar{p}^1)$. Let $\bar{p}$ be the corresponding solution of (3). The Euler-Lagrange equations associated to the minimisation of this functional are

$$\langle DK(\bar{p}^0, \bar{p}^1), (q^0, q^1) \rangle = 0,$$

for any other choice of the initial data (q^0, q^1) of the adjoint system, that is,

$$\sum_{j\geq 1} \int_\omega \phi_j(x)^2\, dx \int_0^T \int_\Omega \bar{p}(t,x)\phi_j(x)\, dx \int_\Omega q(t,x)\phi_j(x)\, dx\, dt + \int_\Omega q^0(x) y^1(x)\, dx - \langle q^1, y^0 \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0.$$

It is then easy to see that the appropriate controls are given by

$$f_j(t) = \int_\Omega \bar{p}(t,x)\phi_j(x)\, dx,$$

for every $j \geq 1$. The estimate (12) follows from (21).

The theorem is proved.

4 Proof of the spectral stabilization

As indicated in the statement of Theorem 2 the energy dissipation law satisfied by system (16) is of the form (18). Thus, in order to achieve the exponential decay rate it is sufficient to prove that the solutions of the dissipated system (16) satisfy the spectral observability inequality

$$E(0) \leq C \sum_{j\geq 1} \int_\omega \phi_j(x)^2\, dx \left| \int_\Omega \partial_t z(t,x)\phi_j(x)\, dx \right|^2. \tag{23}$$

Indeed, combining (18) and (23) is is easy to see that the exists $0 < c < 1$ such that

$$E(T) \leq cE(0)$$

for every solution of the dissipated system (16). And this, together with the semigroup property, yields the exponential stabilization property (15).

Thus, it is sufficient to sow that (23) holds. However, as observed in [4], (23) holds for the damped system, if and only if it holds for the conservative one (3):

$$E(0) \leq C \sum_{j\geq 1} \int_\omega \phi_j(x)^2\, dx \left| \int_\Omega \partial_t p(t,x)\phi_j(x)\, dx \right|^2. \tag{24}$$

And this is a consequence of (19) applied to $\partial_t p$ which, whenever p is a finite-energy solution of the conservative wave equation, is also a solution with data in $L^2(\Omega) \times H^{-1}(\Omega)$.

5 Generalizations and further comments

In this section, we provide several extensions of our main results, and some further comments.

5.1 An abstract result generalizing Theorem 1

The result stated in Theorem 1 holds actually in a more general setting. This is the object of the following result. Several consequences and new examples are then provided in the sequel.

Let X and Y be two Hilbert spaces with inner products denoted $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$ and the induced norms denoted $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let W be a subspace of X. We denote by i_W the canonical injection from W to X. Notice that, endowed with the inner product $\langle\cdot,\cdot\rangle_W$ defined by

$$\langle x, y\rangle_W = \langle i_W(x), i_W(y)\rangle_X,$$

one sees easily that W is a Hilbert space, closed in X.

Denoting by P_W the orthogonal projector from X onto W, there holds $P_W i_W = \mathrm{id}_W$ and $i_W P_W$ is an orthogonal projection operator in X.

Introduce the Hilbert space $U = \ell^2(\mathbb{C})$. In the sequel, the spaces U and X will be identified to their dual space.

Theorem 3. *Let A be the generator of a diagonal[2] C^0 (conservative) group of isometries $\{S_t\}_{t\in\mathbb{R}}$ on X, with purely imaginary eigenvalues $(\lambda_j)_{j\geq 1}$ and with corresponding orthonormal basis of eigenvectors $(\phi_j)_{j\geq 1}$. Let $C \in \mathcal{L}(W,Y)$ be such that*

$$\inf_{j\geq 1} \|C i_W P_W \phi_j\|_Y > 0, \tag{25}$$

Assume moreover that

$$\langle i_W P_W \phi_j, \phi_k\rangle_X = \delta_{jk} \tag{26}$$

for every nonzero integers j and k, where δ_{jk} denotes the Kronecker delta.

[2] It means that

$$Az = \sum_{j\geq 1} \lambda_j \langle z, \phi_j\rangle_X \phi_j \qquad \left(\text{or equivalently } S_t z = \sum_{j\geq 1} \mathrm{e}^{\lambda_j t}\langle z, \phi_j\rangle_X \phi_j\right)$$

for all $z \in \mathcal{D}(A)$.

Let $p > 0$ and let $B \in \mathcal{L}(U, X)$ be defined by

$$Bu = \sum_{j=1}^{+\infty} u_j \|CP_W\phi_j\|_Y^p i_W P_W \phi_j \tag{27}$$

for every $u \in U$.

Then the abstract system

$$\partial_t z = Az + Bu,\ t \in (0, T)$$
$$z(0) = z_0 \in X$$

is exactly controllable in any time $T > 0$ with a control function $u^T(\cdot)$ satisfying

$$\|u^T(\cdot)\|_{L^2(0,T;U)} \leq \frac{\|z_0\|_X}{T \inf_{j\geq 1} \|CP_W\phi_j\|_Y^{2p}}.$$

P r o o f . The proof being very close to the one of Theorem 1, we provide only a sketch.

Introduce the controllability Gramian operator (also called "HUM" operator) $G_T \in \mathcal{L}(X)$, defined by

$$G_T = \int_0^T S_t BB^* S_t^* \, dt.$$

For all $u \in U$ and $z \in X$, one has

$$\begin{aligned}
\langle Bu, z\rangle_X &= \sum_{j\geq 1} u_j \|CP_W\phi_j\|_Y^p \langle i_W P_W \phi_j, z\rangle_X \\
&= \sum_{j\geq 1} u_j \|CP_W\phi_j\|_Y^p \langle i_W P_W \phi_j, i_W P_W z\rangle_X \\
&= \sum_{j\geq 1} u_j \|CP_W\phi_j\|_Y^p \langle \phi_j, i_W P_W z\rangle_X
\end{aligned}$$

by using that $i_W P_W$ is an orthogonal projection operator in $\mathcal{L}(X)$. One thus easily infers that the operator $B^* \in \mathcal{L}(X, U)$ is defined by

$$B^* z = \left(\|CP_W\phi_j\|_Y^p \langle \phi_j, i_W P_W z\rangle_X\right)_{j\geq 1}.$$

As a consequence, one computes

$$\begin{aligned}
G_T z &= \int_0^T S_t BB^* S_t^* z \, dt \\
&= \int_0^T \sum_{j\geq 1}\sum_{k\geq 1}\sum_{m\geq 1} e^{(\overline{\lambda_j}+\lambda_n)t} \|CP_W\phi_j\|_Y^{2p} \langle z, \phi_j\rangle_X \langle i_W P_W\phi_j, \phi_k\rangle_X \langle i_W P_W \phi_j, \phi_m\rangle_X \phi_j \, dt \\
&= T \sum_{j\geq 1} \|CP_W\phi_j\|_Y^{2p} \langle z, \phi_j\rangle_X \phi_j
\end{aligned}$$

for every $z \in X$, by using in particular that

$$S_t^* z = \sum_{j\geq 1} e^{\overline{\lambda_j} t} \langle z, \phi_j\rangle_X \phi_j.$$

From this formula, we infer that

$$\|G_T z\|_X \geq T \|z\|_X \inf_{j\geq 1} \|CP_W\phi_j\|_Y^{2p},$$

and the conclusion follows by using that a pair (A, B) is exactly controllable in time T if and only if G_T is a strictly positive operator. In that case, the control cost (corresponding to the $L^2(0, T; U)$ norm of the associated control) coincides with $\|G_T^{-1}\|_{\mathcal{L}(X)}$. □

Let us roughly comment on this abstract result and the examples that it covers. The following remarks are in order.

The wave equation. The example investigated in Theorem 1 for the wave equation is a particular case of the theorem above. Indeed, it suffices to choose $X = H_0^1(\Omega) \times L^2(\Omega)$, $Y = W = \{0_{H_0^1(\Omega)}\} \times L^2(\Omega)$, and

$$A = \begin{pmatrix} 0 & \mathrm{Id} \\ \triangle & 0 \end{pmatrix} \qquad \text{and} \qquad C \begin{pmatrix} 0 \\ \eta \end{pmatrix} = \eta \chi_\omega,$$

with $\eta \in L^2(\Omega)$ and χ_ω, the characteristic functions of ω. Thus, Theorem 3 applies, leading to the same conclusion as Theorem 1.

Schrödinger equations. The same results apply to a broad class of conservative semigroups, such as the Schrödinger equation. The statement in this case would be that, under the spectral observability inequality, which does not depend on whether we are considering the wave or the Schrödinger equation, the same spectral controllability and stabilization results hold for the Schrödinger equation too.

Boundary observation. The fact that the results of this paper apply in a wide class of abstract problems implies that they also apply to the boundary observability of the wave equation and to its spectral counterpart. Spectral boundary observability leads to a property of spectral controllability, in which the control is distributed everywhere in the domain, weighted by the observed normal derivative instead of the weights $\int_\omega \phi_j(x)^2\, dx$ as above.

It is worth underlying that, contrarily to the classical context of deterministic observability inequalities, that lead to boundary controllability results, in the present setting of spectral boundary observability, the control analog involves controls distributed everywhere in the domain. This is another manifestation of the important gap between spectral observability and the classical one satisfied by all solutions of the evolution model.

5.2 Additional comments and perspectives

We end this article by presenting a series of open problems and perspectives.

Weaker spectral inequalities. Let us assume that (7) is not satisfied. In [12] several examples are provided where this occurs. It is for instance the case when Ω is a disk and ω is any proper subset included in the interior of Ω. Then, due to the whispering gallery phenomenon, a subsequence of the weight factors in the spectral control problem $\int_\omega \phi_j(x)^2\, dx$ tends to zero as j tends to infinity. To simplify the notation, assume that the whole sequence $\int_\omega \phi_j(x)^2 dx$ tends to zero. In this case, of course, the equivalence property (21) fails and then the functional to be minimized for the computation of the control is not coercive in $L^2(\Omega) \times H^{-1}(\Omega)$ but rather in a weaker space, in which the Fourier components are weighted by the vanishing weights $\int_\omega \phi_j(x)^2 dx$. Accordingly, the functional has to be defined and minimized in a weaker space and therefore the data to be controlled have to be taken in a space of smoother data. Roughly, the Fourier coefficients of (y^0, y^1), rather than being in $h^1 \times \ell^2$, need to satisfy a further summability condition, weighted by the factors $(\int_\omega \phi_j(x)^2\, dx)^{-1}$ that tend to infinity. Note however that each of the weights is finite since the eigenfunctions ϕ_j may not vanish in ω because of the classical unique continuation property of elliptic equations. Alternatively, we may control all initial data of finite energy, but the ℓ^2 summability condition is not fulfilled by the controls f_j but rather by the weighted ones $(\int_\omega \phi_j(x)^2\, dx)^{1/2} f_j$.

Control in infinite time. In Sect. 2.1, we have recalled some results of the paper [12], in which it was shown that the spectral observability inequality can also be recast as property of observability of the wave equation in infinite time (see (8)). This allows also to motivate the fact that, at the control level, we come out with a result in which the control is distributed everywhere in the domain Ω as shown in Theorem 1. The heuristic argument is as follows. When observing the wave equation over a very long time interval $[0, T]$, the observed quantity $\int_0^T \int_\omega p(t,x)^2\, dx\, dt$ can be written, roughly, as

$$\int_0^T \int_\omega p(t,x)^2\, dx\, dt \sim \sum_{n=1}^{N} \int_{(n-1)T^0}^{nT^0} \int_\omega p(t,x)^2\, dx\, dt,$$

where $T_0 > 0$ is fixed. But each term $\int_{(n-1)T^0}^{nT^0} \int_\omega p(t,x)^2\, dx\, dt$ of the sum can be rewritten as the observation of p^n over the interval $[0, T_0]$, where p^n is the solution of the wave equation starting from the values $(p(nT^0), p_t(nT^0))$ at time $t = 0$.

The presence of N terms in the observed quantity leads to N control functions. The observation is done in $\omega \times (0, T^0)$ for p^n, whose data are those of the solution p at T^0, having experienced the propagation along bicharacteristic rays during the time interval, nT^0. Thus, the observed quantity in which N terms accumulate can also be interpreted as the superposition of N observations in the interval $(0, T^0)$ but in the N domains ω^n obtained by transporting ω along the bicharacteristic flow (a concept that, of course, would need to be made precise) rather than all being done in ω.

This interpretation is not rigorous, and further analysis would be required to explain why, at the level of controllability, the spectral observability, which can be viewed as an observability inequality in infinite time, can be interpreted as the spectral controllability property above. As we shall see below, things are more clear from the point of view of the exponential stabilization by means of damping.

Optimal spectral stabilization. As indicated above, the property of spectral observability is equivalent to a property of spectral stabilization. However, as indicated, the issue of the optimal spectral stabilization is far from being clear. This is due to the fact that, in particular, the generator of the dissipative semigroup is neither self-adjoint nor skew-adjoint. Accordingly, the spectral decomposition of the dissipated evolution is not as clear as for the conservative dynamics.

There is a rich literature on the optimal location and choice of the damping for the classical velocity damping mechanism even if the number of concluding results is rather limited. It would be interesting to explore whether the system with spectral damping we have introduced is better behaved at this respect.

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