Sparse kinetic Jurdjevic-Quinn control for mean-field equations

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***Abstract*— We consider mean-field equations, that are transport partial differential equations describing the evolution of the density of a crowd of interacting agents when their number grows to infinity. We generalize the Jurdjevic-Quinn control strategy to such mean-field equations to achieve stability in a suitable sense. For instance, in the context of crowds, consensus or alignment of agents are among the most relevant goals and they correspond to stability to a suitable manifold.**
Our main result is the definition of a Jurdjevic-Quinn strategy that is sparse, i.e. that acts on a small set of the configuration space at each time. We show that such a stabilization strategy can be adapted to achieve alignment of the kinetic Cucker-Smale model with a sparse control only.

I. INTRODUCTION

In recent years, the study of collective behavior of a crowd of autonomous agents has drawn a great interest from scientific communities, e.g. in civil engineering (for evacuation problems [15], [6]), robotics (coordination of robots [2], [13], [17], [21]), computer science and sociology (social networks [14]), and biology (animals groups [1], [5], [9]). In particular, it is well known that some simple rules of interaction between agents can promote formation of special patterns, like lines in ants formations and migrating lobsters, V shaped formation in migrating birds etc... This phenomenon is often referred to as *self-organization*. Beside the problem of analyzing the collective behavior of a "closed" system, it is interesting to understand what changes of behavior can be induced by an external agent (e.g. a policy maker) to the crowd. For example, one can try to enforce creation of patterns when they are not formed naturally, or break the formation of such patterns [3], [4], [8], [20]. This is the problem of **control of crowds**, that we address in this article in a specific case.

From the mathematical point of view, problems related to models of crowds are of great interest. From the analysis point of view, one needs to pass from a big set of simple rules for each individual to a model capable of capturing the dynamics of the whole crowd. This can be solved via the so-called mean-field process, that permits to consider the limit of a set of ordinary differential equations (one for each agent) to a partial differential equation (PDE in the following) for the whole crowd [20]. The resulting equation is a transport equation with non-local velocity

$$\partial_t \mu + \nabla \cdot (f[\mu]\,\mu) = 0, \tag{1}$$

where μ is the measure representing the density of agents and $f[\mu]$ is a vector field depending on the measure, taking into account the interactions among agents. We will recall fundamental properties of such equation in Section II. To study the control of (1), we assume to act only on a small part of the configuration space. Since agents are indistinguishable in μ, controls can be state-dependent only and cannot focus on specific agents. For this reason, we model control action with a fixed vector field g describing the available action, and a control gain $u = u(x)$ localized in a small set ω modeling our choice of the gain on the vector field. Both the control function u and the control domain ω can vary in time, while g is fixed and independent from the density μ. The resulting control system is given by:

$$\partial_t \mu + \nabla \cdot [(f[\mu] + \chi_\omega u g)\mu] = 0. \tag{2}$$

We focus on the following **continuous sparsity constraint**: we assume to act on a small subset of the configuration space, and with a finite strength of our control. Then, we assume $|\omega(t)| \leq c$ with a fixed quantity c, where $|\cdot|$ is the Lebesgue measure of a set. Moreover, we assume $\|u(t,.)\|_{L^\infty} \leq 1$. Such constraint was introduced in [20].
We are interested in generalizing Jurdjevic-Quinn results about stabilization, see [16]. In particular, we assume to have a Lyapunov function V for which

- the uncontrolled dynamics $f[\mu]$ gives no increase of V;
- the control permits to increase-decrease V except on a set $\mathcal{Z}$ of the configuration space.

Under this assumption, we prove the natural stabilizability result generalizing the Jurdjevic-Quinn approach. We first prove in Theorem 2 a particular case: if $\mathcal{Z}$ is reduced to a point $\bar{x}$, then one can compute a stabilizing control steering any initial measure μ_0 to such Dirac delta. The main result of the article is then Theorem 3, in which the set $\mathcal{Z}$ is more general: we prove that one can drive any initial measure to a measure in $\mathcal{Z}$. Finally, we apply the Theorem 3 to the mean-field limit of the well known Cucker-Smale model for alignment of birds. The application is nontrivial, since the techniques need to be coupled with some specific invariance properties of the Cucker-Smale system, see Section V.

II. TRANSPORT EQUATION WITH NON-LOCAL VELOCITY

In this section, we recall known results about the transport equation (1) with non-local velocity. Such transport PDE is

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written in terms of measures $\mu \in \mathcal{P}_c(\mathbb{R}^d)$, where $\mathcal{P}_c(\mathbb{R}^d)$ is the space of probability measures on $\mathbb{R}^d$ with compact support. We also deal with $\mathcal{P}_c^{ac}(\mathbb{R}^d)$, the space of probability measures with compact support that are absolutely continuous with respect to the Lebesgue measure.

Recall that the value of a measure at a single point is not a well defined quantity. Then, it is important to observe that the velocity field v is not a function depending on the value of μ in a given point, as it is often the case in the setting of hyperbolic equations in which $f[\mu](x) = f(\mu(x))$. Instead, one has to consider v as an operator taking the whole measure μ and giving a global vector field $f[\mu]$ on the whole space $\mathbb{R}^d$. These operators are often called "non-local", as they consider knowledge of the density not only at a given point, but in a whole neighborhood.

The fundamental tool we use to analyze such PDEs is the $1-$**Wasserstein distance**. We recall that, for $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, the 1-Wasserstein distance is

$$W_1(\mu,\nu) := \inf \left\{ \int |x-y| \, d\pi(x,y) \mid \pi \in \Pi(\mu,\nu) \right\},$$

where $\Pi(\mu,\nu)$ is the set of transference plans for μ, ν, i.e. of the probability densities in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ with marginals μ, ν, respectively. A transference plan π can be interpreted as a choice to move each infinitesimal mass of μ in x to a point y, with density $\pi(x,y)$. If the cost of such infinitesimal displacement is assumed to be proportional to the distance $|x-y|$, then the Wasserstein distance W_1 is the minimal cost of displacement of μ to ν. For a more general description of Wasserstein distances, see e.g [22].

We now recall a result ensuring existence and uniqueness for the Cauchy problem with dynamics (1).

Theorem 1: Let v satisfy the following hypotheses (**V**):

(V)

The function $f[\mu] : \mathcal{P}(\mathbb{R}^d) \to C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfies

- $f[\mu]$ is uniformly Lipschitz and uniformly sublinear, i.e. there exist L, M not depending on μ, such that for all $\mu \in \mathcal{P}(\mathbb{R}^d), x, y \in \mathbb{R}^n$,
$$|f[\mu](x) - f[\mu](y)| \leq L|x-y|, |f[\mu](x)| \leq M.$$
- f is a Lipschitz function, i.e. there exists K such that
$$\|f[\mu] - f[\nu]\|_{C^0} \leq K W_1(\mu,\nu).$$

Then for every $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$ there exists a unique solution of the Cauchy problem:

$$\partial_t \mu + \nabla \cdot (f[\mu]\mu) = 0, \qquad \mu_{|t=0} = \mu_0. \tag{3}$$

Moreover, if $\mu_0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$, then the solution satisfies $\mu(t) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ for all times t. Given two solutions μ, ν of (3) with initial data μ_0, ν_0, respectively, the following stability result holds

$$W_1(\mu(t),\nu(t)) \leq e^{(4K+4L)t} W_1(\mu_0,\nu_0). \tag{4}$$

Proof: See [19]. ■

These results can be generalized to transport PDEs in presence of sources, see [18]. They can also be generalized to velocity fields less regular than what required in **(V)**, see [10].

We now study the particular case of the controlled equation (2). It is clearly required that the vector field $f[\mu] + \chi_\omega u g$ satisfies hypotheses **(V)**. In particular, we require from now to have the following regularity property:

(U)

The function $\chi_\omega u g$ is a Lipschitz vector field.

A. Discretization of measures and transport equations

In this section, we recall some properties of the transport equation (1) and of discretization of measures. The idea is that one can compute solutions of (1) by approximating the initial measure with a sum of Dirac deltas and solve a finite-dimensional dynamical system associated to (1).

Given $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$, one can define

$$\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \tag{5}$$

an approximation of μ_0. For example, in one dimension, one can simply write the cumulative distribution function $h(x) = \mu((-\infty, x])$, define

$$x_i := \inf \left\{ h(x) \leq \frac{i}{N} \right\}, \tag{6}$$

and the corresponding measure (5). Such strategy can be adapted to an arbitrary dimension d. More generally, for a given μ_0, we consider a family of discretizations μ_0^N for $N \in \mathbb{N}$ representing the number of Dirac deltas in (5). The key property we require for discretizations is

$$\lim_{N \to \infty} W_1(\mu_0^N, \mu_0) = 0, \tag{7}$$

i.e. that the discretization converges to the measure in the Wasserstein distance. The discretization (6) satisfies such property in $\mathcal{P}_c(\mathbb{R}^d)$ for $d = 1$. The interest of discretization of measures is that the transport equation (1) is transformed into a finite-dimensional system. Indeed, given μ_0^N of the form (5), then (1) reads as

$$\dot{x}_i = w(x_1, \ldots, x_N)(x_i), \qquad \text{i=1,\ldots,N}, \tag{8}$$

where $w(x_1, \ldots, x_n) := f\left[\mu^N\right]$. One can apply analytical and control tools to study such dynamical system in finite dimension. The corresponding solution is then $\mu^N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$, i.e. with Dirac deltas displaced according to the solution to (8). By construction, $\mu^N(t)$ is also the solution to (1) with initial data μ_0^N, see e.g. [22]. The solution of the transport equation with initial data μ_0 can be then recovered by limit:

$$\mu(t) = \lim_{N \to \infty} \mu^N(t),$$

where the limit is intended for the Wasserstein distance. This is a direct consequence of (4) coupled with (7).

Remark 1: Such estimate shows that the discretization tool is useful for estimation of dynamics on the finite-time horizon. Instead, stabilization is a problem stated on the infinite-time horizon, then one needs some properties to decompose the time horizon for applying discretization. This will be a key aspect of the Jurdjevic-Quinn approach in Sections III and IV.

III. SPARSE KINETIC JURDJEVIC-QUINN STABILIZATION TO A SINGLE POINT

In this section we prove the first result of sparse Jurdjevic-Quinn stabilization for a system of the form (2). We assume strong conditions on the Lie derivative $\mathcal{L}_g V = g \cdot \nabla V$, that needs to be non-zero everywhere except at a point $\bar{x}$. As a result, we have a quite explicit control strategy driving the system asymptotically to $\bar{x}$.

Theorem 2: Let $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$, and $C \in \mathbb{R}$ be such that $X := \{V \le C\}$ is compact. If

- $\{\mathcal{L}_g V(x) = 0\} \cap X = \{\bar{x}\}$ for a given $\bar{x} \in \mathbb{R}^d$;
- $\mathcal{L}_{f[\mu]} V(x) \le 0$ for all $\mu \in \mathcal{P}(X)$ and $x \in X$;

then, there exists a control $\chi_\omega u : [0, +\infty) \to \mathrm{Lip}(\mathbb{R}^d, \mathbb{R})$ such that for each $\mu_0 \in \mathcal{P}(X)$

- the unique solution of (2) with control $\chi_{\omega(t)} u(t)$ converges to $\delta_{\bar{x}}$ for $t \to \infty$;
- the control domain $\omega(t)$ satisfies the sparsity constraint $|\omega(t)| < c$ for a given $c > 0$;
- $|u(t)| \le 1$ for all $t \in [0, +\infty)$.

Proof: First observe that the statement is obviously true if $X = \{\bar{x}\}$. Then, we assume from now on that X is not reduced to a point.

Observe that $\nabla V \neq 0$ for all $x \in X$ with $x \neq \bar{x}$, since $\mathcal{L}_g V = g \cdot \nabla V \neq 0$. This also implies that $\bar{x}$ is the unique global minimum of V in X. Indeed, by $V \in C^\infty(X, \mathbb{R})$ and X compact, there exists a global minimum. Since $\nabla V \neq 0$ for all $x \in X \setminus \{\bar{x}\}$, there exists no local extremum of V in $X \setminus \{\bar{x}\}$, hence the only global minimum is necessarily $\bar{x}$. This also implies $\mathcal{L}_{f[\delta_{\bar{x}}]} V(\bar{x}) = 0$.

As another consequence of $\nabla V \neq 0$ for all $x \in X \setminus \{\bar{x}\}$, the sets $\{V = a\}$ are hypersurfaces of $\mathbb{R}^d$; in particular, they have zero Lebesgue measure. We now define the sets

$$X_a^\delta := \{a - \delta \le V \le a\}$$

and the functions

$$\gamma_a(\delta) := |X_a^\delta|,$$

where $|\cdot|$ indicates the Lebesgue measure of a set. Observe that each set X_a^δ is measurable because it is defined by level sets of a C^∞ function, hence the function γ_a is well defined. By regularity of V, the function γ_a is also continuous, nondecreasing and it satisfies $\gamma_a(0) = 0$.

We now define a sequence of sets on which we will define the control. First choose an open set O containing X and such that $|O \setminus X| < \frac{c}{4}$. Define $\omega_0 := O \setminus X$. Define now the sequence $\{C_i, \delta_i, \omega_i\}$ as follows:

- $C_1 := C$;
- $\delta_i := \max \left\{ \delta \in \left[0, \frac{C_i - V(\bar{x})}{2}\right] \quad \text{s.t.} \quad \gamma_{C_i}(\delta) \le \frac{c}{4} \right\}$;
- $\omega_i := X_{C_i}^{\delta_i}$;
- $C_{i+1} := C_i - \delta_i$.

We now prove that each C_i satisfies

$$C_i > V(\bar{x}), \tag{9}$$

by induction. First observe that $C_i > V(\bar{x})$ implies $\delta_i > 0$. Indeed, $\frac{C_i - V(\bar{x})}{2} > 0$ implies that the interval $\left[0, \frac{C_i - V(\bar{x})}{2}\right]$ is not reduced to 0. Since γ_{C_1} is continuous and nondecreasing, it holds $\delta_i > 0$. We now prove that (9) holds for $i = 1$. Observe that $C_1 = C > V(\bar{x})$ since X is not reduced to a point. We now prove the induction step. Since $C_i > V(\bar{x})$, then

$$C_{i+1} = C_i - \delta_i \ge C_i - \frac{C_i - V(\bar{x})}{2} = \frac{C_i + V(\bar{x})}{2} > V(\bar{x}).$$

By construction, we also have $|\omega_i| \le \frac{c}{4}$.

We also prove $\lim_{i \to \infty} C_i = V(\bar{x})$, by contradiction. Since the sequence C_i is decreasing, then a limit exists. This implies $\lim_{i \to \infty} \delta_i = 0$. By contradiction, assume $\lim_{i \to \infty} C_i > V(\bar{x})$: then, by definition of δ_i, this implies that $|\omega_i| = \gamma_{C_i}(\delta_i) = \frac{c}{4}$ for an infinite number of indices i. Since the ω_i are essentially disjoint and they are all contained in X for $i > 0$, then $|X| = \infty$. This is in contradiction with compactness of X. Then, $\lim_{i \to \infty} C_i = V(\bar{x})$.

For each ω_i, observe that $\mathcal{L}_g V \neq 0$. Then, by compactness, there exists $m_i > 0$ such that $|\mathcal{L}_g V| \ge m_i$.

We now define the control strategy. Define the sequence $T_0 := 0$, $T_{i+1} = T_i + \frac{\delta_i}{m_i}$. On each interval $[T_{i-1}, T_i)$ define $\omega(t) := \omega_{i-1} \cup \omega_i \cup \omega_{i+1}$, that satisfies $|\omega| \le \frac{3c}{4} < c$. Define

$$u(t, x) := -\mathrm{sign}(\mathcal{L}_g V)(x) \cdot \phi(t, x) \tag{10}$$

with ϕ being the Lipschitz function

$$\phi(t, x) \quad := \quad \begin{cases} 1 & \text{on } \omega_i, \\ 0 & \text{on } \Omega_i, \\ \frac{d(x, \Omega_i)}{d(x, \Omega_i) + d(x, \omega_i)} & \text{on } \omega_{i-1} \cup \omega_{i+1}, \end{cases} \tag{11}$$

where $\Omega_i := \mathbb{R}^d \setminus (\omega_{i-1} \cup \omega_i \cup \omega_{i+1})$ and

$$d(x, A) = \inf_{y \in A} |x - y|.$$

We clearly have $\chi_\omega u g$ Lipschitz, by construction.

We now prove that this strategy drives any μ_0 asymptotically to $\delta_{\bar{x}}$. We will prove a stronger result, namely that the solution to (2) at times larger than T_i has support in $\{V \le C_i\}$. We prove such condition by induction. It is clear that $\mathrm{supp}(\mu(T_0)) = \mathrm{supp}(\mu_0) \subset X = \{V \le C_1\}$.

By the induction hypothesis, let $\mathrm{supp}(\mu(T_i)) \subset \{V \le C_i\}$. Consider a family of discretizations $\mu_N(T_i) = \frac{1}{N} \sum \delta_{x_j(T_i)}$ of $\mu(T_i)$ with N particles such that $\mathrm{supp}(\mu_N(T_i)) \subset \{V \le C_i\}$ and $\lim_{N \to \infty} W_1(\mu_N(T_i), \mu(T_i)) = 0$. For each of such discretizations, consider the dynamics of a given particle x_j on the interval $[T_i, T_{i+1}]$. It is given by

$$\dot{x}_j(t) = f[\mu_N(t)](x_j(t)) + u(t, x_j(t)) g(x_j(t)), \tag{12}$$

that is the so-called particle flow associated to (2). Then, the evolution of the quantity $V(x_j(t))$ satisfies

$$\partial_t V(x_j(t)) = \mathcal{L}_{f[\mu_N(t)]}V(x_j(t)) + u(t,x_j(t))\mathcal{L}_g V(x_j(t)). \quad (13)$$

Both terms are nonpositive, by the choice of u. As a consequence both sets $\{V \le C_i\}$ and $\{V \le C_{i+1}\}$ are invariant sets for a given particle x_j under the dynamics (12).

For the second term of (13), we have two possibilities: either $x_j(t) \in \omega_i$, in which case we have $u(t,x_j(t))\mathcal{L}_g V \le -m_i$ by construction, or $x_j(t) \in \{V \le C_{i+1}\}$. Observe that, if $x_j(\bar t) \in \{V \le C_{i+1}\}$ for some $\bar t \in [T_i, T_{i+1}]$, then this is satisfied for all times $t \in [\bar t, T_{i+1}]$, by invariance of $\{V \le C_{i+1}\}$ under the particle flow (12).

Then, consider the dynamics of the particle x_j on the interval $[T_i, T_{i+1}]$: either it satisfies $x_j(\bar t) \in \{V \le C_{i+1}\}$ for some $\bar t$, and in this case we have $x_j(T_{i+1}) \in \{V \le C_{i+1}\}$, or it satisfies $x_j(t) \in \omega_j$ for all $t \in [T_i, T_{i+1}]$. We prove that the second case cannot happen. Indeed, if it would happen, then $V(x_j(T_i)) \le C_i$ and $\partial_t V \le -m_i$ for a time interval of length $\frac{\delta_i}{m_i}$, hence $V(x_j(T_{i+1})) \le C_i - \delta_i = C_{i+1}$.

Since the proof does not depend on the number of particles, this implies that for all N we have $\mathrm{supp}(\mu_N(t)) \subset \{V \le C_i\}$ for all $t \in [T_i, T_{i+1}]$ and $\mathrm{supp}(\mu_N(T_{i+1})) \subset \{V \le C_{i+1}\}$. Since the time interval $[T_i, T_{i+1}]$ is finite, we have

$$\lim_{N\to\infty} W_p(\mu_N(t),\mu(t)) \le \lim_{N\to\infty} e^{K_1(T_{i+1}-T_i)} W_p(\mu_N(T_i),\mu(T_i)) = 0.$$

Here K_1 is a constant bounding the Lipschitz constant of the vector fields $f[\mu] + \chi_\omega u g$, that exists by properties of f and the construction of $\chi_\omega u$ as a Lipschitz function. Convergence in Wasserstein distance with equibounded support implies that the limit has the same support. Hence, $\mathrm{supp}(\mu(t)) \subset \{V \le C_i\}$ for $t \in [T_i, T_{i+1}]$ and $\mathrm{supp}(\mu(T_{i+1})) \subset \{V \le C_i\}$. The induction property is then proved.

It remains to prove that $\mathrm{supp}(\mu(t)) \subset \{V \le C_i\}$ for $t \in [T_i, T_{i+1}]$ for all i implies weak-* convergence as t tends to $+\infty$ of μ to $\delta_{\bar x}$, that we denote

$$\mu(t) \rightharpoonup \delta_{\bar x},$$

We prove that $x(t) \in \mathrm{supp}(\mu(t))$ for $t \in [0,+\infty)$ implies $\lim_{t\to\infty} x(t) = \bar x$. Indeed, $V(x(t)) \le C_i$ for $t \in [T_i, T_{i+1}]$ and $V(x(t)) \ge V(\bar x)$ since $x(t) \in X$. Since $\lim_{i\to\infty} C_i = V(\bar x)$, then $\lim_{t\to\infty} V(x(t)) = V(\bar x)$. By uniqueness of the minimum of V in X, we have $\lim_{t\to\infty} x(t) = \bar x$. Consider now the 1-Wasserstein distance, that satisfies $W(\mu(t),\delta_{\bar x}) = \int |x - \bar x| d\pi(x,\bar x)$ for a given transference plan π. Since $\lim_{t\to\infty}|x - \bar x| = 0$, then $\lim_{t\to\infty} W_p(\mu(t),\delta_{\bar x}) = 0$, hence $\mu(t) \rightharpoonup \delta_{\bar x}$. ■

IV. KINETIC JURDJEVIC-QUINN STABILIZATION

In this section, we state and prove the main general result of this article. We prove the existence of a stabilizing control for (2), that is computed by a generalized Jurdjevic-Quinn method.

We first fix the following notation. For a set $A \subset \mathbb{R}^d$, we denote by $B(A,\varepsilon)$ its neighborhood with respect to the Euclidean distance, i.e. $B(A,\varepsilon) = \cup_{a\in A} B(a,\varepsilon)$. Similarly, for a set $A \subset \mathcal{P}(X)$, we denote by $B_W(A,\varepsilon)$ its neighborhood with respect to the 1-Wasserstein distance, i.e. $B_W(A,\varepsilon) = \cup_{a\in A} B_W(a,\varepsilon)$ with $B_W(a,\varepsilon) = \{\mu \in P(X) \mid W_1(\mu,a) < \varepsilon\}$. Observe that we consider measures in $\mathcal{P}(X)$ only. We recall that compactness of X implies compactness of $\mathcal{P}(X)$ in the Wasserstein topology, see e.g. [23, p. 117].

We now state and prove the main result of this article.

Theorem 3: Let $V \in C^\infty(\mathbb{R}^d,\mathbb{R})$, and let $C \in \mathbb{R}$ be such that $X := \{V \le C\}$ is compact. Define $\mathcal{Z}$ by

$$\mathcal{Z} := \left\{ \begin{array}{l} x \in X \quad \text{s.t. there exists } \mu \in \mathcal{P}(X) \text{ for which} \\ \mathcal{L}^k_{f[\mu]}\mathcal{L}_g V(x) = 0 \quad \text{for all } k \in \mathbb{N} \end{array} \right\}.$$

We assume $\mathcal{Z}$ to be a stratified set in the sense of Whithney (see [11]) of codimension greater or equal to one, and that there exists a neighborhood $B(\mathcal{Z},\varepsilon)$ of $\mathcal{Z}$ that is attractive for all vector fields $f[\mu]$ with $\mu \in P(X)$, i.e. all solutions of the particle flow

$$\dot x = f[\mu](x)$$

converge to $\mathcal{Z}$ for $x(0) \in B(\mathcal{Z},\varepsilon)$. Assume moreover that

$$\mathcal{L}_{f[\mu]}V(x) \le 0$$

for all $\mu \in \mathcal{P}(X)$ and $x \in X$.

Then, there exists a control $\chi_\omega u : [0,+\infty) \to \mathrm{Lip}(\mathbb{R}^d,\mathbb{R})$ such that for each $\mu_0 \in \mathcal{P}(X)$

- the unique solution $\mu(t)$ of (2) with control $\chi_{\omega(t)}u(t)$ converges to $\mathcal{Z}$ in the Wasserstein distance, i.e. there exist a time-dependent measure $\bar\mu(t)$ supported in $\mathcal{Z}$ such that
$$\lim_{t\to+\infty} W_1(\mu(t),\bar\mu(t)) = 0;$$
- the control domain $\omega(t)$ satisfies the sparsity constraint $|\omega(t)| < c$ for a given $c > 0$;
- $|u(t)| \le 1$ for all $t \in [0,+\infty)$.

Proof: Define the sets

$$\mathcal{V}_\alpha = (\{V = \alpha\} \cup \{\mathcal{L}_g V \neq 0\}) \setminus B(\mathcal{Z},\varepsilon).$$

These are similar to level sets $\{V = \alpha\}$, but they do not contain critical points and are separated from $\mathcal{Z}$. Then, they are manifolds (possibly with boundary), and in particular they have zero Lebesgue measure.

Similarly to the proof of Theorem 2, we define the sets

$$X^\delta_a := \cup_{\alpha\in[a-\delta,a]} \mathcal{V}_\alpha$$

and the functions

$$\gamma_a(\delta) := |X^\delta_a|.$$

Their regularity properties are the same as in the proof of Theorem 2.

Denote $V_* = \inf_{x\in X} V(x)$. We now define a sequence of sets on which we will define the control. Choose an open set O containing X and such that $|O \setminus X| < \frac{c}{4}$. Define $\omega_0 := O \setminus X$. Define now the sequence $\{C_i, \delta_i, \tilde\omega_i\}$ as follows:

- $C_1 := C$;
- $\delta_i := \max\left\{\delta \in \left[0, \frac{C_i - V_*}{2}\right] \quad \text{s.t.} \quad \gamma_{C_i}(\delta) \leq \frac{c}{2}\right\}$;
- $\tilde{\omega}_i := X_{C_i}^{\delta_i}$;
- $C_{i+1} := C_i - \delta_i$.

With the same techniques used in the proof of Theorem 2, we prove that each C_i satisfies $C_i > V_*$ and $\lim_{i\to\infty} C_i = V_*$. Replace now the sets $\tilde{\omega}_i$ with their closure $\omega_i = \text{cl}(\tilde{\omega}_i)$. By construction, such sets contain $\{\mathcal{L}_g V = 0\} \setminus B(\mathcal{Z}, \varepsilon)$. Notice that $\{\mathcal{L}_g V = 0\}$ has zero measure since it is contained in $\mathcal{Z}$, which has codimension greater or equal to one. By regularity of the $\tilde{\omega}_i$ we have $|\omega_i| = |\tilde{\omega}_i| \leq \frac{c}{2}$.

We now want to prove the following property: given a measure μ_k with support in $B(\mathcal{Z}, \varepsilon) \cup (\cup_{i=k}^{\infty} \omega_i)$, we find a time T_k and a control u such that the corresponding solution μ_{k+1} of (2) at time T_k is supported in $B(\mathcal{Z}, \varepsilon) \cup (\cup_{i=k+1}^{\infty} \omega_i)$. For simplicity of notation, we prove such property for $k = 1$. Define $\xi > 0$ such that $|B(\omega_1, 4\xi)| < c$ and $4\xi < \frac{\varepsilon}{2}$, that exists by regularity of ω_i. Define the control

$$u(x) = -\text{sign}(\mathcal{L}_g V(x)) \cdot \phi(x) \tag{14}$$

with ϕ being the Lipschitz function

$$\phi(t, x) \quad := \quad \begin{cases} 1 & \text{on } B(\omega_1, 3\xi), \\ 0 & \text{on } \mathbb{R}^d \setminus B(\omega_1, 4\xi), \\ \tilde{\phi}(x) & \text{on } B(\omega_1, 4\xi) \setminus B(\omega_1, 3\xi), \end{cases}$$

with $\tilde{\phi}(x) = \frac{d(x, \mathbb{R}^d \setminus B(\omega_1, 4\xi))}{d(x, \mathbb{R}^d \setminus B(\omega_1, 4\xi)) + d(x, B(\omega_1, 3\xi))}$. Observe that the resulting vector field $\chi_\omega u g$ is Lipschitz, then existence and uniqueness for (2) is ensured. Then, for a given $\mu_0 \in \mathcal{P}(X)$, the vector field $f[\mu(t)] + \chi_\omega u g$ is well defined. We now want to prove that for some finite time T_1 we have $\text{supp}(\mu(T_1)) \cap \omega_1 = \emptyset$. To prove it, we use the following technique: we consider any particle x_0, eventually coming from a discretization of μ, and the dynamics given by

$$\dot{x} = f[\mu(t)] + \chi_\omega u g. \tag{15}$$

By a sequence of steps based on compactness arguments, we aim to prove the existence of the global T_1 such that all particles $x_0 \in \overline{B(\omega_1, \xi)}$ exit such set in time T_1 under the dynamics (15).

First consider the larger set $\overline{B(\omega_1, 3\xi)}$ and observe that it satisfies $\overline{B(\omega_1, 3\xi)} \cap \mathcal{Z} = \emptyset$ by construction. Then, consider the controlled system $\dot{x} = f[\mu(t)] + ug$ and apply the Jurdjevic-Quinn strategy [16] with Lyapunov function V, that does not depend on μ: such strategy coincides with the control (14) on the set $\overline{B(\omega_1, 3\xi)}$, by construction. We have two possibilities: either the solution of (15) exits $\overline{B(\omega_1, 3\xi)}$, or it satisfies the Jurdjevic-Quinn condition, hence it converges to the maximal invariant subset of

$$\left\{\mathcal{L}_{f[\mu(t)]}^k \mathcal{L}_g V(x) = 0 \quad \text{for all } k \in \mathbb{N}\right\}.$$

Even though such set varies in time, it is included in $\mathcal{Z}$, hence the solution of (15) converges to $\mathcal{Z}$. Since $\mathcal{Z}$ and $\overline{B(\omega_1, 3\xi)}$ have distance at least $\frac{\varepsilon}{2} + \xi$ by construction, we have that in this case too the solution exits $\overline{B(\omega_1, 3\xi)}$. Observe that, since $\dot{V} \leq 0$, either the solution enters $B(\mathcal{Z}, \varepsilon)$ or it enters $\cup_{i=2}^{\infty} \omega_i$.

Then there exists a finite time $T^3(\mu_0, x_0)$ for which the particle $x_0 \in \overline{B(\omega_1, 3\xi)}$ exits $\overline{B(\omega_1, 3\xi)}$ under the dynamics (15) with $\mu(t)$ being the unique solution of (2) with initial data μ_0. Such function is not continuous, in general. Restrict now the initial data $x_0 \in \overline{B(\omega_1, 2\xi)}$ and consider the time $T^2(\mu_0, x_0)$ to exit $\overline{B(\omega_1, 2\xi)}$ itself. By continuity of the solution of (15), we have that for each ε^2 there exists $\delta^2(x_0)$ such that for all $x \in B(x_0, \delta^2(x_0))$ it holds

$$T^2(\mu_0, x) \leq T^3(\mu_0, x_0) + \varepsilon^2.$$

Since open sets of the form $B(x_0, \delta^2(x_0))$ with $x_0 \in \overline{B(\omega_1, 2\xi)}$ cover the compact set $\overline{B(\omega_1, 2\xi)}$ itself, then one can extract a finite covering $\left\{B(x_0^i, \delta^2(x_0^i))\right\}_{i=1}^{I}$. This implies $T^2(\mu_0, x) \leq T_*^2(\mu_0)$ for all $x \in \overline{B(\omega_1, 2\xi)}$, with $T_*^2 = \max_{i=1,\ldots,I} T^3(\mu_0, x_0^i) + \varepsilon^2$.

Restrict now the initial data $x_0 \in \overline{B(\omega_1, \xi)}$ and consider the time $T^1(\mu_0, x_0)$ to exit $\overline{B(\omega_1, \xi)}$ itself. By continuity of the solution of (2), and of the corresponding solution of (15) with respect to the vector field, we have that for each ε^1 there exists $\delta^1(\mu_0)$ such that for all $\mu \in B_W(\mu_0, \delta^1(\mu_0))$ it holds

$$T^1(\mu, x) \leq T^2(\mu_0, x) + \varepsilon^1 \leq T_*^2(\mu_0) + \varepsilon^1.$$

Recall that $\mathcal{P}(X)$ is compact for the Wasserstein topology and that open sets $B_W(\mu_0, \delta^1(\mu_0))$ cover $\mathcal{P}(X)$. By extracting a finite covering indexed by $\left\{\mu_0^j\right\}_{j=1}^{J}$, we have that

$$T^1(\mu, x) \leq \max_{j=1,\ldots,J} T_*^2(\mu_0^j) + \varepsilon^1.$$

For simplicity of notation, we denote $T_1 = \max_{j=1,\ldots,J} T_*^2(\mu_0^j) + \varepsilon^1$, with reference to the initial set ω_1.

As a consequence, we have found a time T_1 independent on the initial data μ_0, x_0 for which the control defined by (14) drives each particle from ω_1 to $B(\mathcal{Z}, \varepsilon) \cup (\cup_{i=2}^{\infty} \omega_i)$. Consider now the initial measure μ_0 with support in $\mathcal{P}(X)$ and discretize it with a sequence of initial data μ_0^N. Then, the previous study for particles shows that, for each discretized initial data, the corresponding solution satisfies $\text{supp}(\mu^N(T_1)) \subset B(\mathcal{Z}, \varepsilon) \cup (\cup_{i=2}^{\infty} \omega_i)$. Then, it also holds $\text{supp}(\mu(T_1)) \subset B(\mathcal{Z}, \varepsilon) \cup (\cup_{i=2}^{\infty} \omega_i)$.

Apply now the previous proof to $\omega_2, \omega_3, \ldots$ and observe that all particles entering $B(\mathcal{Z}, \varepsilon)$ do not exit by local attractivity of this set. Then, by construction of control, we have the existence of times $T_2, T_3, \ldots$ such that the solution of (2) with initial data μ_0 satisfies $\text{supp}(\mu(T_1 + T_2 + \ldots + T_{k-1})) \subset B(\mathcal{Z}, \varepsilon) \cup (\cup_{i=k}^{\infty} \omega_i)$. This observation, together with local attractivity of $B(\mathcal{Z}, \varepsilon)$, implies that the solution converges to a solution supported in $\mathcal{Z} \cup \{V = V_*\}$. It is now sufficient to observe that $\{V = V_*\} \subset \mathcal{Z}$ to have the result. ■

V. SPARSE STABILIZATION OF THE KINETIC CUCKER-SMALE MODEL

In this section, we study the problem of sparse stabilization of the kinetic Cucker-Smale model. It is the mean-field

limit of the well-known model for crowds dynamics, called the Cucker-Smale model, first proposed in [7]. The kinetic version was presented in [12]. We presented in [20] the concept of sparse control for such kinetic model. Such model may, for instance, reproduce the behavior of a group of birds, in which each bird tries to align its velocity with the velocities of its neighbors. For this reason, the standard goal of control strategies is to enforce alignment of velocities. In particular, in [3], [4] the authors present a sparse strategy enforcing alignment for the original Cucker-Smale. In [20] we instead showed techniques to enforce alignment of the kinetic Cucker-Smale model.
In this section, we prove alignment by adapting the strategy presented in Section 3. For simplicity of notation, we consider a second-order system on the real line, i.e. $(x, v) \in \mathbb{R}^2$. Given a set of N birds, the dynamics of the i-th bird is given by

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{1}{N}\sum_{j=1}^{N} \phi(|x_j - x_i|)(v_j - v_i), \quad i = 1, \dots N, \end{cases} \tag{16}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a nonincreasing positive function, accounting for the influence between two individuals, that depends on their distance only. Depending on the strength of the interaction ϕ for large distances, the crowd can either exhibit asymptotic alignment of the velocities or it disperses into separate clusters without aligning.

In [12], the authors have formally proved the existence of the mean-field limit of the Cucker-Smale model when the number of agents tends to infinity. The limit of the dynamics (16) is given by the transport PDE

$$\partial_t \mu + v \cdot \nabla_x \mu + \nabla_v \cdot [\xi[\mu]\,\mu] = 0, \tag{17}$$

where $\mu = \mu(t, x, v)$ is the density of crowd and $\xi[\mu](x, v) := \int \phi(|x-y|)(w-v)\, d\mu(x, v)$ is the interaction kernel. Its expression satisfies hypothesis $(\mathbf{V})$ for ϕ Lipschitz, see e.g. [19].

Consider now the add of a control on the velocity variable in (17). We write the dynamics in the form

$$\partial_t \mu + \nabla \cdot ((f[\mu] + \chi_\omega u g)\mu) = 0 \tag{18}$$

where $f[\mu] = (v, \xi[\mu])$ and $g = (0, 1)$. We want to apply Theorem 3 to this equation. For this reason, we consider the following Lyapunov function

$$V(x, v; x_0, v_0) = (v - v_0)^2 + (x - x_0)^2 \psi(-v(x - x_0)),$$

where $(x_0, v_0) \in \mathbb{R}^2$ is a fixed position-velocity, and $\psi : \mathbb{R} \to \mathbb{R}$ is a C^∞ function satisfying the following properties:

1) $\psi(x) = 0$ for $x \leq 0$;
2) ψ is strictly increasing on $[0, +\infty)$;
3) $\lim_{x \to +\infty} \psi(x) = +\infty$;
4) $\psi^{(n)}(0) = 0$ for all order $n \in \mathbb{N}$ of derivatives.

A possible choice for ψ is $\psi(x) = xe^{-\frac{1}{x}}\chi_{[0,+\infty)}$. It is clear that ψ cannot be analytic.

We now aim to prove that $V(\cdot, \cdot; x_0, v_0)$ satisfies hypotheses of Theorem 3. Fix an initial distribution $\mu_0 \in \mathcal{P}_c(\mathbb{R}^2)$ and remark that its mean velocity $\bar{v} = \int v\, d\mu(x, v)$ is a constant for solutions of (17). By translating the velocity variable to have $\bar{v} = 0$, one can define a box $[a, b] \times [-V, +V]$ containing the support of μ_0. Fix a large enough final time T and observe that the solution $\mu(t)$ of (17) is contained in $X_T := [a-TV, b+TV] \times [-V, V]$, that is compact. Observe that $\{V \leq C\}$ is not compact, since $V(x, v; x_0, v_0) = 0$ for all $v = v_0$ and $(x - x_0)v < 0$. Nevertheless, by changing its definition outside X_T, the set $\{V \leq C\}$ can be chosen to be compact.

Fix now a pair $(x_0, v_0) \in \mathbb{R}^2$. On the whole space $\mathcal{P}(X_T)$ the condition

$$\begin{aligned} &\mathcal{L}_{f[\mu]} V(x, v; x_0, v_0) = 2v(x - x_0) \cdot \psi(-(x - x_0)v) - \\ &v^2(x - x_0)^2 \psi'(-v(x - x_0)) + \\ &\xi[\mu](x, v) \cdot (2(v - v_0) - (x - x_0)^3 \psi'(-v(x - x_0))) \leq 0 \end{aligned} \tag{19}$$

is not satisfied in general. Nevertheless, it holds $v(x - x_0)\psi(-(x - x_0)v) \leq 0$ by definition of ψ. All other terms satisfy similar conditions, due to definition of ψ, except for the term $\xi[\mu](x, v) \cdot 2(v - v_0)$. For such last term, observe that the interaction kernel satisfies $\xi[\mu](x, v) \cdot 2(v - \bar{v}) < 0$ for $v \neq \bar{v}$ where $\bar{v}$ is the mean velocity of μ, see e.g. [7], [12]. Then, by choosing $v_0 = \bar{v} = 0$ with μ_0, we have $L_{f[\mu_0]} V(\cdot, \cdot, x_0, 0) < 0$. Observe also that $-(x - x_0)^3 \psi'(-v(x - x_0))$ and v have the same sign, hence the term $-\xi[\mu](x, v) \cdot (x - x_0)^3 \psi'(-v(x - x_0))$ is negative too. By Lipschitzianity of $f[\mu]$, this implies that there exists a whole neighborhood $B_W(\mu_0, \varepsilon)$ for which the condition $L_{f[\mu]} V(\cdot, \cdot, x_0, 0) \leq 0$ is satisfied.
We also compute $\mathcal{L}_g V(x, v; x_0, v_0) = 2(v - v_0) - (x - x_0)^3 \psi'(-v(x - x_0)) \neq 0$ except for

$$\mathcal{Z} = \{(x, v_0) | \mid v_0(x - x_0) \geq 0\}.$$

Then, observe that $\mathcal{L}_{f[\mu]} \mathcal{L}_g V = 2\xi[\mu] = 0$ on $\mathcal{Z}$, and similarly for higher-order derivatives.

As a consequence, we can apply the Theorem 3 as follows. First define the functional

$$\mathcal{W}(\mu, x_0, v_0) = \int V(x, y, x_0, v_0)\, d\mu(x, v).$$

For μ_0 fixed, with zero mean velocity, compute the control $\chi_\omega u$ according to the Theorem and find a whole time interval $[0, T_1]$ on which the solution of (18) satisfies $L_{f[\mu(t)]} V(\cdot, \cdot, x_0, 0) \leq 0$. On such time interval, the function $V(x, v; x_0, 0)$ is decreasing. Moreover, by approximating $\mu(t)$ with particles, it is also possible to prove that $\mathcal{W}(\mu, x_0, v_0)$ is decreasing too. With the same technique, one can prove that the control does not increase the support of the solution, hence that the support is contained in X_T.

Denote now by μ_1 the solution of (18) at time T_1 and v_1 its mean velocity. One of the properties of the mean is that it minimizes the mean square displacement, and in particular it holds $\int (v - v_1)^2\, d\mu_1 \leq \int (v - v_0)^2\, d\mu_2$. This implies that $\mathcal{W}(\mu_1, x_0, v_1) \leq \mathcal{W}(\mu_0, x_0, v_0)$. Also observe that the mean velocity v_1 is contained in the support of velocities of μ_0, that is invariant with respect to the solution of (18).

We now repeat the procedure: for the initial data μ_i at time T_i, consider the function $V(\cdot, \cdot, x_0, v_i)$ and build a control based on the Jurdjevic-Quinn strategy in Theorem 3. Then find a time T_{i+1} and a solution μ_{i+1} satisfying $\mathcal{W}(\mu_{i+1}, x_0, v_{i+1}) \leq \mathcal{W}(\mu_i, x_0, v_i)$. We now prove that such strategy drives the solution to $\mathcal{Z}$. Since the sequence T_i is increasing, it admits a limit T_*. If such limit is finite, then the sequence μ_i admits a limit μ_*. If this is the case, such limit is in X_T for $T > T_*$. Then, one can always restart the procedure. Hence, the sequence T_i grows to infinity.

Denote now with $m_x(\mu), m_v(\mu)$ the Lebesgue measure of the smallest interval containing the projection of the support of μ in the x and v variable, respectively. Invariance of the support in the velocity variables implies that m_v is nonincreasing, hence it admits a limit. If the limit is zero, then we have achieved alignment, as required. Otherwise, m_v admits a strictly positive limit m_v^*. We prove that this is impossible. Indeed, by construction of the control, the rate decrease of m_v is smaller than $-\frac{c}{3m_x}$ and the rate of increase of m_x is smaller than m_v. Then, starting from a sufficiently large time T, we have m_v close to m_v^* and m_x^T given. The function m_v then satisfies

$$\dot{m}_v(t) \leq -\frac{c}{3(m_x^T + (t-T)(v^* - \varepsilon))},$$

that gives by integration

$$\begin{aligned} m_v(t) &\leq m_v(T) - \int_T^t \frac{c}{3(m_x^T + s(v^* - \varepsilon)} \, ds \\ &\leq m_v(T) - \frac{c(v^* - \varepsilon)}{3} \log(m_x^T + s(v^* - \varepsilon)). \end{aligned}$$

This implies that m_v converges to $-\infty$ which gives a contradiction.

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