Robust optimal stabilization of the Brockett integrator via a hybrid feedback

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Abstract

The problem of semi-global robust stabilization of the Brockett integrator (also called Heisenberg system) in minimal time is addressed and solved by means of a hybrid feedback law. It is shown that the solutions of the closed-loop system converge to the origin in minimal time (for a given bound on the control) with a robustness property with respect to small measurement noise, external disturbance and actuator noise.

Keywords: Brockett integrator, optimal control, hybrid feedback, robust stabilization, measurement errors, actuator noise, external disturbances.

1 Introduction

Let M be a n-dimensional manifold. We consider on M a control system of the form

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{1}$$

where $f_1, \ldots, f_m$ are smooth vector fields on M, and where the control $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$ satisfies the constraint

$$\sum_{i=1}^{m} u_i(t)^2 \leq 1. \tag{2}$$

Let x_0 be a point of the manifold M. The system (1), together with the constraint (2), is said to be globally asymptotically stabilizable at the point x_0, if, for each point x of M, there exists a control satisfying the constraint (2) such that the solution of (1) associated to this control and starting from x tends to x_0 as t tends to $+\infty$.

This asymptotic stabilization problem has a long history and has been widely investigated. Note that as soon as $m < n$ the Brockett's condition [10, Theorem 1, (iii)] is not satisfied by (1), and thus there does not exist a continuous stabilizing feedback law for (1). However several control laws have been derived for such a driftless control systems, see e.g. [19, 15, 5] and references therein.

The robust asymptotic stabilization is under actual and very active research. There exists a large variety of control laws that solve the robust asymptotic stabilization problem, such as discontinuous sampling feedback [11, 29], time-varying control laws [12, 18, 20, 21], patchy feedbacks (as in [3]), SRS

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feedbacks [28], ..., yielding different robustness properties depending on the errors under consideration in these papers.

The class of feedback laws under consideration in this paper consists of feedbacks mixing discrete and continuous components. It gives rise to a closed-loop system with a hybrid term studied e.g. in [31, 8]. The use of such a class of feedbacks for the stabilization of nonlinear systems (*a priori* without discrete state) appeared first in [23]. It allows to design a switching strategy between different smooth control laws defined on a partition of the state space. This idea of defining the control smoothly part by part and switch between the different components is very usual in nonlinear control theory see eg [30, 4]. In this paper we compute an optimal control that is smooth on a part of the state space and another control law defined on the complement of this part. We use a hybrid term to define the switching strategy between both control laws, which ensures robustness with respect to (small) measurement noise, acturator errors and external disturbances.

More precisely the first step of our procedure consists in solving the time-optimal control problem, for the system (1) submitted to the constraint (2), of steering a point x to the point x_0. Of course, on the one hand this problem is in general very difficult to solve, and on the other hand due to the Brockett condition such controls are not smooth functions of x whenever $m < n$. This raises the problem of the regularity of optimal controls in a closed-loop form. The literature on this subject is immense. The problem of determining the analytic regularity of the value function for a given (analytic) optimal control problem, has been, among others, investigated by [30]. For systems of the form (1), the time-optimal problem under the constraint (2) is equivalent to the sub-Riemannian problem associated to the vector fields $f_1, \ldots, f_m$. In this framework, the time-minimal function to x_0 is equal to the sub-Riemannian distance to x_0, see for instance [7]. The analytic regularity of the sub-Riemannian distance is related to the existence of singular minimizing trajectories, see [1, 2, 32]. More precisely, if the vector fields $f_1, \ldots, f_m$ are analytic and if there does not exist any nontrivial singular minimizing trajectory starting from x_0, then the sub-Riemannian distance to x_0 is subanalytic outside x_0 (see [13, 14] for a general definition of subanalytic sets).

In the present article we focus on the so-called Brockett system in $\mathbb{R}^3$

$$\dot{x}(t) = u_1(t) f_1(x(t) + u_2(t) f_2(x(t)), \tag{3}$$

where, denoting $x = (x_1, x_2, x_3)$,

$$f_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \ f_2 = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}, \tag{4}$$

and the control satisfies the constraint

$$u_1(t)^2 + u_2(t)^2 \leq 1. \tag{5}$$

Our aim is twofold, and consists in achieving a robust stabilization process for the system (3) under the constraint (5). To do this we first solve the corresponding time-optimal control problem and then define a hybrid feedback law using a suitable switching strategy (more precisely an hysteresis) between this time-optimal control and another control defined on a neighborhood of the discontinuity set of the optimal control.

In the present proceedings paper, we do not give the proofs, but we explicit our hybrid feedback law and its construction. First of all, we recall a notion of solution adapted to hybrid controls, and make precise the notion of stabilization via a hybrid feedback law in minimal time in Section 2. We then state the main result, namely that there exists a hybrid time-minimal control stabilizing semi-globally the origin of (3), (5) (see Section 3). The rest of the paper is then devoted to the contruction of the hybrid feecback. We define our "local" minimal-time control law in Section 4.1, the "global"one in Section 4.2 and the hybrid feedback law by making the hysteresis between the both components (see Section 5). For a complete proof of the robust optimal stabilization see [27].

2 Class of controllers and notion of solutions

In this section we introduce the notions of controller and of solutions of differential equations that will be used throughout the paper.

The controllers under consideration admit the following description (see [31, 8])

$$u = u(x, s_d), \quad s_d = k_d(x, s_d^-), \tag{6}$$

where s_d evolves in the finite set $\{1,2\}$, $k : \mathbb{R}^n \times \{1,2\} \to \mathbb{R}^2$ is continuous in x for each fixed s_d, $k_d : \mathbb{R}^n \times \{1,2\} \to \{1,2\}$ is a function, and s_d^- is defined, at this stage only formally, as

$$s_d^-(t) = \lim_{s<t} s_d(s). \tag{7}$$

The set $\{1,2\}$ is endowed with the discrete topology, *i.e.* every set is an open set. The above controller is hybrid due to the presence of the discrete dynamics of s_d. It gives rise to a non-classical ordinary differential equation describing the dynamics of the closed-loop system.

Denoting $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ the function defining the right hand-side of the differential equation (3), we can rewrite (3) as

$$\dot{x} = f(x, u). \tag{8}$$

In this paper we are interested in a notion of robustness to small noise. Consider two functions e and d satisfying the following *regularity assumptions*:

$$\begin{aligned} &e(\cdot,\cdot), d(\cdot,\cdot) \in L^\infty_{loc}(\mathbb{R}^n \times [0,+\infty); \mathbb{R}^n), \\ &e(\cdot,t), d(\cdot,t) \in C^0(\mathbb{R}^n, \mathbb{R}^n), \ \forall t \in [0,+\infty). \end{aligned} \tag{9}$$

We introduce these functions as a measurement noise e and an external noise d, and define the perturbed system[1] with u given by (6), *i.e.*

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(x(t) + e(x(t),t), s_d(t))) + d(x(t),t), \\ s_d(t) &= k_d(x(t) + e(x(t),t), s_d^-(t)). \end{aligned} \tag{10}$$

The notion of solution of such hybrid perturbed systems has been well-studied in the literature, see *e.g.* [8, 9, 17, 31, 25, 26]. To be self-contained, let us recall the definition of a solution of (10).

Definition 2.1. *Given $T > 0$, $(x_0, s_0) \in \mathbb{R}^n \times \{1,2\}$, and a non-empty set $\mathcal{RC}$ strictly contained in $\mathbb{R}^n \times \{1,2\}$, we say that (X, S_d) is a* solution*, starting from (x_0, s_0), of (10) on $[0,T)$ if the following conditions hold:*

1. *the map X is absolutely continuous on $[0,T)$;*
2. *there holds, for almost all t in $[0,T)$,*
$$\dot{X}(t) = f\left(X(t), k(X(t) + e(X(t),t), S_d(t))\right) + d(X(t),t);$$
3. *for all $t \in [0,T)$ such that $(X(t), S_d(t))$ is in $\mathcal{RC}$, the mapping S_d is right-continuous at t;*
4. *for all $t \in (0,T)$ such that $S_d^-(t)$ exists, one has*
$$S_d(t) = k_d(X(t) + e(X(t),t), S_d^-(t)); \tag{11}$$
5. *there hold $X(0) = x_0$ and $S_d(0) = k_d(x_0 + e(x_0, 0), s_0)$.*

[1] we can also consider an actuator noise, see *e.g.* [16, 23].

In this context, let us define the concept of stabilization of (8) by a hybrid feedback law in minimal time with a robustness property with respect to measurement noise and external disturbance. The usual Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$ and we recall that a function of class $\mathcal{K}_\infty$ is a function δ: $[0,+\infty) \to [0,+\infty)$ which is continuous, strictly increasing, satisfying $\delta(0) = 0$ and $\lim_{\varepsilon\to+\infty} \delta(\varepsilon) = +\infty$.

Definition 2.2. *Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying*

$$\rho(x) > 0, \ \forall x \neq 0. \tag{12}$$

We say that the completeness assumption for ρ *holds if, for all (e, d) satisfying the regularity assumptions (9), and so that,*

$$\sup_{[0,+\infty)}|e(x,\cdot)| \leq \rho(x), \ \operatorname{esssup}_{[0,+\infty)}|d(x,\cdot)| \leq \rho(x), \ \forall x \in \mathbb{R}^n, \tag{13}$$

for all $(x_0, s_0) \in \mathbb{R}^n \times \{1, 2\}$, there exists a maximal solution on $[0,+\infty)$ of (10) starting from (x_0, s_0).

Definition 2.3. *We say that the* uniform finite time convergence property *holds if there exists a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfying (12), such that the completeness assumption for ρ holds, and if there exists a function $\delta : [0,+\infty) \to [0,+\infty)$ of class $\mathcal{K}_\infty$ such that, for all $R > 0$, there exists $\tau = \tau(R) > 0$ such that, for all functions e, d satisfying the regularity assumptions (9) and inequalities (13) for this function ρ, all $x_0 \in \mathbb{R}^n$, $|x_0| \leq R$, and all $s_0 \in \{1, 2\}$, the maximal solution (x, s_d) of (10) starting from (x_0, s_0) satisfies*

$$|x(t)| \leq \delta(R), \ \forall t \geq 0, \tag{14}$$

and

$$x(t) = 0, \ \forall t \geq \tau. \tag{15}$$

We are now able to define the main concept of the paper.

Definition 2.4. *The origin is said to be a* semi-global minimal time robust stable equilibrium *for the system (8) if, for all $\varepsilon > 0$ and all compact subset $K \subset \mathbb{R}^n$, there exists a hybrid feedback law $(u, k_d) : \mathbb{R}^n \times \{1, 2\} \to \mathbb{R}^m \times \{1, 2\}$ satisfying the constraint*

$$\|u(x, s_d)\| \leq 1, \tag{16}$$

where $\|\cdot\|$ stands for the Euclidian norm in $\mathbb{R}^m$, such that:

- *the uniform finite time convergence property holds;*
- *there exists a continuous function $\rho_{\varepsilon,K} : \mathbb{R}^n \to \mathbb{R}$ satisfying (12) for $\rho = \rho_{\varepsilon,K}$, such that, for all functions e, d satisfying the regularity assumptions (9) and inequalities (13) for $\rho = \rho_{\varepsilon,K}$, all $x_0 \in K$, the maximal solution of (10) starting from x_0 reaches the origin within time $T(x_0)+\varepsilon$, where $T(x_0)$ denotes the minimal time to steer the system (8) from x_0 to the origin, under the constraint $\|u\| \leq 1$.*

3 Main result

Theorem 1. *There exists a hybrid feedback law (u, k_d), $u : \mathbb{R}^3 \times \{1, 2\} \to \mathbb{R}^2$ and $k_d : \mathbb{R}^3 \times \{1, 2\} \to \{1, 2\}$, such that the origin is a semi-global minimal-time robust stable equilibrium for the system (3), under the constraint (5).*

Remark 3.1 In Section 5 we give an explicit expression of the hybrid feedback law (u, k_d). ◇

Remark 3.2 We can state a dual result for the stabilization to the origin of the Brockett integrator with minimal energy when fixing the final time. ◇

Intuitively, the strategy is as follows. For $x \in \mathbb{R}^n$, let $T(x)$ denote the minimal time needed to steer the system (3) from x to the origin, under the constraint (5). The corresponding minimal time feedback law, called *local controller*, happens to be continuous (even analytic) on $\mathbb{R}^n \setminus \{x_1 = x_2 = 0\}$. It is therefore necessary to use another controller, called *global controller*, in a neigborhood Ω of the line $\{x_1 = x_2 = 0\}$. More precisely, Ω will be constructed so as to be cylindric around this line, and conic near the origin (see Fig. 3 further). In this neighborhood we have to define an adequate switching strategy. Notice that Ω is arbitrarily thin, and thus the time ε needed for the traversing of Ω is arbitrarily small, uniformly with respect to the initial condition. Therefore, starting from an initial point x, the time needed to join the origin, using this hybrid strategy, is equal to $T(x) + \varepsilon$.

The rest of the paper is organized as follows. We define the local controller in Section 4.1 and the global one in Section 4.2. The switching strategy between these feedbacks by means of a hysteresis is explained in Section 5. For a complete proof of our main result see [27].

4 The components of the hysteresis

4.1 The local controller

In this section we define and compute the *local controller* and we give some properties of the Carathéodory solutions of (8) with such a control law.

Consider the Brockett system (3). It is a standard fact that the minimum time problem for the system (3), with the constraint $u_1^2 + u_2^2 \leq 1$, is equivalent to the *sub-Riemannian problem* in $\mathbb{R}^3$ associated to the vector fields f_1 and f_2 (see for instance [7]), and moreover the minimal time $T(x)$ needed to steer the origin to a point $x \in \mathbb{R}^3$ is equal to the sub-Riemannian distance of x to the origin. Using this fact, the function T may be computed explicitly, and we recall the following result of [6] (see also [27] for a computation)

Proposition 4.1. *Let us consider the minimum time problem for the system (3) under the constraint $u_1^2 + u_2^2 = 1$. The minimal time $T(x)$ needed to steer the origin to a point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is given by*

$$T(x_1, x_2, x_3) = \frac{\theta}{\sqrt{\theta + \sin^2\theta - \sin\theta\cos\theta}} \sqrt{x_1^2 + x_2^2 + 2|x_3|}, \tag{17}$$

where $\theta = \theta(x_1, x_2, x_3)$ is the unique solution in $[0, \pi[$ of

$$\frac{\theta - \sin\theta\cos\theta}{\sin^2\theta}(x_1^2 + x_2^2) = 2|x_3|. \tag{18}$$

Moreover the function T is continuous on $\mathbb{R}^3$, and is analytic outside the line $x_1 = x_2 = 0$.

A level set $\{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = r\}$, where $r > 0$, is drawn on Fig. 1. In the language of sub-Riemannian geometry it represents the sub-Riemannian sphere centered at the origin, with radius r, in the Heisenberg case. Observe that it is axial symmetric, with respect to the axis $(0x_3)$.

On Fig. 2 are drawn intersections of different level sets of T with a plane containing the axis $(0x_3)$.

We can give an explicit expression of the optimal controller (local controller), as follows (see [27])

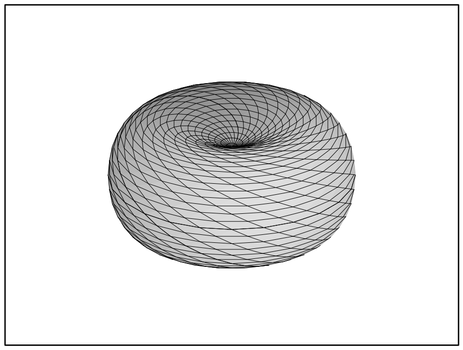

Figure 1: Level set $\{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = r\}$.

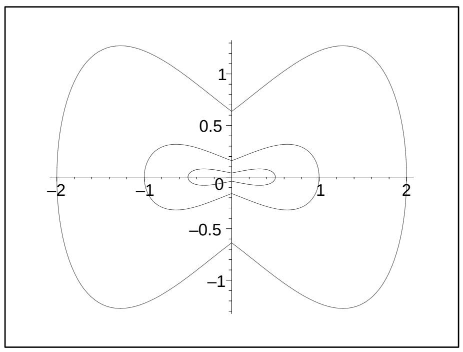

Figure 2: Intersection of different level sets with a vertical plane.

Proposition 4.2. *The time-minimal controller* $u_l = (u_{l1}, u_{l2})$ *steering a point* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ *such that* $x_1^2 + x_2^2 \neq 0$ *to the origin writes*

$$\begin{aligned} u_{l1}(x) &= -\frac{1}{2}\left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}\cos\left(g^{-1}\left(\frac{2|x_3|}{x_1^2+x_2^2}\right)\right) + sign(x_3)\, x_2 \sin\left(g^{-1}\left(\frac{2|x_3|}{x_1^2+x_2^2}\right)\right)\right), \\ u_{l2}(x) &= -\frac{1}{2}\left(\frac{x_2}{\sqrt{x_1^2+x_2^2}}\cos\left(g^{-1}\left(\frac{2|x_3|}{x_1^2+x_2^2}\right)\right) - sign(x_3)\, x_1 \sin\left(g^{-1}\left(\frac{2|x_3|}{x_1^2+x_2^2}\right)\right)\right), \end{aligned} \tag{19}$$

where the function g*, defined by*

$$g(\theta) = \frac{\theta - \sin\theta\cos\theta}{\sin^2\theta}, \ \theta \in]0, \pi[, \quad g(0) = 0,$$

is a monotone increasing diffeomorphism of $[0, \pi[$ *onto* $\mathbb{R}^+$*.*

Now this local controller has been defined, we investigate the robustness properties of the system in closed-loop with this controller.

Given $e, d : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$, the perturbed closed-loop system under consideration in this section is of the form

$$\dot{x}(t) = f(x(t), u_l(x(t) + e(x(t), t))) + d(x(t), t). \tag{20}$$

For all $M > 0$ and $r > 0$, we introduce the subset of $\mathbb{R}^3$

$$\Omega_{M,r} = \{(x_1, x_2, x_3) \in \mathbb{R}^3,\ x_1^2 + x_2^2 \leq \min(r, M|x_3|)\} \tag{21}$$

and let us denote its complementary in $\mathbb{R}^3$ by $\Gamma_{M,r}$. Near the origin, $\Omega_{M,r}$ is a cone, otherwise it is a cylinder around the axis $(0x_3)$, see Fig. 3.

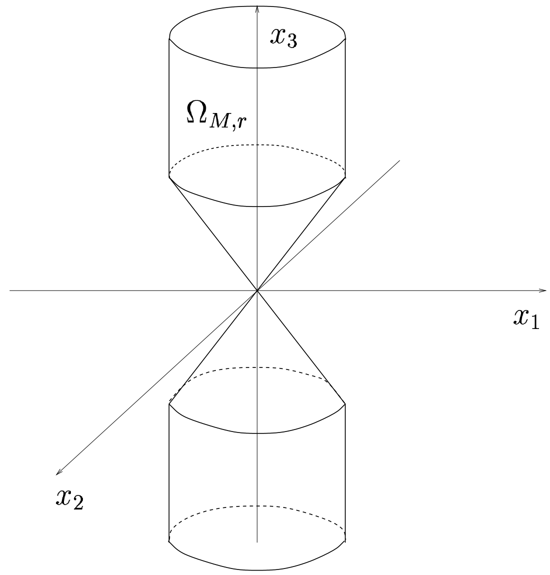

Figure 3: Shape of $\Omega_{M,r}$.

The following lemma is clear from Fig. 2.

Lemma 4.3. *There exist $M_0 > 0$ and $r_0 > 0$ such that, for all M and r satisfying $0 < M < M_0$ and $0 < r < r_0$, the subset $\Gamma_{M,r}$ is invariant by the feedback optimal control u_l.*

A robust version of Lemma 4.3 can be stated for all noise vanishing at the discontinuous set of the local controller. More precisely all properties needed to state our main result are summarized in the following lemma.

Lemma 4.4. *There exist a continuous function $\rho_l : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\rho_l(\xi) > 0,\ \forall \xi \neq 0, \tag{22}$$

and δ_l: $[0, +\infty) \to [0, +\infty)$ a continuous function of class $\mathcal{K}_\infty$ such that, for all $0 < M < M_0$, all $0 < r < r_0$, all $e, d : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ satisfying the regularity assumptions (9) and

$$\sup_{\mathbb{R}_{\geq 0}} |e(x, \cdot)| \leq \rho_l(x_1^2 + x_2^2),\ \operatorname{esssup}_{\mathbb{R}_{\geq 0}} |d(x, \cdot)| \leq \rho_l(x_1^2 + x_2^2), \tag{23}$$

for all x in $\mathbb{R}^n$, and all $x_0 \in \Gamma_{M,r}$, there exists a unique Carathéodory solution X of (20) starting from x_0, maximally defined on $[0, +\infty)$, and satisfying $X(t) \in \Gamma_{M,r}$, for all $t > 0$.

Moreover, for all $R > 0$, there exists $T = T(R) > 0$ such that, for all x_0 in $\mathbb{R}^3$ with $|x_0| \leq R$ and all maximal solution X of (20) starting from x_0, one has

$$|X(t)| \leq \delta_l(R),\ \forall t \geq 0\ , \tag{24}$$

$$|X(t)| = 0,\ \forall t \geq T\ , \tag{25}$$

and

$$\|u_l(X(t))\| \leq 1,\ \forall t \geq 0\ . \tag{26}$$

4.2 The global controller

In this section we define the second component of the hysteresis, called *global controller* and denoted u_g. Moreover we give some basic properties of the Carathéodory solutions of the closed–loop system (8) with such a control law u_g.

Let us consider the following control law:

$$\begin{array}{rcl} u_{g1}(x) & = & 1 \\ u_{g2}(x) & = & 0 \end{array} \tag{27}$$

The closed–loop system considered in this section is of the form

$$\dot{x} = f(x(t), u_g(x(t) + e(x(t), t))) + d(x(t), t) \ . \tag{28}$$

Consider the constants $M_0 > 0$ and $r_0 > 0$ given by Lemma 4.3.

The following result, whose proof is obvious using (27), states that, for all $M > 0$, the trajectories of the system (28) enter the region $\Gamma_{M,r}$ in finite time, while remaining bounded up to this time.

Lemma 4.5. *There exists a continuous function ρ_g: $\mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\rho_g(x) > 0 \ , \ \forall x \neq 0 \ , \tag{29}$$

such that, for all initial condition, the perturbed system (28), where e and d are two arbitrary functions satisfying the regularity assumptions (9) and equations (13) with $\rho = \rho_g$, admits a unique Carathéodory solution, defined for all $t \geq 0$.

Moreover there exists a function δ_g of class $\mathcal{K}_\infty$ such that, for all $R > 0$, all $0 < M < M_0$ and all $0 < r < r_0$, there exists a time $T_g = T_g(M, r, R)$ such that all Carathéodory solution X of (28) starting from x_0, with $|x_0| \leq R$, satisfies

$$|X(t)| \leq \delta_g(R), \ \forall t \leq T_g, \tag{30}$$

$$X(t) \in \Gamma_{M,r}, \ \forall t \geq T_g, \tag{31}$$

and

$$\|u_g(X(t))\| \leq 1, \ \forall t \geq 0. \tag{32}$$

5 Definition of the hybrid controller

In this section we define the hybrid controller by using a hysteresis to join the controllers defined in Sections 4.1 and 4.2.

For all $i \in \{1, \ldots, 6\}$, let M_i and r_i be such that

$$\begin{array}{c} 0 < M_6 < M_5 < M_4 < M_3 < M_2 < M_1 < M_0, \\ 0 < r_6 < r_5 < r_4 < r_3 < r_2 < r_1 < r_0. \end{array} \tag{33}$$

For the sake of simplicity, in what follows we set $\Gamma_i := \Gamma_{M_i,r_i}$ and $\Omega_i := \Omega_{M_i,r_i}$, for all $i \in \{1, \ldots, 6\}$. The hybrid controller (u, k_d) is defined using the following hysteresis between u_l and u_g on Γ_5 and Γ_2:

$$\begin{array}{rcll} u : \{1,2\} \times \mathbb{R}^n & \to & \mathbb{R}^2 & \\ (s_d, x) & \mapsto & u_l(x) & \text{if} \quad s_d = 1, \\ & & u_g(x) & \text{if} \quad s_d = 2, \end{array} \tag{34}$$

and

$$\begin{array}{rcll} k_d : \mathbb{R}^n \times \{1,2\} & \to & \{1,2\} & \\ (x, s_d) & \mapsto & 1 & \text{if} \quad x \in \Gamma_2, \\ & & s_d & \text{if} \quad x \in \Gamma_5 \setminus \Gamma_2, \\ & & 2 & \text{if} \quad x \notin \Gamma_5 \cup \{0\}. \end{array} \tag{35}$$

This hybrid controller is such that such that the origin is a global minimal-time robust stable equilibrium for the system (3), under the constraint (5) as claimed in Theorem 1.

References

[1] A. Agrachev, Compactness for SR minimizers and subanalyticity, Rend. Semin. Mat. Torino 56, 1998.

[2] A. Agrachev, J. P. Gauthier, On subanalyticity of Carnot-Carathéodory distances, Ann. Inst. H. Poincaré Anal. Non Linéaire 18, no. 3, 2001.

[3] Ancona, Fabio; Bressan, Alberto Flow stability of patchy vector fields and robust feedback stabilization. SIAM J. Control Optim. 41 (2002), no. 5, 1455–1476

[4] Ancona, Fabio; Bressan, Alberto Patchy vector fields and asymptotic stabilization, *ESAIM Control Optim. Calc. Var.* 4 (1999), 445–471

[5] A. Astolfi, Discontinuous control of the Brockett integrator, *European J. Control* 4 (1998), 49-53.

[6] R. Beals, B. Gaveau, P. C. Greiner, Hamilton-Jacobi theory and the heat kernel on Heisenberg groups, J. Math. Pures Appl. 79, 7 (2000), 633-689.

[7] A. Bellaïche, *Tangent space in sub-Riemannian geometry, Sub-Riemannian geometry*, Birkhäuser, 1996.

[8] A. Bensoussan and J.L. Menaldi, Hybrid control and dynamic programming, *Dyn. Cont. Discrete Impulsive Syst.* 3(**4**) (1997), 395-442.

[9] M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, *IEEE Trans. Automat. Control* 43 (1998), 475-482.

[10] R.W. Brockett, Asymptotic stability and feedback stabilization, in *Differential geometric control theory*, R.W. Brockett, R.S. Millman and H.J. Sussmann, ed., Boston, Birkhäuser, 1983, pp. 181-191.

[11] F.H Clarke, Yu.S. Ledyaev, L. Rifford and R.J. Stern, *Feedback stabilization and Lyapunov functions*, SIAM J. Control Opt., 39(**1**) (2000), pp. 25-48.

[12] J.-M. Coron, Global asymptotic stabilization for controllable systems without drift, *Math. Control Signals Systems*, 5 (1992) pp. 295-312.

[13] R.M. Hardt, Stratification of real analytic mappings and images, *Invent. Math.*, 28, 1975.

[14] H. Hironaka, Subanalytic sets, Number theory, algebraic geometry and commutative algebra, in honor of Y. Akizuki, Tokyo, 1973.

[15] I. Kolmanovsky and N. H. McClamroch, Developments in nonholonomic control problems, *IEEE Control Systems* 15 (1995), 20-36.

[16] Yu. S. Ledyaev and E. D . Sontag, A remark on robust stabilization of general asymptotically controllable systems, *Proc. Conf. on Information Sciences and Systems* (CISS 97), Johns Hopkins, Baltimore, MD, 246-251, 1997.

[17] E. Litsyn, Y.V. Nepomnyashchikh and A. Ponosov, Stabilization of linear differential systems via hybrid feedback controls, *SIAM J. Control Optim.* 38 (2000), 1468-1480.

[18] M'Closkey, Robert T., Murray, Richard M. Exponential stabilization of driftless nonlinear control systems using homogeneous feedback, *IEEE Trans. Automat. Control* 42 (1997), no. 5, 614–628.

[19] P. Morin, J.-B. Pomet and C. Samson, Developments in time-varying feedback stabilization of nonlinear systems, *IFAC Symp. on Nonlinear Control System Design, Enschede, The Netherlands*, Vol. 3, 1998.

[20] Morin, Pascal; Samson, Claude Practical stabilization of driftless systems on Lie groups: the transverse function approach, *IEEE Trans. Automat. Control* 48 (2003), no. 9, 1496–1508.

[21] Morin, Pascal; Samson, Claude Exponential stabilization of nonlinear driftless systems with robustness to unmodeled dynamics. ESAIM Control Optim. Calc. Var. 4 (1999), 1–35

[22] L. Pontryagin et al., Théorie mathématique des processus optimaux, Eds Mir, Moscou, 1974.

[23] C. Prieur, Uniting local and global controllers with robustness to vanishing noise, *Math. Control Signals Systems* 14 (2001), 143-172.

[24] C. Prieur, A robust globally asymptotically stabilizing feedback: The example of the Artstein's circles, (A. Isidori, F. Lamnabhi-Lagarrigue and W. Respondek, eds.), in "Nonlinear Control in the Year 2000" (NCN), Lecture Notes in Control and Information Sciences, Vol. 258, Springer Verlag, London, 279-300, 2000.

[25] C. Prieur, Asymptotic controllability and robust asymptotic stabizability, *SIAM J. Control Opt.*, to appear, 2004.

[26] C. Prieur et A. Astolfi, Robust stabilization of chained systems via hybrid control, *IEEE Trans. Auto. Contr.*, 48 (10), pp. 1768- 1772, 2003.

[27] C. Prieur and E. Trélat, Robust optimal stabilization of the Brockett integrator via a hybrid feedback, preprint, 2004.

[28] L. Rifford, The stabilization problem: AGAS and SRS feedbacks, to be published.

[29] E.D. Sontag, Clocks and insensitivity to small measurement errors, *ESAIM Control Optim. Calc. Var.*, 4 (1999), pp. 537-557.

[30] H. J. Sussmann, Subanalytic sets and feedback control, *J. Differential Equations*, 31 (1979), no. 1, 31-52.

[31] L. Tavernini, Differential automata and their discrete simulators, *Nonlinear Anal.* 11 (1997), 665-683.

[32] E. Trélat, Etude asymptotique et transcendance de la fonction valeur en contrôle optimal ; catégorie log-exp en géométrie sous-riemannienne dans le cas Martinet. PhD, Univ. de Bourgogne, Dijon, France, 2000.