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QUASI-OPTIMAL ROBUST STABILIZATION OF CONTROL SYSTEMS*

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Abstract. In this paper, we investigate the problem of semiglobal minimal time robust stabilization of analytic control systems with controls entering linearly, by means of a hybrid state feedback law. It is shown that in the absence of minimal time singular trajectories, the solutions of the closed-loop system converge to the origin in quasi-minimal time (for a given bound on the controller) with a robustness property with respect to small measurement noise, external disturbances, and actuator noise.

Key words. hybrid feedback, robust stabilization, measurement errors, actuator noise, external disturbances, optimal control, singular trajectory, sub-Riemannian geometry

AMS subject classifications. 93B52, 93D15

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1. Introduction.

Let m and n be two positive integers. Consider on $\mathbb{R}^n$ the control system

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{1.1}$$

where $f_1, \dots, f_m$ are analytic vector fields on $\mathbb{R}^n$, and where the control function $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$ satisfies the constraint

$$\sum_{i=1}^{m} u_i(t)^2 \leqslant 1. \tag{1.2}$$

All results of this paper still hold on a Riemannian analytic manifold of dimension n, which is connected and complete. However, for the sake of simplicity, our results are stated in $\mathbb{R}^n$. Let $\bar{x} \in \mathbb{R}^n$. The system (1.1), together with the constraint (1.2), is said to be *globally asymptotically stabilizable* at the point $\bar{x}$ if, for each point $x \in \mathbb{R}^n$, there exists a control law satisfying the constraint (1.2) such that the solution of (1.1) associated to this control law with the initial condition x tends to $\bar{x}$ as t tends to $+\infty$.

This asymptotic stabilization problem has a long history and has been widely investigated. Note that, due to *Brockett's condition* [16, Theorem 1 (iii)], if $m < n$, and if the vector fields $f_1, \dots, f_m$ are independent, then there does not exist any continuous stabilizing feedback law for (1.1). However several control laws have been derived for such control systems (see, for instance, [8, 29] and the references therein).

The *robust asymptotic stabilization problem* is under current and active research. Many notions of controllers have been introduced to treat this problem, such as discontinuous sampling feedbacks [19, 45], time varying control laws [20, 21, 33, 34],

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patchy feedbacks (as in [5]), and SRS feedbacks [43], enjoying different robustness properties depending on the errors under consideration.

In the present paper, we consider feedback laws having both discrete and continuous components, which generate closed-loop systems with *hybrid* terms (see, for instance, [11, 49]). Such feedback appeared first in [37] to stabilize nonlinear systems having a priori no discrete state. They consist in defining a *switching strategy* between several smooth control laws defined on a partition of the state space. Many results on the stabilization problem of nonlinear systems by means of hybrid controllers have been recently established (see, for instance, [14, 22, 25, 26, 30, 32, 35, 53]). The notion of solution, connected with the robustness problem, is by now well defined in the hybrid context (see [25, 39], among others). Specific conditions for the optimization can be found in the literature (see, e.g., [9, 24]).

The strategy of our paper is to combine a minimal time controller that is smooth on a part of the state space, and other controllers defined on the complement of this part, so as to provide a *quasi-minimal time hybrid controller* by defining a switching strategy between all control laws. The resulting hybrid law enjoys a quasi-minimal time property, and robustness with respect to (small) measurement noise, actuator errors, and external disturbances.

More precisely, in a first step, we consider the minimal time problem for the system (1.1) with the constraint (1.2) of steering a point $x \in \mathbb{R}^n$ to the point $\bar{x}$. Note that this problem is solvable as soon as the *Lie algebra rank condition* holds for the m-tuple of vector fields $(f_1, \dots, f_m)$. Of course, in general, it is impossible to compute explicitly the minimal time feedback controllers for this problem. Moreover, Brockett's condition implies that such control laws are not smooth whenever $m < n$ and the vector fields $f_1, \dots, f_m$ are independent. This raises the problem of the regularity of optimal feedback laws. The literature on this subject is immense. In an analytic setting, the problem of determining the analytic regularity of the minimal time function has been investigated in among others, [47]. For systems of the form (1.1), it follows from [1, 2, 50] that the minimal time function to $\bar{x}$ is *subanalytic*, provided there does not exist any nontrivial singular minimal time trajectory starting from $\bar{x}$ (see [27, 28] for a general definition of subanalytic sets). This assumption holds generically for systems (1.1), whenever $m \geqslant 3$ (see [18]). In particular, this function is analytic outside a stratified submanifold $\mathcal{S}$ of $\mathbb{R}^n$ of codimension greater than or equal to 1 (see [48]). As a consequence, outside this submanifold it is possible to provide an analytic minimal time feedback controller for the system (1.1), (1.2). This optimal controller gives rise to trajectories never crossing again the singular set $\mathcal{S}$.

Note that the analytic context is used so as to ensure stratification properties, which do not hold a priori if the system is smooth only. These properties are related to the notion of an *o-minimal category* (see [23]).

In a neighborhood of $\mathcal{S}$, we prove the existence of a set of controllers steering the system (1.1), (1.2) outside of this neighborhood in small time.

Then, in order to achieve a minimal time robust stabilization procedure, using a hybrid feedback law, we define a suitable switching strategy (more precisely, a hysteresis) between the minimal time feedback controller and other controllers defined in a neighborhood of $\mathcal{S}$. The resulting hybrid system has the following property: If the state is close to the singular submanifold $\mathcal{S}$, the feedback controller will push the state far enough from $\mathcal{S}$, in small time; if the state is not too close to $\mathcal{S}$, then the feedback controller will steer the system to $\bar{x}$ in minimal time. Hence, the stabilization is quasi-optimal and is proved to enjoy robustness properties.

Note that we thus give an alternative solution, in the context of hybrid systems using hysteresis, to a conjecture of [15, Conjecture 1, p. 101] concerning patchy feedbacks for smooth control systems.[1]

In a previous paper [41], this program was achieved on the so-called Brockett system, for which $n = 3$, $m = 2$, and, denoting $x = (x_1, x_2, x_3)$,

$$f_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \quad f_2 = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}.$$

In this case, there does not exist any nontrivial singular trajectory, and the manifold $\mathcal{S}$ coincides with the axis $(0x_3)$. A simple explicit hybrid strategy was described. In contrast, in the present paper, we derive a general result that requires a countable number of components in the definition of the hysteresis hybrid feedback law.

The paper is organized as follows. In section 2, we first recall some facts about the minimal time problem for the system (1.1) with (1.2) and recall the definition of a singular trajectory. Then we give a notion of solution adapted to hybrid feedback laws and define the concept of stabilization via a minimal time hybrid feedback law. The main result, Theorem 2.10 in section 2.3, states that if there does not exist any nontrivial singular minimal time trajectory of (1.1), (1.2), starting from $\bar{x}$, then there exists a minimal time hybrid feedback law stabilizing semiglobally the point $\bar{x}$ for the system (1.1), (1.2). Section 2.4 describes the main ideas of the proof of the main result and, in particular, contains two key lemmas. Section 3 is then devoted to the detailed proof of Theorem 2.10 and gathers all technical aspects needed to deal with hybrid systems: the components of the hybrid feedback law and a switching strategy between both components are defined, and properties of the closed-loop system are investigated.

The results in this work were announced in [42].

2. Definitions and main result.

2.1. The minimal time problem. Consider the minimal time problem for the system (1.1) with the constraint (1.2).

Throughout the paper, we assume that the *Lie algebra rank condition* holds, that is, the Lie algebra spanned by the vector fields $f_1, \ldots, f_m$ is equal to $\mathbb{R}^n$, at every point x of $\mathbb{R}^n$.

It is well known that, under this condition, any two points of $\mathbb{R}^n$ can be joined by a minimal time trajectory of (1.1), (1.2).

Let $\bar{x} \in \mathbb{R}^n$ be fixed. We denote by $T_{\bar{x}}(x)$ the minimal time needed to steer the system (1.1) with the constraint (1.2) from a point $x \in \mathbb{R}^n$ to the point $\bar{x}$.

Remark 2.1. Obviously, the control function u associated to a minimal time trajectory of (1.1), (1.2), actually satisfies $\sum_{i=1}^{m} u_i^2 = 1$.

For $T > 0$, let $\mathcal{U}_T$ denote the (open) subset of $u(\cdot)$ in $L^\infty([0,T], \mathbb{R}^m)$ such that the solution of (1.1), starting from $\bar{x}$ and associated to a control $u(\cdot) \in \mathcal{U}_T$, is well defined on $[0,T]$. The mapping

$$\begin{array}{rccc} E_{\bar{x},T} : & \mathcal{U}_T & \longrightarrow & \mathbb{R}^n \\ & u(\cdot) & \longmapsto & x(T), \end{array}$$

[1]This conjecture on patchy feedbacks has been recently considered in [6]. In this preprint, written during the review process of the present work, the authors prove, using a penalization method, a general result on stabilization by means of patchy feedback of nonlinear control systems in quasi-minimal time.

which to a control $u(\cdot)$ associates the end-point $x(T)$ of the corresponding solution $x(\cdot)$ of (1.1) starting at $\bar{x}$, is called *end-point mapping* at the point $\bar{x}$, in time T; it is a smooth mapping.

DEFINITION 2.2. *A trajectory $x(\cdot)$ of (1.1), so that $x(0) = \bar{x}$, is said to be singular on $[0, T]$ if its associated control $u(\cdot)$ is a singular point of the end-point mapping $E_{\bar{x},T}$ (i.e., if the Fréchet derivative of $E_{\bar{x},T}$ at $u(\cdot)$ is not onto). The control $u(\cdot)$ is said to be singular.*

Remark 2.3. If $x(\cdot)$ is singular on $[0, T]$, then it is singular on $[t_0, t_1]$ for all $t_0, t_1 \in [0, T]$ such that $t_0 < t_1$.

Remark 2.4. It is a standard fact that the minimal time control problem for the system (1.1) with the constraint (1.2) is equivalent to the sub-Riemannian problem associated to the m-tuple of vector fields $(f_1, \ldots, f_m)$ (see [10] for a general definition of a sub-Riemannian distance). In this context, there holds $T_{\bar{x}}(x) = d_{SR}(\bar{x}, x)$, where d_{SR} is the sub-Riemannian distance. This implies that the functions $T_{\bar{x}}(\cdot)$ and $d_{SR}(\bar{x}, \cdot)$ share the same regularity properties. In particular, the function $T_{\bar{x}}(\cdot)$ is continuous.

2.2. Class of controllers and notion of hybrid solution. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be defined by $f(x, u) = \sum_{i=1}^m u_i f_i(x)$. The system (1.1) is written as

$$\dot{x}(t) = f(x(t), u(t)). \tag{2.1}$$

Let $\bar{x} \in \mathbb{R}^n$ be fixed.

The controllers under consideration in this paper depend on the continuous state $x \in \mathbb{R}^n$ and also on a discrete variable $s_d \in \mathcal{N}$, where $\mathcal{N}$ is a nonempty subset of $\mathbb{N}$. According to the concept of a hybrid system of [25], we introduce the following definition.

DEFINITION 2.5. *A hybrid feedback is a 4-tuple (C, D, k, k_d), where*
- *C and D are subsets of $\mathbb{R}^n \times \mathcal{N}$;*
- *$k : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$ is a function;*
- *$k_d : \mathbb{R}^n \times \mathcal{N} \to \mathcal{N}$ is a function.*

The sets C and D are, respectively, called the controlled continuous evolution set *and the* controlled discrete evolution set.

We next recall the notion of robustness to small noise (see [46]). Consider two functions e and d satisfying the following *regularity assumptions*:

$$\begin{aligned} &e(\cdot,\cdot), d(\cdot,\cdot) \in L^\infty_{loc}(\mathbb{R}^n \times [0, +\infty); \mathbb{R}^n), \\ &e(\cdot, t), d(\cdot, t) \in C^0(\mathbb{R}^n, \mathbb{R}^n) \ \forall t \in [0, +\infty). \end{aligned} \tag{2.2}$$

We introduce these functions as a measurement noise e and an external disturbance d.

Formally, the k-component of a hybrid feedback (k, k_d, C, D) governs the differential equation

$$\dot{x} = f(x, k(x + e)) + d \ \ \forall (x, s_d) \in C,$$

whereas the k_d-component governs the jump equation

$$s_d^+ = k_d(x, s_d) \ \ \forall (x, s_d) \in D.$$

The set C indicates where in the state space flow may occur, while the set D indicates where in the state space jumps may occur. The combination of this flow equation and this jump equation, under the perturbations e and d, is a perturbed hybrid system

$\mathcal{H}_{(e,d)}$, as considered, e.g., in [26]. We next provide a precise definition of the notion of solutions considered here.

This concept is well studied in the literature (see, e.g., [11, 14, 31, 38, 39, 49]). Here, we consider the notion of solution given in [25, 26].

DEFINITION 2.6. *Let* $S = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$, *where* $J \in \mathbb{N} \cup \{+\infty\}$ *and* $(x_0, s_0) \in \mathbb{R}^n \times \mathcal{N}$. *The domain* S *is said to be a hybrid time domain. A map* $(x, s_d) : S \to \mathbb{R}^n \times \mathcal{N}$ *is said to be a solution of* $\mathcal{H}_{(e,d)}$ *with the initial condition* (x_0, s_0) *if*

- *the map* x *is continuous on* S;
- *for every* j, $0 \leqslant j \leqslant J-1$, *the map* $x : t \in (t_j, t_{j+1}) \mapsto x(t, j)$ *is absolutely continuous;*
- *for every* j, $0 \leqslant j \leqslant J-1$, *and almost every* $t \geqslant 0$, $(t, j) \in S$, *we have*

$$(x(t,j) + e(x(t,j), t), s_d(t,j)) \in C \tag{2.3}$$

and

$$\dot{x}(t,j) = f(x(t), k(x(t,j) + e(x(t,j), t), s_d(t,j))) + d(x(t,j), t), \tag{2.4}$$

$$\dot{s}_d(t,j) = 0 \tag{2.5}$$

(where the dot stands for the derivative with respect to the time variable t*);*

- *for every* $(t, j) \in S$, $(t, j+1) \in S$, *we have*

$$(x(t,j) + e(x(t,j), t), s_d(t,j)) \in D \tag{2.6}$$

and

$$x(t, j+1) = x(t, j), \tag{2.7}$$

$$s_d(t, j+1) = k_d(x(t,j) + e(x(t,j), t), s_d(t,j)); \tag{2.8}$$

- $(x(0,0), s_d(0,0)) = (x_0, s_0)$.

In this context, we next define the concept of stabilization of (2.1) by a minimal time hybrid feedback law sharing a robustness property with respect to measurement noise and external disturbances. The usual Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$, and the open ball centered at $\bar{x}$ with radius R is denoted $B(\bar{x}, R)$. Recall that a function of class $\mathcal{K}_\infty$ is a function δ: $[0, +\infty) \to [0, +\infty)$ which is continuous, increasing, and satisfies $\delta(0) = 0$ and $\lim_{R \to +\infty} \delta(R) = +\infty$.

As usual, the system is said to be complete if all solutions are maximally defined in $[0, +\infty)$ (see, e.g., [7]). More precisely, we have the following definition.

DEFINITION 2.7. *Let* $\rho : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function satisfying*

$$\rho(x) > 0 \ \forall x \neq \bar{x}. \tag{2.9}$$

We say that the completeness assumption *for* ρ *holds if, for all* (e, d) *satisfying the regularity assumptions* (2.2)*, and such that*

$$\sup_{[0,+\infty)} |e(x, \cdot)| \leqslant \rho(x), \ \operatorname{esssup}_{[0,+\infty)} |d(x, \cdot)| \leqslant \rho(x) \ \forall x \in \mathbb{R}^n, \tag{2.10}$$

for every $(x_0, s_0) \in \mathbb{R}^n \times \mathcal{N}$, *there exists a maximal solution on* $[0, +\infty)$ *of* $\mathcal{H}_{(e,d)}$ *starting from* (x_0, s_0).

Roughly speaking, the finite time convergence property means that all solutions reach $\bar{x}$ within finite time. A precise definition of this concept follows.

DEFINITION 2.8. *We say that the* uniform finite time convergence property *holds if there exists a continuous function* $\rho : \mathbb{R}^n \to \mathbb{R}$ *satisfying* (2.9)*, such that the completeness assumption for* ρ *holds, and if there exists a function* $\delta : [0, +\infty) \to [0, +\infty)$ *of class* $\mathcal{K}_\infty$ *such that, for every* $R > 0$*, there exists* $\tau = \tau(R) > 0$*, so that for all functions* e, d *satisfying the regularity assumptions* (2.2) *and inequalities* (2.10) *for this function* ρ*, for every* $x_0 \in B(\bar{x}, R)$*, and every* $s_0 \in \mathcal{N}$*, the maximal solution* (x, s_d) *of* $\mathcal{H}_{(e,d)}$ *starting from* (x_0, s_0) *satisfies*

$$|x(t,j) - \bar{x}| \leqslant \delta(R) \;\forall t \geqslant 0, \;(t,j) \in S, \tag{2.11}$$

and

$$x(t,j) = \bar{x} \;\forall t \geqslant \tau, \;(t,j) \in S. \tag{2.12}$$

We are now in a position to introduce our main definition. It deals with closed-loop systems whose trajectories converge to the equilibrium within quasi-minimal time and with a robustness property with respect to measurement noise and external disturbances.

DEFINITION 2.9. *The point* $\bar{x}$ *is said to be a* semiglobally quasi-minimal time robustly stabilizable equilibrium *for the system* (2.1) *if, for every* $\varepsilon > 0$ *and every compact subset* $K \subset \mathbb{R}^n$*, there exists a hybrid feedback law* (C, D, k, k_d) *satisfying the constraint*

$$\|k(x, s_d)\| \leqslant 1, \tag{2.13}$$

where $\|\cdot\|$ *stands for the Euclidean norm in* $\mathbb{R}^m$*, such that*

- *the uniform finite time convergence property holds;*
- *there exists a continuous function* $\rho_{\varepsilon,K} : \mathbb{R}^n \to \mathbb{R}$ *satisfying* (2.9) *for* $\rho = \rho_{\varepsilon,K}$*, such that, for every* $(x_0, s_0) \in K \times \mathcal{N}$*, all functions* e, d *satisfying the regularity assumptions* (2.2) *and inequalities* (2.10) *for* $\rho = \rho_{\varepsilon,K}$*, the maximal solution of* $\mathcal{H}_{(e,d)}$ *starting from* (x_0, s_0) *reaches* $\bar{x}$ *within time* $T_{\bar{x}}(x_0) + \varepsilon$*, where* $T_{\bar{x}}(x_0)$ *denotes the minimal time to steer the system* (2.1) *from* x_0 *to* $\bar{x}$*, under the constraint* $\|u\| \leqslant 1$.

2.3. Main result. The main result of this article is the following.

THEOREM 2.10. *Let* $\bar{x} \in \mathbb{R}^n$*. If there exists no nontrivial minimal time singular trajectory of* (1.1), (1.2)*, starting from* $\bar{x}$*, then* $\bar{x}$ *is a semiglobally quasi-minimal time robustly stabilizable equilibrium for the system* (1.1)*, under the constraint* (1.2).

Remark 2.11. The problem of global quasi-minimal time robust stabilization (i.e., $K = \mathbb{R}^n$ in Definition 2.9) cannot be achieved a priori because measurement noise may then accumulate and slow down the solution reaching $\bar{x}$ (compare with [15]).

Remark 2.12. The assumption of the absence of nontrivial singular minimizing trajectory is crucial. Notice the following facts, which show the relevance of this assumption:

- If $m \geqslant n$ and if the vector fields $f_1, \dots, f_m$, are everywhere linearly independent, then there exists no singular trajectory. In this case, we are actually in the framework of Riemannian geometry (see Remark 2.4).
- Let $\mathcal{F}_m$ be the set of m-tuples of linearly independent vector fields $(f_1, \dots, f_m)$, endowed with the C^∞ Whitney topology. If $m \geqslant 3$, there exists an open dense subset of $\mathcal{F}_m$, such that any control system of the form (1.1), associated to a m-tuple of this subset, admits no nontrivial singular minimizing trajectory

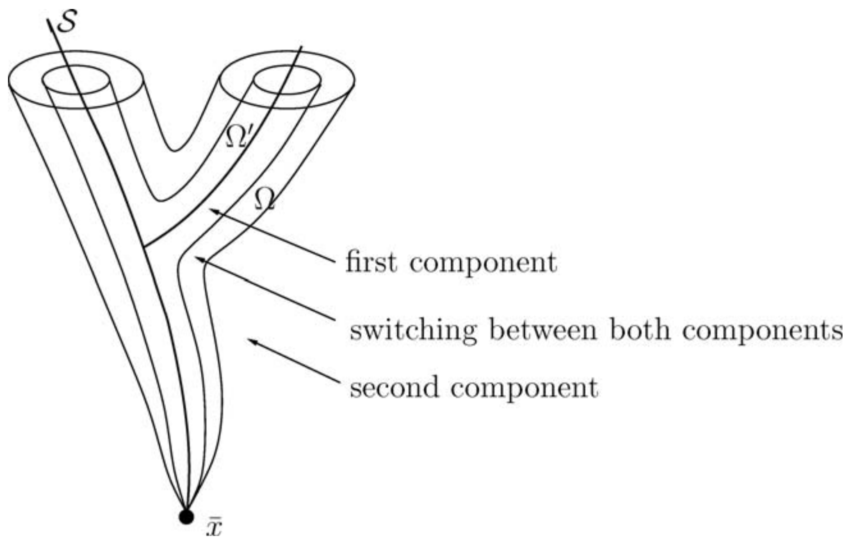

FIG. 2.1. *Switching strategy.*

(see [17, 18]; see also [2] for the existence of a dense set only). Hence generically the conclusion of Theorem 2.10 holds without assuming the absence of nontrivial singular minimizing trajectories.

- If there exist singular minimizing trajectories, then the conclusion on subanalyticity of the function T may fail (see [13, 50]), and we cannot a priori prove that the set $\mathcal{S}$ of singularities of T is a stratifiable manifold, which is the crucial fact in order to define a hybrid strategy.

2.4. Short description of the proof. The strategy of the proof of Theorem 2.10 is the following.

Under the assumption of the absence of nontrivial singular minimal time trajectory, the minimal time function $T_{\bar{x}}$ associated to the system (1.1), (1.2) is subanalytic, and hence is analytic outside a stratified submanifold $\mathcal{S}$ of $\mathbb{R}^n$, of codimension greater than or equal to one. Therefore, the corresponding minimal time feedback controller (further precisely defined in section 3.2.3) is continuous (even analytic) on $\mathbb{R}^n \setminus \mathcal{S}$ (see Figure 2.1). In a neighborhood of $\mathcal{S}$, it is therefore necessary to use other controllers, and then to define an adequate switching strategy.

More precisely, the proof of Theorem 2.10 relies on both of the following key lemmas.

LEMMA 2.13. *For every $\varepsilon > 0$, there exists a neighborhood Ω of $\mathcal{S}$ such that, for every stratum*[2] *M_i of $\mathcal{S}$, there exist a nonempty subset $\mathcal{N}_i$ of $\mathbb{N}$, a locally finite family $(\Omega_{i,p})_{p\in\mathcal{N}_i}$ of open subsets of Ω, a sequence of smooth controllers $u_{i,p}$ defined in a neighborhood of $\Omega_{i,p}$, satisfying $\|u_{i,p}\| \leqslant 1$, and there exists a continuous function $\rho_{i,p} : \mathbb{R}^n \to [0, +\infty)$ satisfying $\rho_{i,p}(x) > 0$ whenever $x \neq \bar{x}$, such that every solution of*

$$\dot{x}(t) = f(x(t), u_{i,p}(x(t) + e(x(t), t))) + d(x(t), t), \tag{2.14}$$

[2] Since $\mathcal{S}$ is a stratified submanifold of $\mathbb{R}^n$ of codimension greater than or equal to one, there exists a partition $(M_i)_{i\in\mathbb{N}}$ of $\mathcal{S}$, where M_i is a stratum, i.e., a locally closed submanifold of $\mathbb{R}^n$. Recall that if $M_i \cap \partial M_j \neq \emptyset$, then $M_i \subset M_j$ and $\dim(M_i) < \dim(M_j)$.

where $e, d : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ *are two functions satisfying the regularity assumptions* (2.2) *and*

$$\sup_{[0,+\infty)}|e(x,\cdot)| \leqslant \rho_{i,p}(x), \quad \operatorname{esssup}_{[0,+\infty)}|d(x,\cdot)| \leqslant \rho_{i,p}(x), \tag{2.15}$$

starting from $\Omega_{i,p}$ *and maximally defined on* $[0, T)$, *leaves* Ω *within time* ε*; moreover, there exists a function* $\delta_{i,p}$ *of class* $\mathcal{K}_\infty$ *such that, for every* $R > 0$, *every such solution starting from* $\Omega_{i,p} \cap B(\bar{x}, R)$ *satisfies*

$$|x(t) - \bar{x}| \leqslant \delta_{i,p}(R) \quad \forall t \in [0, T). \tag{2.16}$$

According to this lemma, in a neighborhood Ω of $\mathcal{S}$, there exist controllers steering the system outside Ω. Moreover, since this neighborhood can be chosen arbitrarily thin, the time ε needed for its traversing is arbitrarily small.

Outside Ω, the optimal controller is analytic. The following lemma shows that this controller shares an invariance property; in brief, it gives rise to trajectories never again crossing the singular set $\mathcal{S}$.

LEMMA 2.14. *For every neighborhood* Ω *of* $\mathcal{S} \setminus \{\bar{x}\}$ *in* $\mathbb{R}^n$, *there exists a neighborhood* Ω' *of* $\mathcal{S} \setminus \{\bar{x}\}$ *in* $\mathbb{R}^n$, *satisfying*

$$\Omega' \subsetneq \operatorname{clos}(\Omega') \subsetneq \Omega, \tag{2.17}$$

such that every trajectory of the closed-loop system (1.1) *with the optimal controller, starting from any point* $x \in \mathbb{R}^n \setminus \Omega$, *reaches* $\bar{x}$ *in minimal time, and is contained in* $\mathbb{R}^n \setminus \Omega'$.

Finally, our hybrid strategy is the following. For every $\varepsilon > 0$, there exists a neighborhood Ω of the singular set $\mathcal{S}$, and there exist controllers which steer the system outside this neighborhood in time less than ε. Outside Ω, there exists a continuous controller giving rise to trajectories never again crossing $\mathcal{S}$ and joining $\bar{x}$ in minimal time.

It is therefore necessary to define an adequate switching strategy connecting both controllers (see Figure 2.1). This is achieved in the context of hybrid systems, using a hysteresis strategy. The first component consists of controllers which are defined in Ω, and whose existence is stated in Lemma 2.13. The second component of the hysteresis is defined by the optimal controller, outside Ω. Both components are united using an hysteresis, by adding a dynamical discrete variable s_d and using a hybrid feedback law. With this resulting hybrid controller, the time needed to join $\bar{x}$, from any point x_0 of $\mathbb{R}^n$, is less than $T_{\bar{x}}(x_0) + \varepsilon$.

The next section, devoted to the detailed proof of Theorem 2.10, is organized as follows.

The first component of the hysteresis is defined in section 3.1, and Lemma 2.13 is proved.

Section 3.2 concerns the definition and properties of the second component of the hysteresis, defined by the minimal time controller. In section 3.2.1, we recall how to compute minimal time trajectories of the system (1.1), (1.2), using the Pontryagin maximum principle. We then provide in section 3.2.2 a crucial result on the cut locus (Proposition 3.6). The optimal feedback controller is defined in section 3.2.3; basic facts on subanalytic functions are recalled, permitting us to define the singular set $\mathcal{S}$. Invariance properties of this optimal controller are then investigated: Lemma 2.14 is proved in section 3.2.4; robustness properties are given and proved in section 3.2.5.

The hybrid controller is then defined in section 3.3. A definition of a hybrid control system, and properties of solutions, are given in sections 3.3.1 and 3.3.2. A precise description of the switching strategy is provided in section 3.3.3. Theorem 2.10 is proved in section 3.4.

3. Proof of Theorem 2.10.

In what follows, let $\bar{x} \in \mathbb{R}^n$ be fixed.

3.1. The first component of the hysteresis.

The first component of the hysteresis consists of a set of controllers, defined in a neighborhood of $\mathcal{S}$, whose existence is stated in Lemma 2.13. Hereafter, we provide a proof of this lemma.

Proof of Lemma 2.13. First of all, recall that, on the one hand, the minimal time function coincides with the sub-Riemannian distance associated to the m-tuple $(f_1, \dots, f_m)$ (see Remark 2.4); on the other, since the Lie algebra rank condition holds, the topology defined by the sub-Riemannian distance d_{SR} coincides with the Euclidean topology of $\mathbb{R}^n$, and, since $\mathbb{R}^n$ is complete, any two points of $\mathbb{R}^n$ can be joined by a minimizing path (see [10]).

Let $\varepsilon > 0$ fixed. Since $\mathcal{S}$ is a stratified submanifold of $\mathbb{R}^n$, there exists a neighborhood Ω of $\mathcal{S}$ satisfying the following property: for every $y \in \mathcal{S}$, there exists $z \in \mathbb{R}^n \setminus \mathrm{clos}(\Omega)$ such that $d_{SR}(y, z) < \varepsilon$.

Consider a stratum M_i of $\mathcal{S}$. For every $y \in M_i$, let $z \in \mathbb{R}^n \setminus \mathrm{clos}(\Omega)$ such that $d_{SR}(y, z) < \varepsilon$. The Lie algebra rank condition implies that there exists an open-loop control $t \mapsto u_y(t)$, defined on $[0, T)$ for a $T > \varepsilon$, satisfying the constraint $\|u_y\| \leqslant 1$, such that the associated trajectory $x_y(\cdot)$ (which can be assumed to be one-to-one), solution of (1.1), starting from y, reaches z (and thus leaves $\mathrm{clos}(\Omega)$) within time ε. Using a density argument, the control u_y can, moreover, be chosen as a smooth function (see [10, Theorem 2.8, p. 21] for the proof of this statement). Since the trajectory is one-to-one, the open-loop control $t \mapsto u_y(t)$ can be considered as a feedback $t \mapsto u_y(x_y(t))$ along $x_y(\cdot)$. Consider a smooth extension of u_y on $\mathbb{R}^n$, still denoted u_y, satisfying the constraint $\|u_y(x)\| \leqslant 1$ for every $x \in \mathbb{R}^n$. By continuity, there exists a neighborhood Ω_y of y, and positive real numbers δ_y and ρ_y, such that every solution of

$$\dot{x}(t) = f(x(t), u_y(x(t) + e(x(t), t))) + d(x(t), t), \tag{3.1}$$

where $e, d : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ are two functions satisfying the regularity assumptions (2.2) and

$$\sup_{[0,+\infty)} |e(x, \cdot)| \leqslant \rho_y, \quad \mathrm{esssup}_{[0,+\infty)} |d(x, \cdot)| \leqslant \rho_y,$$

starting from Ω_y and maximally defined on $[0, T)$, leaves Ω within time ε; moreover,

$$|x(t) - \bar{x}| \leqslant \delta_y \ \forall t \in [0, T).$$

Repeat this construction for each $y \in M_i$.

Now, on the one hand, let $(y_p)_{p \in \mathcal{N}_i}$ be a sequence of points of M_i such that the family $(\Omega_{y_p})_{p \in \mathcal{N}_i}$ is a locally finite covering of M_i, where $\mathcal{N}_i$ is a subset of $\mathbb{N}$. Define $\Omega_{i,p} = \Omega_{y_p}$ and $u_{i,p} = u_{y_p}$.

On the other hand, the existence of a continuous function $\rho_{i,p} : \mathbb{R}^n \to [0, +\infty)$, satisfying $\rho_{i,p}(x) > 0$ whenever $x \neq x$, follows for the continuity of solutions with respect to disturbances. The existence of a function $\delta_{i,p}$ of class $\mathcal{K}_\infty$ such that (2.16) holds is obvious.

Repeat this construction for every stratum M_i of $\mathcal{S}$. Then the statement of the lemma follows. ∎

3.2. The second component of the hysteresis.

3.2.1. Computation of minimal time trajectories. Let $x_1 \in \mathbb{R}^n$, and let $x(\cdot)$ be a minimal time trajectory, associated to a control $u(\cdot)$, steering the system (1.1), (1.2), from $\bar{x}$ to x_1, in time $T = T_{\bar{x}}(x_1)$. According to Pontryagin's maximum principle (see [36]), the trajectory $x(\cdot)$ is the projection of an *extremal*, i.e., a triple $(x(\cdot), p(\cdot), u(\cdot))$ solution of the constrained Hamiltonian system

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t)),$$

$$H(x(t), p(t), p^0, u(t)) = \max_{\|v\| \leqslant 1} H(x(t), p(t), p^0, v),$$

almost everywhere on $[0, T]$, where

$$H(x, p, u) = \sum_{i=1}^{m} u_i \langle p, f_i(x) \rangle$$

is the *Hamiltonian* of the optimal control problem, and $p(\cdot)$ (called the *adjoint vector*) is an absolutely continuous mapping on $[0, T]$ such that $p(t) \in \mathbb{R}^n \setminus \{0\}$. Moreover, the function $t \mapsto \max_{\|v\| \leqslant 1} H(x(t), p(t), p^0, v)$ is Lipschitzian and everywhere constant on $[0, T]$. If this constant is not equal to zero, then the extremal is said to be *normal*; otherwise it is *abnormal*.

Remark 3.1. Any singular trajectory is the projection of an abnormal extremal, and conversely.

Controls associated to normal extremals can be computed as

$$u_i(t) = \frac{\langle p(t), f_i(x(t) \rangle}{\sqrt{\sum_{j=1}^{m} \langle p(t), f_j(x(t) \rangle^2}}, \quad i = 1, \dots, m. \tag{3.2}$$

Indeed, note that, by definition of normal extremals, the denominator of (3.2) cannot vanish. It follows that normal extremals are solutions of the Hamiltonian system

$$\dot{x}(t) = \frac{\partial H_1}{\partial p}(x(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H_1}{\partial x}(x(t), p(t)), \tag{3.3}$$

where

$$H_1(x, p) = \sqrt{\sum_{i=1}^{m} \langle p, f_i(x) \rangle^2}.$$

Notice that $H_1(x(t), p(t))$ is constant, nonzero, along each normal extremal. Since $p(0)$ is defined up to a multiplicative scalar, it is usual to normalize it so that $H_1(x(t), p(t)) = 1$. Hence, we introduce the set

$$X = \{p_0 \in \mathbb{R}^n \mid H_1(\bar{x}, p_0) = 1\}.$$

It is a submanifold of $\mathbb{R}^n$ of codimension one, since $\frac{\partial H_1}{\partial p}(\bar{x}, p_0) = \dot{x}(0) \neq 0$ (see [12] for a similar construction in the general case). There exists a connected open subset

U of $[0,+\infty) \times X$ such that, for every $(t^*, p_0) \in X$, the differential system (3.3) has a well-defined smooth solution on $[0, t^*]$ such that $x(0) = \bar{x}$ and $p(0) = p_0$.

DEFINITION 3.2. *The smooth mapping*

$$\begin{array}{rccc} \exp_{\bar{x}} : & U & \longrightarrow & \mathbb{R}^n \\ & (t, p_0) & \longmapsto & x(t), \end{array}$$

where $(x(\cdot), p(\cdot))$ *is the solution of the system* (3.3) *such that* $x(0) = \bar{x}$ *and* $p(0) = p_0 \in X$*, is called* exponential mapping *at the point* $\bar{x}$*.*

The exponential mapping parameterizes normal extremals. Note that the domain of $\exp_{\bar{x}}$ is a subset of $\mathbb{R} \times X$ which is locally diffeomorphic to $\mathbb{R}^n$ (since we are in the normal case).

DEFINITION 3.3. *A point* $x \in \exp_{\bar{x}}(U)$ *is said to be* conjugate *to* $\bar{x}$ *if it is a critical value of the mapping* $\exp_{\bar{x}}$*, i.e., if there exists* $(t_c, p_0) \in U$ *such that* $x = \exp_{\bar{x}}(t_c, p_0)$ *and the differential* $d\exp_{\bar{x}}(t_c, p_0)$ *is not onto. The* conjugate locus *of* $\bar{x}$*, denoted by* $\mathcal{C}(\bar{x})$*, is defined as the set of all points conjugate to* $\bar{x}$*.*

With the previous notation, define $\mathcal{C}_{min}(\bar{x})$ *as the set of points* $x \in \mathcal{C}(\bar{x})$ *such that the trajectory* $t \mapsto \exp_{\bar{x}}(t, p_0)$ *is minimizing between* $\bar{x}$ *and* x*.*

3.2.2. The cut locus. A standard definition is the following.

DEFINITION 3.4. *A point* $x \in \mathbb{R}^n$ *is not a* cut point *with respect to* $\bar{x}$ *if there exists a minimal time trajectory of* (1.1), (1.2)*, joining* $\bar{x}$ *to* x*, which is the strict restriction of a minimal time trajectory starting from* $\bar{x}$*. The* cut locus *of* $\bar{x}$*, denoted by* $\mathcal{L}(\bar{x})$*, is defined as the set of all cut points with respect to* $\bar{x}$*.*

In other words, a cut point is a point at which a minimal time trajectory ceases to be optimal.

Remark 3.5. In the analytic case, it follows from the theory of conjugate points that every nonsingular minimal time trajectory ceases to be minimizing beyond its first conjugate point (see, for instance, [3, 13]). Hence, if there exists no singular minimal time trajectory starting from $\bar{x}$, then $\mathcal{C}_{min}(\bar{x}) \subset \mathcal{L}(\bar{x})$.

The following result on the cut locus is crucial for the proof of Theorem 2.10.

PROPOSITION 3.6. *Assume that the vector fields* $f_1, \dots, f_m$ *are analytic, and that there exists no singular minimal time trajectory starting from* $\bar{x}$*. Then the set of points of* $\mathbb{R}^n$ *where the function* $T_{\bar{x}}(\cdot)$ *is not analytic is equal to the cut locus of* $\bar{x}$*, that is,*

$$\text{Sing } T_{\bar{x}}(\cdot) = \mathcal{L}(\bar{x}). \tag{3.4}$$

Remark 3.7. Under the previous assumptions, one can prove that the set of points of $\mathbb{R}^n$ where $T_{\bar{x}}(\cdot)$ is analytic is equal to the set of points where $T_{\bar{x}}(\cdot)$ is of class C^1.

Proof. Let $x \in \mathbb{R}^n$ so that $T_{\bar{x}}(\cdot)$ is analytic at x. Then there exists a neighborhood V of x in $\mathbb{R}^n$ such that $T_{\bar{x}}(\cdot)$ is analytic on V. Let us prove that $x \notin \mathcal{L}(\bar{x})$. It follows from the maximum principle and the Hamilton–Jacobi theory (see [36]) that, for every $y \in V$, there exists a unique minimal time trajectory joining $\bar{x}$ to y having, moreover, a normal extremal lift $(x(\cdot), p(\cdot), u(\cdot))$ satisfying

$$p(T_{\bar{x}}(y)) = \nabla T_{\bar{x}}(y)$$

(compare with [44, Proposition 2.3]). Set $U_1 = \exp_{\bar{x}}^{-1}(V)$. It follows easily from the Cauchy–Lipschitz theorem that the mapping $\exp_{\bar{x}}$ is an analytic diffeomorphism from U_1 into V. Hence, obviously, the point x is not in the cut locus of $\bar{x}$.

Conversely, let $x \notin \mathcal{L}(\bar{x})$. To prove that $T_{\bar{x}}(\cdot)$ is analytic at x, we need the following two lemmas.

LEMMA 3.8. *The point x is not conjugate to $\bar{x}$ and is joined from $\bar{x}$ by a unique minimal time trajectory.*

Proof of Lemma 3.8. From the assumption of the absence of a singular minimal time trajectory, there exists a nonsingular minimal time trajectory joining $\bar{x}$ to x. From Remark 3.5, the point x is not conjugate to $\bar{x}$.

By contradiction, suppose that x is joined from $\bar{x}$ by at least two minimal time trajectories. By assumption, these two trajectories must admit normal extremal lifts. Since the structure is analytic, their junction at the point x is necessarily not smooth. This implies that both trajectories lose their optimality at the point x (indeed if not, there would exist a nonsmooth normal extremal, which is absurd), and thus $x \in \mathcal{L}(\bar{x})$. This is a contradiction. ◻

LEMMA 3.9. *There exists a neighborhood V of x in $\mathbb{R}^n$ such that every point $y \in V$ is not conjugate to $\bar{x}$, and there exists a unique (nonsingular) minimal time trajectory joining $\bar{x}$ to y.*

Proof of Lemma 3.9. Let $p_0 \in X$ so that $x = \exp_{\bar{x}}(T_{\bar{x}}(x), p_0)$. Since x is not conjugate to $\bar{x}$, the exponential mapping $\exp_{\bar{x}}$ is a diffeomorphism from a neighborhood U_1 of $(T_{\bar{x}}(x), p_0)$ in U into a neighborhood V of x in $\mathbb{R}^n$. Set $U_2 = \exp_{\bar{x}}^{-1}(V)$.

Let us prove that $\exp_{\bar{x}}$ is proper from U_2 into V. We argue by contradiction and suppose that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of V converging towards x, such that for each integer n there exists $p_n \in X$, satisfying $(T_{\bar{x}}(x_n), p_n) \in U_2$ and $x_n = \exp_{\bar{x}}(T_{\bar{x}}(x_n), p_n)$, such that $(p_n)_{n\in\mathbb{N}}$ is not bounded. It then follows from [51, Lemmas 4.8 and 4.9] (see also [52, Fact 1, p. 378]) that x is joined from $\bar{x}$ by a singular control u. In particular, x is conjugate to $\bar{x}$; this is a contradiction.

Therefore, the set $\{p \mid \exp_{\bar{x}}(T_{\bar{x}}(x), p) = x\}$ is compact in U_2. Moreover, since x is not conjugate to $\bar{x}$, this set has no cluster point, and thus is finite. As a consequence, up to reducing V, we assume that V is a connected open subset of $\exp_{\bar{x}}(U_2)$, and that U_2 is a finite union of disjoint connected open sets, all of which are diffeomorphic to V by the mapping $\exp_{\bar{x}}$. We infer that every point $y \in V$ is not conjugate to $\bar{x}$. Hence, the mapping $\exp_{\bar{x}}$ is a proper submersion from U_2 into V, and thus is a fibration with finite degree. Since, from Lemma 3.8, there exists a unique minimal time trajectory joining $\bar{x}$ to x, this degree is equal to one, that is, $\exp_{\bar{x}}$ is a diffeomorphism from U_2 into V. The conclusion follows. ◻

It follows from the previous lemma that $(T_{\bar{x}}(y), p_0) = \exp_{\bar{x}}^{-1}(y)$ for every $y \in V$, and hence $T_{\bar{x}}(\cdot)$ is analytic on V. ◻

3.2.3. Definition of the optimal controller. By assumption, there does not exist any nontrivial singular minimal time trajectory starting from $\bar{x}$. Under these assumptions, the function $T_{\bar{x}}(\cdot)$ is *subanalytic* outside $\bar{x}$ (see [1, 2, 50], combined with Remark 2.4).

For the sake of completeness, we recall below the definition of a subanalytic function (see [27, 28]) and some properties that are used in a crucial way in the present paper (see [48]).

Let M be a real analytic finite-dimensional manifold. A subset A of M is said to be *semianalytic* if and only if, for every $x \in M$, there exists a neighborhood U of x in M and $2pq$ analytic functions g_{ij}, h_{ij} ($1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$), such that

$$A \cap U = \bigcup_{i=1}^{p} \{y \in U \mid g_{ij}(y) = 0 \text{ and } h_{ij}(y) > 0,\ j = 1, \ldots, q\}.$$

Let SEM(M) denote the set of semianalytic subsets of M. The image of a semianalytic subset by a proper analytic mapping is not in general semianalytic, and thus this class has to be enlarged.

A subset A of M is said to be *subanalytic* if and only if, for every $x \in M$, there exist a neighborhood U of x in M and $2p$ couples $(\Phi_i^\delta, A_i^\delta)$ ($1 \leqslant i \leqslant p$ and $\delta = 1, 2$), where $A_i^\delta \in SEM(M_i^\delta)$, and where the mappings $\Phi_i^\delta : M_i^\delta \to M$ are proper analytic, for real analytic manifolds M_i^δ, such that

$$A \cap U = \bigcup_{i=1}^{p} (\Phi_i^1(A_i^1) \backslash \Phi_i^2(A_i^2)).$$

Let SUB(M) denote the set of subanalytic subsets of M.

The subanalytic class is closed by union, intersection, complementary, inverse image by an analytic mapping, and image by a proper analytic mapping. In brief, the subanalytic class is *o-minimal* (see [23]). Moreover subanalytic sets are *stratifiable* in the following sense. A *stratum* of a differentiable manifold M is a locally closed submanifold of M. A locally finite partition $\mathcal{S}$ of M is a *stratification* of M if any $S \in \mathcal{S}$ is a stratum such that

$$\forall T \in \mathcal{S} \quad T \cap \partial S \neq \emptyset \Rightarrow T \subset \partial S \quad \text{and} \quad \dim T < \dim S.$$

Finally, a mapping $f : M \to N$ between two analytic manifolds is said to be *subanalytic* if its graph is a subanalytic subset of $M \times N$.

Let M be an analytic manifold, and let f be a subanalytic function on M. The *analytic singular support* of f is defined as the complement of the set of points x in M such that the restriction of f to some neighborhood of x is analytic. The following property is of great interest in the present paper (see [48]): the analytic singular support of f is subanalytic (and thus, in particular, is stratifiable). If f is, moreover, locally bounded on M, then it is of codimension greater than or equal to one.

Let us turn back to our problem. The function $T_{\bar{x}}(\cdot)$ is subanalytic outside $\bar{x}$, and, hence, its singular set $\mathcal{S} = \text{Sing}\ T_{\bar{x}}(\cdot)$ (i.e., the analytic singular support of $T_{\bar{x}}(\cdot)$) is a stratified submanifold of $\mathbb{R}^n$, of codimension greater than or equal to 1.

Remark 3.10. Note that the point $\bar{x}$ belongs to the adherence of $\mathcal{S}$ (see [1]).

Outside the singular set $\mathcal{S}$, it follows from the dynamic programming principle (see [36]) that the minimal time controllers steering a point $x \in \mathbb{R}^n \setminus \mathcal{S}$ to $\bar{x}$ are given by the closed-loop formula

$$u_i(x) = -\frac{\langle \nabla T_{\bar{x}}(x), f_i(x) \rangle}{\sqrt{\sum_{j=1}^m \langle \nabla T_{\bar{x}}(x), f_j(x) \rangle^2}}, \ i = 1, \dots, m. \tag{3.5}$$

The objective is to construct neighborhoods of $\mathcal{S} \setminus \{\bar{x}\}$ in $\mathbb{R}^n$ whose complements share invariance properties for the optimal flow. This is the contents of Lemma 2.14, proved next.

3.2.4. Proof of Lemma 2.14. It suffices to prove that, for every compact subset K of $\mathbb{R}^n$ and for every neighborhood Ω of $\mathcal{S} \setminus \{\bar{x}\}$ in $\mathbb{R}^n$, there exists a neighborhood Ω' of $\mathcal{S} \setminus \{\bar{x}\}$ in $\mathbb{R}^n$, satisfying (2.17), such that every trajectory of the closed-loop system (1.1) with the optimal controller, joining a point $x \in (\mathbb{R}^n \setminus \Omega) \cap K$ to $\bar{x}$, is contained in $\mathbb{R}^n \setminus \Omega'$.

By definition of the cut locus, and using Proposition 3.6, every optimal trajectory joining a point $x \in (\mathbb{R}^n \setminus \Omega) \cap K$ to $\bar{x}$ does not intersect $\mathcal{S}$, and thus has a positive distance to the set $\mathcal{S}$. Using the assumption of the absence of nontrivial singular minimizing trajectories starting from $\bar{x}$, a reasoning similar to the proof of Lemma 3.9 proves that the optimal flow joining points of the compact set $(\mathbb{R}^n \setminus \Omega) \cap K$ to $\bar{x}$ is parameterized by a compact set. Hence, there exists a positive real number $\delta > 0$ so that every optimal trajectory joining a point $x \in (\mathbb{R}^n \setminus \Omega) \cap K$ to $\bar{x}$ has a distance to the set $\mathcal{S}$ which is greater than or equal to δ. The existence of Ω' follows.

3.2.5. Robustness properties of the optimal controller. In this section, we prove robustness properties of the Carathéodory solutions of system (2.1) in closed-loop with this feedback optimal controller. Given $e, d : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$, the perturbed system in closed-loop with the optimal controller (denoted u_{opt}) is

$$\dot{x}(t) = f(x(t), u_{opt}(x(t) + e(x(t), t))) + d(x(t), t). \tag{3.6}$$

Since the optimal controller is continuous outside the singular set $\mathcal{S}$, it enjoys a natural robustness property, stated below. In the next result, the notation $d(x, \mathcal{S})$ stands for the Euclidean distance from x to $\mathcal{S}$.

LEMMA 3.11. *There exist a continuous function* $\rho_{opt} : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\rho_{opt}(\xi) > 0 \ \forall \xi \neq 0 \tag{3.7}$$

and a continuous function $\delta_{opt} : [0, +\infty) \to [0, +\infty)$ *of class* $\mathcal{K}_\infty$ *such that the following three properties hold:*

- Robust Stability.
  For every neighborhood Ω *of* $\mathcal{S}$*, there exists a neighborhood* $\Omega' \subset \Omega$ *of* $\mathcal{S}$*, such that, for all* $e, d : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ *satisfying the regularity assumptions* (2.2) *and, for every* $x \in \mathbb{R}^n$*,*

  $$\sup_{[0,+\infty)} |e(x, \cdot)| \leqslant \rho_{opt}(d(x, \mathcal{S})), \ \operatorname{esssup}_{[0,+\infty)} |d(x, \cdot)| \leqslant \rho_{opt}(d(x, \mathcal{S})), \tag{3.8}$$

  and for every $x_0 \in \mathbb{R}^n \setminus \Omega$*, there exists a unique Carathéodory solution* $x(\cdot)$ *of* (3.6) *starting from* x_0*, maximally defined on* $[0, +\infty)$*, and satisfying* $x(t) \in \mathbb{R}^n \setminus \Omega'$ *for every* $t > 0$*.*
- Finite time convergence.
  For every $R > 0$*, there exists* $\tau_{opt} = \tau_{opt}(R) > 0$ *such that, for all* $e, d : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ *satisfying the regularity assumptions* (2.2) *and* (3.8)*, for every* $x_0 \in \mathbb{R}^n$ *with* $|x_0 - \bar{x}| \leqslant R$*, and every maximal solution* $x(\cdot)$ *of* (3.6) *starting from* x_0*, one has*

  $$|x(t) - \bar{x}| \leqslant \delta_{opt}(R) \ \forall t \geqslant 0, \tag{3.9}$$

  $$x(t) = \bar{x} \ \forall t \geqslant \tau_{opt}, \tag{3.10}$$

  and

  $$\|u_{opt}(x(t))\| \leqslant 1 \ \forall t \geqslant 0. \tag{3.11}$$

- Optimality.
  For every neighborhood Ω *of* $\mathcal{S}$*, every* $\varepsilon > 0$*, and every compact subset* K *of* $\mathbb{R}^n$*, there exists a continuous function* $\rho_{\varepsilon,K} : \mathbb{R}^n \to \mathbb{R}$ *satisfying* (2.9) *such*

that, for all $e, d : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ *satisfying the regularity assumptions* (2.2) *and*

$$\begin{aligned}\sup_{[0,+\infty)}|e(x,\cdot)| &\leqslant \min(\rho_{opt}(d(x,\mathcal{S})), \rho_{\varepsilon,K}(x)),\\ \operatorname{esssup}_{[0,+\infty)}|d(x,\cdot)| &\leqslant \min(\rho_{opt}(d(x,\mathcal{S}), \rho_{\varepsilon,K}(x)) \ \forall x \in \mathbb{R}^n,\end{aligned} \tag{3.12}$$

and for every $x_0 \in K \cap (\mathbb{R}^n \setminus \Omega)$*, the solution of* (3.6)*, starting from* x_0*, reaches* $\bar{x}$ *within time* $T_{\bar{x}}(x_0) + \varepsilon$*.*

Proof. Since Carathéodory conditions hold for the system (3.6), the existence of a unique Carathéodory solution of (3.6), for every initial condition, is ensured. Inequality (3.11) follows from the constraint (1.2). Since the optimal controller u_{opt} defined by (3.5) is continuous on $\mathbb{R}^n \setminus \mathcal{S}$, Lemma 2.14 implies the existence of $\rho_{opt} : \mathbb{R}^n \to [0, +\infty)$ so that the *robust stability* and the *finite time convergence* properties hold.

The so-called *optimality* property follows from the definition of u_{opt}, from the continuity of solutions with respect to disturbances, and from the compactness of the set of all solutions starting from $K \cup (\mathbb{R}^n \setminus \Omega)$. ◻

3.3. Definition of the hybrid feedback law.

A switching strategy must be defined in order to connect the first component (optimal controller), and the second component (consisting of a set of controllers, stated in Lemma 2.13). The switching strategy is achieved by adding a dynamical discrete variable s_d and using a hybrid feedback law, described next.

3.3.1. A class of hybrid feedbacks.

Let $\mathcal{F} = \{1, \ldots, 6\}$, and let $\mathcal{N}$ be a countable set. In what follows, Greek letters refer to elements of $\mathcal{N}$. Fix ω as an element of $\mathcal{N}$. We emphasize that we do not introduce any order in $\mathcal{N}$.

Given a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we define the solutions $x(\cdot)$ of the differential inclusion $\dot{x} \in F(x)$ as all absolutely continuous functions satisfying $\dot{x}(t) \in F(x(t))$ almost everywhere.

DEFINITION 3.12. *The family* $(\mathbb{R}^n \setminus \{\bar{x}\}, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{N}})$ *is said to satisfy the property* $(\mathcal{P})$ *if*

1. *for every* $(\alpha, l) \in \mathcal{N} \times \mathcal{F}$*, the set* $\Omega_{\alpha,l}$ *is an open subset of* $\mathbb{R}^n$*;*
2. *for every* $\alpha \in \mathcal{N}$ *and every* $m > l \in \mathcal{F}$*,*

$$\Omega_{\alpha,l} \subsetneq \operatorname{clos}(\Omega_{\alpha,l}) \subsetneq \Omega_{\alpha,m}; \tag{3.13}$$

3. *for every* α *in* $\mathcal{N}$*,* g_α *is a smooth vector field, defined in a neighborhood of* $\operatorname{clos}(\Omega_{\alpha,6})$*, taking values in* $\mathbb{R}^n$*;*
4. *for every* $(\alpha, l) \in \mathcal{N} \times \mathcal{F}$*,* $l \leqslant 5$*, there exists a continuous function* $\rho_{\alpha,l} : \mathbb{R}^n \to [0, +\infty)$ *satisfying* $\rho_{\alpha,l}(x) \neq 0$ *whenever* $x \neq \bar{x}$ *such that every maximal solution* $x(\cdot)$ *of*

$$\dot{x} \in g_\alpha(x) + B(0, \rho_{\alpha,l}(x)), \tag{3.14}$$

defined on $[0, T)$ *and starting from* $\partial\Omega_{\alpha,l}$*, is such that*

$$x(t) \in \operatorname{clos}(\Omega_{\alpha,l+1}) \ \forall t \in [0, T);$$

5. *for every* $l \in \mathcal{F}$*, the sets* $(\Omega_{\alpha,l})_{\alpha\in\mathcal{N}}$ *form a locally finite covering of* $\mathbb{R}^n \setminus \{\bar{x}\}$*.*

Remark 3.13. Some observations are in order.

- First note that this notion is close to the notion of a family of nested patchy vector fields defined in [38]. However, note that, in general, the sets $(\Omega_{\alpha,l}, g_\alpha)$ may not be a patch as defined in [4, 38]. Indeed, due to property 4, the set $\Omega_{\alpha,l}$ may not be invariant for the system (3.14). Since the notion of a patch is one of the main ingredients of the proofs of [40], we cannot apply [40] directly, even though some notions are similar (see in particular Definition 3.14 below).
- To state our main result, we need a family of six nested patchy vector fields (see Remark 3.21 for comments on the necessity of six families). The patches 1, 2, 4, and 6 define the dynamics of the discrete component of our hybrid controller (see Definition 3.14 below). The patches 3 and 5 are used for technical reasons to handle the measurement noise.

We next define a class of hybrid controllers as those considered in section 2 (see also [40]).

DEFINITION 3.14. *Let* $(\mathbb{R}^n \setminus \{\bar{x}\}, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{N}})$ *satisfy the property* $(\mathcal{P})$ *as in Definition* 3.12*. Assume that, for every* α *in* $\mathcal{N}$*, there exists a smooth function* k_α *defined in a neighborhood of* $\Omega_{\alpha,6}$ *and taking values in* $\mathbb{R}^m$*, such that, for every* x *in a neighborhood of* $\Omega_{\alpha,6}$*,*

$$g_\alpha(x) = f(x, k_\alpha(x)). \tag{3.15}$$

Set

$$D_1 = \Omega_{\omega,2}, \tag{3.16}$$

$$D_{\alpha,2} = \mathbb{R}^n \setminus \Omega_{\alpha,6}. \tag{3.17}$$

Let (C, D, k, k_d) *be the hybrid feedback defined by*

$$C = \{(x,\alpha) \mid x \in (\mathrm{clos}(\Omega_{\alpha,4}) \setminus \Omega_{\omega,1})\}, \tag{3.18}$$

$$D = \{(x,\alpha) \mid x \in D_1 \cup D_{\alpha,2}\}, \tag{3.19}$$

$$\begin{array}{rcl} k : \mathbb{R}^n \times \mathcal{N} & \to & \mathbb{R}^m \\ (x,\alpha) & \mapsto & k_\alpha(x) \quad \text{if} \quad x \in \Omega_{\alpha,6}, \\ & & 0 \qquad \text{else}, \end{array} \tag{3.20}$$

and

$$\begin{array}{rcl} k_d : \mathbb{R}^n \times \mathcal{N} & \rightrightarrows & \mathcal{N} \\ (x,\alpha) & \mapsto & \omega \quad \text{if } x \in \mathrm{clos}(\Omega_{\omega,1} \cap D_1) \text{ and if } x \notin D_{\alpha,2}, \\ & & \alpha' \quad \text{if } x \in \mathrm{clos}(\Omega_{\alpha',1} \cap D_{\alpha,2}). \end{array} \tag{3.21}$$

The 4*-tuple* (C, D, k, k_d) *is a hybrid feedback law on* $\mathbb{R}^n$ *as considered in section* 2.2*. We denote by* $\mathcal{H}_{(e,d)}$ *the system* (2.1) *in closed-loop with such feedback with the perturbations* e *and* d *as measurement noise and external disturbance, respectively.*

In this definition, we do not use any order in $\mathcal{N}$. However, the element ω has a particular role in what follows, since it will refer to the optimal controller in the hybrid feedback law.

3.3.2. Properties of solutions. In this section, we investigate some properties of the solutions of the system in closed-loop with the hybrid feedback law defined above.

DEFINITION 3.15. *Let* $\chi : \mathbb{R}^n \to \mathbb{R}$ *be a continuous map such that* $\chi(x) > 0$ *for every* $x \neq \bar{x}$*.*

- *We say that χ is an admissible radius for the measurement noise if, for every $x \in \mathbb{R}^n$ and every $\alpha \in \mathcal{N}$, such that $x \in \Omega_{\alpha,6}$,*

$$\chi(x) < \frac{1}{2} \min_{l \in \{1,\dots,5\}} d(\mathbb{R}^n \setminus \Omega_{\alpha,l+1}, \Omega_{\alpha,l}). \tag{3.22}$$

- *We say that χ is an admissible radius for the external disturbances if, for every $x \in \mathbb{R}^n$, we have*

$$\chi(x) \leqslant \max_{(\alpha,l),\, x \in \Omega_{\alpha,l}} \rho_{\alpha,l}(x).$$

There exists an admissible radius for the measurement noise and for the external disturbances (note that, from (3.13), the right-hand side of inequality (3.22) is positive).

Consider an admissible radius χ for the measurement noise and the external disturbances. Let e and d be a measurement noise and an external disturbance respectively, such that, for all $(x,t) \in \mathbb{R}^n \times [0,+\infty)$,

$$e(x,t) \leqslant \chi(x), \quad d(x,t) \leqslant \chi(x). \tag{3.23}$$

The properties of the solutions of the system in closed-loop with the hybrid feedback law defined in Definition 3.14 are similar to those of [40]. Hence, we skip the proof of the following three lemmas, which do not use statement 4 of Definition 3.12 but only the definition of the hybrid feedback law.

LEMMA 3.16. *For all $(x_0, s_0) \in \mathbb{R}^n \times \mathcal{N}$, there exists a solution of $\mathcal{H}_{(e,d)}$ starting from (x_0, s_0).*

Recall that a Zeno solution is a complete solution whose domain of definition is bounded in the t-direction. A solution (x, s_d), defined on a hybrid domain S, is an instantaneous Zeno solution if there exist $t \geqslant 0$ and an infinite number of $j \in \mathbb{N}$ such that $(t,j) \in S$.

The Zeno solutions do not require a special treatment.

LEMMA 3.17. *There do not exist instantaneous Zeno solutions, although a finite number of switches may occur at the same time.*

We note, as usual, that maximal solutions of $\mathcal{H}_{(e,d)}$ blow up if their domain of definition is bounded. Since Zeno solutions are avoided, the blow-up phenomenon only concerns the t-direction of the domain of definition, and we get the following result (see also [25, Proposition 2.1]).

LEMMA 3.18. *Let (x, s_d) be a maximal solution of $\mathcal{H}_{(e,d)}$ defined on a hybrid time S. Suppose that the supremum T of S in the t-direction is finite. Then*

$$\limsup_{t \to T, (t,l) \in S} |x(t,l)| = +\infty.$$

We conclude this series of technical lemmas by studying the behavior of solutions between two jumps. For every $\alpha \in \mathcal{N}$, set

$$\tau_\alpha = \sup\big\{T \mid x \text{ is a Carathéodory solution of } \dot{x} = f(x, k_\alpha) + B(0, \chi(x)) \text{ with } x(t) \in \Omega_{\alpha,6}\ \forall t \in [0,T)\big\}. \tag{3.24}$$

Note that, at this stage, there may exist $\alpha \in \mathcal{N}$ such that $\tau_\alpha = +\infty$.

LEMMA 3.19. *Let (x, s_d) be a solution of $\mathcal{H}_{(e,d)}$ defined on a hybrid time domain S and starting in $(\mathbb{R}^n \setminus \{\bar{x}\}) \times \mathcal{N}$. Let T be the supremum in the t-direction of S. Then one of the two following cases may occur:*

- *Either there exists no positive jump time—more precisely, there exists* $\alpha \in \mathcal{N}$ *such that*
  1. *for almost every* $t \in (0,T)$ *and for every* l *such that* $(t,l) \in S$*, one has* $k(s_d(t,l)) = k_\alpha$*;*
  2. *for every* $t \in (0,T)$ *and every* l *such that* $(t,l) \in S$*, one has* $x(t,l) + e(x(t,l),t) \in \mathrm{clos}(\Omega_{\alpha,4}) \setminus \Omega_{\omega,1}$*;*
  3. *for all* $(t,l) \in S$*,* $t > 0$*, one has* $x(t,l) + e(x(t,l),t) \notin D$*, where* D *is defined by* (3.19)*;*
  4. *the inequality* $T < \tau_\alpha$ *holds;*
- *or there exists a unique positive jump time—more precisely, there exist* $\alpha \in \mathcal{N} \setminus \{\omega\}$ *and* $t_1 \in (0,T)$ *such that, letting* $t_0 = 0$*,* $t_2 = T$*,* $\alpha_0 = \alpha$*,* $\alpha_1 = \omega$*, for every* $j = 0, 1$*, the following properties hold:*
  5. *for almost every* $t \in (t_j, t_{j+1})$ *and for every* l *such that* $(t,l) \in S$*, one has* $k(s_d(t,l)) = k_{\alpha_j}$*;*
  6. *for every* $t \in (t_0, t_1)$ *and every* l *such that* $(t,l) \in S$*, one has* $x(t,l) + e(x(t,l),t) \in \mathrm{clos}(\Omega_{\alpha,4}) \setminus \Omega_{\omega,1}$*;*
  7. *for every* t *in* (t_j, t_{j+1}) *and every* l *such that* $(t,l) \in S$*, one has* $x(t,l) + e(x(t,l),t) \notin D_{\alpha_j,2}$*, where* $D_{\alpha_j,2}$ *is defined by* (3.17)*;*
  8. *the inequality* $t_1 < \tau_{\alpha_j}$ *holds.*

Proof. Consider the sequence $(t_j)_{j\in\overline{m}}$ of jump times, i.e., the times such that $t_0 = 0$ and, for every $j \in \overline{m}$, $j \leqslant m-1$,

$$t_j \leqslant t_{j+1}, \tag{3.25}$$

$$(x(t_{j+1}, j) + e(x(t_{j+1}, j), t_{j+1}), s_d(t_{j+1}, j)) \in D, \tag{3.26}$$

and

$$(x(t_{j+1}, j+n_j) + e(x(t_{j+1}, j), t_{j+1}), s_d(t_{j+1}, j+n_j)) \in C, \tag{3.27}$$

where n_j is the finite number of instantaneous switches[3] (see Lemma 3.17). Let $\sigma : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $t_{\sigma(j)} < t_{\sigma(j+1)}$.

Between two jumps, $s_d(t)$ is constant, and thus there exists a sequence (α_j) in $\mathcal{N}$ such that, for every $t \in (t_{\sigma(j)}, t_{\sigma(j+1)})$, except for a finite number of t, we have

$$s_d(t, \sigma(j)) = \alpha_j, \tag{3.28}$$

$$x \text{ is a Carathéodory solution of } \dot{x} = f(x, k_{\alpha_j}) + d \text{ on } (t_{\sigma(j)}, t_{\sigma(j+1)}), \tag{3.29}$$

and

$$k(s_d(t, \sigma(j))) = k_{\alpha_j}. \tag{3.30}$$

From (3.18), (3.27), and (3.28), we have, for every $t \in [t_{\sigma(j)}, t_{\sigma(j+1)}]$,

$$x(t,j) + e(x(t,j),t) \in \mathrm{clos}(\Omega_{\alpha_j,4}) \setminus \Omega_{\omega,1}. \tag{3.31}$$

Note that, from (3.30), (3.31), and statement 4 of Definition 3.12, for every $t > 0$ such that $t \in [t_{\sigma(j)}, t_{\sigma(j+1)}]$, one has

$$x(t, \sigma(j)) \notin D_{\alpha_j,2}. \tag{3.32}$$

[3]We emphasize that we may have jumps at time $t = 0$ (see also Remark 3.21).

Therefore, the positive jump time may occur only at time t_j where the point $x(t_j, l) + e(x(t_j, l), t_j)$ belongs to D_1. Thus, there exists at most one positive jump time. From (3.29) and (3.30), Statements 1 and 5 hold. Statements 2 and 6 are deduced from (3.31). Equation (3.32) implies Statements 3 and 7 Finally, Statements 4 and 8 are a consequence of (3.24) and (3.29). □

3.3.3. Definition of the hybrid feedback law, and switching strategy. We next define our hybrid feedback law. Let $\varepsilon > 0$ and K be a compact subset of $\mathbb{R}^n$. Let Ω be the neighborhood of $\mathcal{S}$ given by Lemma 2.13. For this neighborhood Ω, let $\Omega' \subset \Omega$ be the neighborhood of $\mathcal{S}$ yielded by Lemma 2.14.

Let $\mathcal{N}$ be the countable set defined by

$$\mathcal{N} = \{(i, p),\ i \in \mathbb{N},\ p \in \mathcal{N}_i\} \cup \{\omega\},$$

where ω is an element of $\mathbb{N} \times \mathbb{N}$, distinct from every (i, p), $i \in \mathbb{N}$, $p \in \mathcal{N}_i$.

We proceed in two steps. We first define k_α and $\Omega_{\alpha,l}$, where $\alpha \in \mathcal{N} \setminus \{\omega\}$ and $l \in \mathcal{F}$. Then we define k_ω and $\Omega_{\omega,l}$, where $l \in \mathcal{F}$.

1. Let $i \in \mathbb{N}$. Lemma 2.13, applied with the stratum M_i, implies the existence of a family of smooth controllers $(k_{i,p})_{p\in\mathcal{N}_i}$ satisfying the constraint (1.2) and of a family of neighborhoods $(\Omega_{i,p,7})_{p\in\mathcal{N}_i}$. The existence of the families $(\Omega_{i,p,1})_{p\in\mathcal{N}_i}, \dots, (\Omega_{i,p,6})_{p\in\mathcal{N}_i}$, satisfying

$$\Omega_{i,p,l} \subsetneq \mathrm{clos}(\Omega_{i,p,l}) \subsetneq \Omega_{i,p,m}$$

for every $m > l \in \mathcal{F}$, follows from a finite induction argument, using Lemma 2.13.

We have thus defined $k_{i,p}$ and $\Omega_{i,p,l}$, where $(i, p) \in \mathcal{N} \setminus \{\omega\}$ and $l \in \mathcal{F}$.

Remark 3.20. It follows from [1] that, near the point $\bar{x}$, the cut locus $\mathcal{S}$ is contained in a conic neighborhood $\mathcal{C}$ centered at $\bar{x}$ (as illustrated in Figure 2.1), the axis of the cone being transversal to the subspace $\mathrm{Span}\{f_1(\bar{x}), \dots, f_m(\bar{x})\}$. Moreover, we may modify slightly the previous construction, and assume that, near $\bar{x}$, the set $\bigcup_{\alpha\in\mathcal{N}\setminus\{\omega\},\, l\in\mathcal{F}} \Omega_{\alpha,l}$ is contained in this conic neighborhood.

2. It remains to define the sets $\Omega_{\omega,l}$, where $l \in \mathcal{F}$, and the controller k_ω. Let $\Omega_{\omega,1}$ be an open set of $\mathbb{R}^n$ containing $\mathbb{R}^n \setminus \bigcup_{\alpha\in\mathcal{N}\setminus\{\omega\}} \Omega_{\alpha,1}$ and contained in $\mathbb{R}^n \setminus \mathcal{S}$. From the previous remark, the point $\bar{x}$ belongs to $\mathrm{clos}(\Omega_{\omega,1})$. Lemma 2.14, applied with $\Omega = \mathbb{R}^n \setminus \mathrm{clos}(\Omega_{\omega,1})$, allows to define k_ω as k_{opt}, and Ω' a closed subset of $\mathbb{R}^n$ such that

$$\Omega' \subsetneq \Omega, \tag{3.33}$$

and such that Ω' is a neighborhood of $\mathcal{S}$. Set $\Omega_{\omega,2} = \mathbb{R}^n \setminus \Omega'$; it is an open subset of $\mathbb{R}^n$, contained in $\mathbb{R}^n \setminus \mathcal{S}$. Moreover, from (3.33), $\Omega_{\omega,1} \subsetneq \mathrm{clos}(\Omega_{\omega,1}) \subsetneq \Omega_{\omega,2}$. The existence of the sets $\Omega_{\omega,3}, \dots, \Omega_{\omega,6}$ follows from a finite induction argument, using Lemma 2.14. Moreover, from Lemma 3.11, we have the following property: for every $l \in \{1, \dots, 6\}$, for every $x_0 \in \Omega_{\omega,l}$, the unique Carathéodory solution $x(\cdot)$ of (3.6), with $x(0) = x_0$, satisfies $x(t) \in \Omega_{\omega,l+1}$ for every $t \geqslant 0$.

Therefore, $(\mathbb{R}^n \setminus \{\bar{x}\}, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{N}})$ is a family satisfying the property $(\mathcal{P})$ as in Definition 3.12, where g_α is a function defined in a neighborhood of $\Omega_{\alpha,6}$ by $g_\alpha(x) = f(x, k_\alpha)$. The hybrid feedback law (C, D, k, k_d) is then defined according to Definition 3.14.

3.4. Proof of Theorem 2.10. Let $\varepsilon > 0$, and let K be a compact subset of $\mathbb{R}^n$. Consider the hybrid feedback law (C, D, k, k_d) defined previously. Let χ be an admissible radius for the external disturbances and the measurement noise (see Definition 3.15). We may reduce this function, and assume that, for every $\alpha \in \mathcal{N}\setminus\{\omega\}$,

$$\chi(x) \leqslant \rho_{opt}(d(x, \mathcal{S})) \quad \forall x \in \Omega_{\omega,6},$$
$$\chi(x) \leqslant \rho_\alpha(x) \quad \forall x \in \Omega_{\alpha,6}.$$

Note that, from the choice of the components of the hybrid feedback law, and from Lemmas 2.13 and 3.19, for every $\alpha \in \mathcal{N} \setminus \{\omega\}$, the constant τ_α defined by (3.24) is such that $\tau_\alpha < \varepsilon$.

Let us prove that the point $\bar{x}$ is a semiglobal quasi-minimal time robust stable equilibrium for the system (1.1) in closed-loop with the hybrid feedback law (C, D, k, k_d) as stated in Theorem 2.10.

Step 1: *Completeness and global stability.* Let $R > 0$ and $\delta : [0, +\infty) \to [0, +\infty)$ of class $\mathcal{K}_\infty$ be such that, for every $\alpha \in \mathcal{N}$ and every x, with $|x - \bar{x}| < R$ and $x \in \Omega_{\alpha,6}$, we have

$$\delta(R) \leqslant \delta_{opt}(\delta_\alpha(R)),$$

where the functions δ_α are defined in Lemma 2.13. Let e, d be two functions satisfying the regularity assumptions and (3.23). Let (x, s_d) be a maximal solution of $\mathcal{H}_{(e,d)}$ on a hybrid domain S with the initial condition (x_0, s_0), where $|x_0 - \bar{x}| < R$. From Lemmas 2.13, 3.11, and 3.19, we have, for every $(t, l) \in S$,

$$|x(t, l) - \bar{x}| \leqslant \delta(R). \tag{3.34}$$

Therefore, the conclusion of Lemma 3.18 cannot hold (since $\limsup_{t\to T,(t,l)\in S} |x(t, l)| \neq +\infty$), and thus the supremum T of S in the t-direction is infinite, and the maximality property follows. The stability property follows from (3.34).

Step 2: *Uniform finite time convergence property.* Let $x_0 \in B(\bar{x}, R)$, and $s_0 \in \mathcal{N}$. Let (x, s_d) denote the solution of $\mathcal{H}_{(e,d)}$ with the initial condition (x_0, s_0). If $x_0 = \bar{x}$, then, using (3.20) and the fact that $\chi(\bar{x}) = 0$, the solution remains at the point $\bar{x}$. We next assume that $x_0 \neq \bar{x}$. Let $\alpha_0 \in \mathcal{N}$ such that $x(\cdot)$ is a solution of $\dot{x} = f(x, k_{\alpha_0}(x)) + d$ on $(0, t_1)$ for a $t_1 > 0$ given by Lemma 3.19.

If $\alpha_0 = \omega$, then the feedback law under consideration coincides with the optimal controller and, from statement 3 of Lemma 3.19, there does not exist any switching time $t > 0$. Then, from Lemma 3.11, $x(\cdot)$ reaches $\bar{x}$ within time $T_{\bar{x}}(x_0) + \varepsilon$.

If $\alpha_0 \neq \omega$, then, from Lemmas 2.13 and 3.19, the solution $x(\cdot)$ leaves $\Omega_{\alpha_0,6}$ within time ε and then enters the set $\Omega_{\omega,4}$. Therefore, since $\tau_\alpha < \varepsilon$, $x(\cdot)$ reaches $\bar{x}$ within time $T_{\bar{x}}(x_1) + \varepsilon$, where x_1 denotes the point of $x(\cdot)$ when entering $\Omega_{\omega,4}$.

Let $\tau(R) = \max_{x\in\delta(R)} T(x) + \varepsilon$. With (3.34), we get (2.12) and the uniform finite time property. Note that, from Lemma 2.13, the constraint (2.13) is satisfied.

Step 3: *Quasi-optimality.* Let K be a compact subset of $\mathbb{R}^n$, and $(x_0, s_0) \in K \times \mathcal{N}$. Let $R > 0$ such that $K \subset B(0, R)$. From the previous arguments, two cases occur:

- The solution starting from (x_0, s_0) reaches $\bar{x}$ within time $T_{\bar{x}}(x_0) + \varepsilon$ whenever $\alpha_0 = \omega$.
- The solution starting from (x_0, s_0) reaches $\bar{x}$ within time $T_{\bar{x}}(x_1) + \varepsilon$, whenever $\alpha_0 \neq \omega$, where x_1 denotes the point of $x(\cdot)$ when entering $\Omega_{\omega,6}$. We may reduce the neighborhoods $\Omega_{\alpha,l}$, one has $|T_{\bar{x}}(x_0) - T_{\bar{x}}(x_1)| \leqslant \varepsilon$. Indeed, from Remark 2.4, the function $T_{\bar{x}}(\cdot)$ is uniformly continuous on the compact K.

Hence, the maximal solution starting from (x_0, s_0) reaches $\bar{x}$ within time $T_{\bar{x}}(x_0) + 2\varepsilon$. This is the quasi-optimality property.

Theorem 2.10 is proved.

Remark 3.21. We deduce from Lemma 3.19 and from the proof of Theorem 2.10 that the following two cases may occur:

- If $x_0 \in \Omega_{\omega,2}$, then the solutions (x, s_d) of $\mathcal{H}_{(e,d)}$ defined on the hybrid time domain S and with the initial condition (x_0, ω) satisfy $x(t,l) + e(x(t,l), t) \in \Omega_{\omega,4}$ for all $(t,l) \in S$.
- If $x_0 \in \Omega_{\alpha,2}$ for $\alpha \neq \omega$, then there exists a jump time $\bar{t} \geqslant 0$, $(\bar{t}, \bar{l}) \in S$ such that $x(\bar{t}, \bar{l}) + e(x(\bar{t}, \bar{l}), \bar{t}) \in \Omega_{\omega,2}$, and for all $(t,l) \in S$, $t \geqslant \bar{t}$, we have $x(t, \bar{l}) + e(x(t,l), t) \in \Omega_{\omega,4}$.

Hence, for every $x_0 \in \mathbb{R}^n \setminus \{\bar{x}\}$, by considering only the initial conditions (x_0, s_0) with $x_0 \in \Omega_{s_0,2}$, the solutions of $\mathcal{H}_{(e,d)}$ depend only on the sets $\Omega_{\alpha,2}$, $\Omega_{\alpha,3}$, and $\Omega_{\alpha,4}$ for some $\alpha \in \mathcal{N}$. In other words, for such initial conditions, we only need three nested families of open sets to state a quasi-optimality property. However, six nested families of open sets $(\Omega_{\alpha,l})_{\alpha \in \mathcal{N},\, l \in \mathcal{F}}$ are required to establish the quasi-optimality property stated in Theorem 2.10 *for all initial conditions* $(x_0, s_0) \in \mathbb{R}^n \setminus \{\bar{x}\} \times \mathcal{N}$, and *not only* for initial conditions $\{(x_0, s_0),\ x_0 \in \Omega_{s_0,2}\}$.

Since there is no restriction on the s_0 variable, a jump may occur at time $t = 0$ and the second claim of (3.21) is active for some initial conditions. We emphasize that Lemma 3.19 asserts that there exists at most one positive jump time, but another jump may occur at time $t = 0$.

Intuitively, two patches are required for defining both kinds of jumps: from the controllers built in Lemma 2.13 to the controller considered in Lemma 2.14, and conversely. The other four patches are used to define neighborhoods of these jump sets and to bound the admissible measurement noise (see Definition 3.15). This explains the necessity of considering at least six patches to establish our main result.

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