SIAM J. CONTROL OPTIM.
Vol. 60, No. 1, pp. 1–21
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PROPORTIONAL INTEGRAL REGULATION CONTROL OF A ONE-DIMENSIONAL SEMILINEAR WAVE EQUATION*

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Abstract. This paper is concerned with the proportional integral (PI) regulation control of the left Neumann trace of a one-dimensional semilinear wave equation. The control input is selected as the right Neumann trace. The control design goes as follows. First, a preliminary (classical) velocity feedback is applied in order to shift all but a finite number of the eivenvalues of the underlying unbounded operator into the open left half-plane. We then leverage the projection of the system trajectories into an adequate Riesz basis to obtain a truncated model of the system capturing the remaining unstable modes. The controller is computed by applying a classical PI control design scheme to this truncated model. Local stability of the resulting closed-loop infinite-dimensional system is obtained through the study of an adequate Lyapunov function. Finally, an estimate assessing the set point tracking performance of the left Neumann trace is derived.

Key words. 1-D semilinear wave equation, PI regulation control, Neumann trace, partial differential equations (PDEs)

AMS subject classifications. 35L05, 35L71, 93C20

DOI. 10.1137/20M1346857

1. Introduction.

1.1. Historical context.

Due to its widespread adoption by industry [2, 3], the stabilization and regulation control of finite-dimensional systems by means of proportional integral (PI) controllers has been intensively studied. For this reason, the opportunity of extending PI control strategies to infinite-dimensional systems, and in particular to systems modeled by partial differential equations (PDEs), has attracted much attention in the recent years. Efforts in this research direction were originally devoted to the case of bounded control operators [25, 26] and then extended to unbounded control operators [35]. The study of PI control design combined with high-gain conditions was reported in [23]. More recently, the problem of PI boundary control of linear hyperbolic systems has been reported in a number of works [6, 12, 17, 36]. This research direction has then been extended to the case of nonlinear transport equations [5, 8, 13, 16, 27, 33]. The case of the state-feedback boundary regulation control of the Neumann trace for a linear reaction-diffusion PDE in the presence of an input delay was considered in [21] while extensions to output feedback strategies were reported in [18, 20]. The case of the boundary regulation control of the boundary velocity for linear damped wave equations, in the presence of a nonlinearity in the

*Received by the editors June 19, 2020; accepted for publication (in revised form) August 11, 2021; published electronically January 4, 2022.
https://doi.org/10.1137/20M1346857
Funding: The work of H. Lhachemi has been partially supported by a research grant from Science Foundation Ireland (SFI) under grant number 16/RC/3872 and is co-funded under the European Regional Development Fund and by I-Form industry partners. The work of H. Lhachemi has also been supported by ANR PIA funding: ANR-20-IDEES-0002. The work of C. Prieur has been partially supported by MIAI@Grenoble Alpes (ANR-19-P3IA-0003).
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boundary conditions, has been considered in [4, 32]. A general procedure allowing the addition of an integral component for regulation control to open-loop exponentially stable semigroups with unbounded control operators has been proposed in [30, 31].

This paper is concerned with the PI regulation control of the left Neumann trace of a one-dimensional semilinear (undamped) wave equation. The selected control input takes the form of the right Neumann trace. The control design procedure goes as follows. First, inspired by [10], a preliminary (classical) velocity feedback is applied in order to shift all but a finite number of the eigenvalues of the underlying unbounded operator into the open left half-plan. Second, inspired by the early work [28] later extended in [9, 10, 29] to semilinear heat and wave PDEs, we leverage the projection of the system trajectories into a Riesz basis formed by the generalized eigenstructures of the unbounded operator in order to obtain a truncated model capturing the remaining unstable modes. Finally, similarly to [21] in the context of a reaction-diffusion PDE, we adapt to this infinite-dimensional model a classical PI control design procedure as the one recalled in subsection 1.2 for finite-dimensional systems. The local stability of the resulting closed-loop infinite-dimensional system and the subsequent setpoint regulation performance is assessed by a Lyapunov-based argument. The theoretical results are illustrated based on the simulation of an open-loop unstable semilinear wave equation.

1.2. The rationale behind PI control. Consider the case of a finite-dimensional linear time invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t),$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the control input, and $y(t) \in \mathbb{R}$ is the to-be-regulated output. Here one has $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, and $C \in \mathbb{R}^{1\times n}$. Assuming that the control design objective is to (i) stabilize the closed-loop system and (ii) ensure that the system output $y(t)$ achieves the setpoint tracking of a reference signal $r(t) \in \mathbb{R}$, that is, $y(t) \to r_e$ as $t \to +\infty$ as soon as $r(t) \to r_e \in \mathbb{R}$, the PI approach consists of the addition of the integral component $\zeta(t) \in \mathbb{R}$ whose dynamics are described by

$$\dot{\zeta}(t) = y(t) - r(t) = Cx(t) - r(t).$$

Augmenting the state vector as $X = \begin{bmatrix} x^\top & \zeta \end{bmatrix}^\top \in \mathbb{R}^{n+1}$ and defining $\mathbf{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$, $\mathbf{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \in \mathbb{R}^{n+1}$, and $\mathbf{B}_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$, the system dynamics read

$$\dot{X}(t) = \mathbf{A}X(t) + \mathbf{B}u(t) - \mathbf{B}_r r(t).$$

The pair $(\mathbf{A}, \mathbf{B})$ satisfies the Kalman condition if and only if the pair (A, B) satisfies the Kalman condition and the matrix $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is full rank. In that case, by setting the state feedback $u(t) = \mathbf{K}X(t)$, where $\mathbf{K} \in \mathbb{R}^{1\times(n+1)}$ is selected such that the matrix $\mathbf{A}_K = \mathbf{A} + \mathbf{B}\mathbf{K}$ is Hurwitz, the closed-loop dynamics read

$$\dot{X}(t) = \mathbf{A}_K X(t) - \mathbf{B}_r r(t).$$

We can now characterize the equilibrium condition of the system associated with a constant reference signal $r(t) = r_e \in \mathbb{R}$. Since $\mathbf{A}_K$ is Hurwitz and thus invertible, we can define $X_e = \begin{bmatrix} x_e^\top & \zeta_e \end{bmatrix}^\top = \mathbf{A}_K^{-1}\mathbf{B}_r r_e$ which satisfies $0 = \mathbf{A}_K X_e - \mathbf{B}_r r_e$. From the latter identity, we obtain in particular from the last line that $Cx_e = r_e$. Introducing $\Delta X = X - X_e$ and $\Delta r = r - r_e$, we infer that

$$\Delta\dot{X}(t) = \mathbf{A}_K \Delta X(t) - \mathbf{B}_r \Delta r(t), \quad y(t) - r_e = \begin{bmatrix} C & 0 \end{bmatrix} \Delta X(t).$$

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When $r(t) = r_e$, this ensures the exponential decrease of both the system trajectories $\Delta X(t)$ and the regulation error $y(t) - r_e$ to zero. More generally, this achieves the setpoint tracking of the reference signal in the sense that if $r(t) \to r_e$ as $t \to +\infty$, then $y(t) \to r_e$.

1.3. Contribution. Let $L > 0$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$. We consider the wave equation on $(0, L)$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + f(y), \qquad y(t,0) = 0, \qquad \frac{\partial y}{\partial x}(t,L) = u(t), \tag{1.1a}$$

$$y(0,x) = y_0(x), \qquad \frac{\partial y}{\partial t}(0,x) = y_1(x) \tag{1.1b}$$

for $t > 0$ and $x \in (0, L)$, where the state is $y(t,\cdot) : [0, L] \to \mathbb{R}$ and the control input is $u(t)$ and applies to the right Neumann trace. The control objective is to locally stabilize the closed-loop system and locally regulate the system output selected as the left Neumann trace $z(t) = \frac{\partial y}{\partial x}(t, 0)$.

DEFINITION 1.1. *A function $y_e \in \mathcal{C}^2([0, L])$ is a steady state of* (1.1) *with associated constant control input $u_e \in \mathbb{R}$ and constant system output $z_e \in \mathbb{R}$ if $y_e''(x) + f(y_e(x)) = 0$ for all $x \in (0, L)$, $y_e(0) = 0$, $y_e'(L) = u_e$ and $z_e = y_e'(0)$.*

Remark 1.2. Introducing $F(y) = \int_0^y f(s)\,\mathrm{d}s$ for any $y \in \mathbb{R}$, assume either that $F(y) \to +\infty$ when $|y| \to +\infty$, or that, for any $a > 0$, the integral $\int \frac{\mathrm{d}y}{\sqrt{a-F(y)}}$ diverges at $-\infty$ and $+\infty$. Then we have the existence of a steady state $y_e \in \mathcal{C}^2([0, L])$ of (1.1) associated with any given value of the system output $z_e \in \mathbb{R}$. Indeed, define $y \in \mathcal{C}^2([0, l))$ with $0 < l \le +\infty$ as the maximal solution of $y'' + f(y) = 0$ with $y(0) = 0$ and $y'(0) = z_e$. We only need to assess that $l > L$. Multiplying by y' both sides of the ODE satisfied by y and then integrating over $[0, x]$, we observe that y satisfies the conservation law $y'(x)^2 + 2F(y(x)) = z_e^2$ for all $x \in [0, l)$. Hence any of the two above assumptions implies that y and y' are bounded on $[0, l)$. Thus $l = +\infty$ and the associated steady state control input is given by $u_e = y'(L)$.

Let $y_e \in \mathcal{C}^2([0, L])$ be a steady state of (1.1) with associated output $z_e \in \mathbb{R}$. This steady-state is fixed for the rest of the study. The objective addressed in this paper is the design of a controller ensuring for any given steady state $y_0 \in \mathcal{C}^2([0, L])$ of (1.1) located in a neighborhood[1] of y_e, and with associated output $z_0 \in \mathbb{R}$, the local stability of the closed-loop system and the regulation $z(t) = \frac{\partial y}{\partial x}(t, 0) \to z_0$ when $t \to +\infty$. To achieve this objective, we propose to apply a PI-type control design procedure on the PDE model. We introduce the following deviations: $y_\delta(t, x) = y(t, x) - y_e(x)$ and $u_\delta(t) = u(t) - u_e$. A Taylor expansion with integral remainder shows that (1.1) can equivalently be rewritten under the form

$$\frac{\partial^2 y_\delta}{\partial t^2} = \frac{\partial^2 y_\delta}{\partial x^2} + f'(y_e) y_\delta + y_\delta^2 \int_0^1 (1-s) f''(y_e + s y_\delta)\,\mathrm{d}s, \tag{1.2a}$$

$$y_\delta(t,0) = 0, \qquad \frac{\partial y_\delta}{\partial x}(t,L) = u_\delta(t), \tag{1.2b}$$

$$y_\delta(0,x) = y_0(x) - y_e(x), \qquad \frac{\partial y_\delta}{\partial t}(0,x) = y_1(x), \tag{1.2c}$$

for $t > 0$ and $x \in (0, L)$, while the output to be regulated is now expressed as

$$z_\delta(t) = \frac{\partial y_\delta}{\partial x}(t,0) = z(t) - z_e. \tag{1.3}$$

[1]In the sense of the $L^\infty(0, 1)$-norm.

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Finally, following classical PI control design schemes, we introduce the integral component on the tracking error

$$\dot{\zeta}(t) = \frac{\partial y_\delta}{\partial x}(t,0) - z_r(t) = z(t) - (z_e + z_r(t)), \tag{1.4}$$

where $z_e \in \mathbb{R}$ stands for the output value of the steady state y_e while $z_r(t) \in \mathbb{R}$ represents deviations of the reference signal around z_e.

Remark 1.3. It was shown in [21] for a linear reaction-diffusion equation with Dirichlet boundary control that a simple PI controller can be used to successfully control a Neumann trace. The control design was performed on a finite-dimensional truncated model capturing the unstable modes of the infinite dimensional system while assessing the stability of the full infinite-dimensional system via a Lyapunov-based argument. Such an approach cannot be directly applied to the case of the wave equation studied in this paper due to the fact that, even in the case of a linear function f, the open-loop system might exhibit an infinite number of unstable modes. To avoid this pitfall, we borrow the following remark from [10]. In the case $f = 0$, the control input $u_\delta(t) = -\alpha \frac{\partial y_\delta}{\partial t}(t,L)$, with $\alpha > 0$, ensures the exponential decay of the energy function defined by $E(t) = \int_0^L (\frac{\partial y_\delta}{\partial t}(t,x))^2 + (\frac{\partial y_\delta}{\partial x}(t,x))^2 \,\mathrm{d}x$. As suggested in [10], a suitable control input candidate for (1.2) takes the form

$$u_\delta(t) = -\alpha \frac{\partial y_\delta}{\partial t}(t,L) + v(t), \tag{1.5}$$

where $\alpha > 0$ is to be selected and $v(t)$ is an auxiliary command input. In particular, it was shown in [10] that, in the presence of the nonlinear term f, the velocity feedback can be used to locally stabilize all but possibly a finite number of the modes of the system. Then the authors showed that the design of the auxiliary control input v can be performed by pole shifting on a finite-dimensional truncated model to achieve the stabilization of the remaining unstable modes. The stability of the resulting closed-loop system was assessed via the introduction of a suitable Lyapunov function.

We propose to take advantage of the preliminary control input (1.5) in order to achieve the regulation of the left Neumann trace by means of a PI control design scheme via the introduced integral component (1.4). The main result of this paper can be informally stated as follows.

THEOREM 1.4. *Considering a steady state y_e of the wave equation* (1.1) *and the preliminary control input* (1.5)*, there exists a dynamical controller taking the form of a state feedback of y_δ that achieves the exponential stabilization of the wave equation* (1.1) *in a neighborhood of the steady state y_e as well as the local regulation control of the left Neumann trace.*

The rest of the paper is organized as follows. The proposed control design procedure, with its whole explicit construction, is presented in details in section 2. The subsequent stability theorems, which formalize Theorem 1.4, are stated in subsection 2.6. The proofs of the main results are reported in section 3. The theoretical results are numerically illustrated in section 4. Finally, concluding remarks are formulated in section 5.

2. Explicit control design.

2.1. Equivalent homogeneous problem.

Making the change of variable

$$w^1(t,x) = y_\delta(t,x), \qquad w^2(t,x) = \frac{\partial y_\delta}{\partial t}(t,x) - \frac{x}{\alpha L} v(t), \tag{2.1}$$

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we obtain from the wave equation (1.2), the integral component (1.4), and the control strategy (1.5) that

$$\frac{\partial w^1}{\partial t} = w^2 + \frac{x}{\alpha L} v(t), \tag{2.2a}$$

$$\frac{\partial w^2}{\partial t} = \frac{\partial^2 w^1}{\partial x^2} + f'(y_e) w^1 + r(t,x) - \frac{x}{\alpha L}\dot{v}(t), \tag{2.2b}$$

$$\dot{\zeta}(t) = \frac{\partial w^1}{\partial x}(t,0) - z_r(t), \tag{2.2c}$$

$$w^1(t,0) = 0, \qquad \frac{\partial w^1}{\partial x}(t,L) + \alpha w^2(t,L) = 0, \tag{2.2d}$$

$$w^1(0,x) = y_0(x) - y_e(x), \qquad w^2(0,x) = y_1(x) - \frac{x}{\alpha L} v(0), \qquad \zeta(0) = \zeta_0 \tag{2.2e}$$

for $t > 0$ and $x \in (0,L)$, with the residual term

$$r(t,x) = (w^1(t,x))^2 \int_0^1 (1-s) f''(y_e(x) + s w^1(t,x))\, \mathrm{d}s. \tag{2.3}$$

Remark 2.1. A more classical change of variable for (1.2) with control input u given by (1.5) is generally obtained by setting

$$w(t,x) = y_\delta(t,x) - \frac{x(x-L)}{L} v(t). \tag{2.4}$$

In that case, (1.2) with u given by (1.5) yields

(2.5)

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + f'(y_e) w - \frac{x(x-L)}{L}\ddot{v}(t) + \left(\frac{x(x-L)}{L} f'(y_e) + \frac{2}{L} \right) v(t) + r(t,x),$$

$$w(t,0) = 0, \qquad \frac{\partial w}{\partial x}(t,L) + \alpha \frac{\partial w}{\partial t}(t,L) = 0,$$

$$w(0,x) = y_0(x) - y_e(x) - \frac{x(x-L)}{L} v(0), \qquad \frac{\partial w}{\partial t}(0,x) = y_1(x) - \frac{x(x-L)}{L}\dot{v}(0),$$

with

$$r(t,x) = \left(w(t,x) + \frac{x(x-L)}{L} v(t)\right)^2 \int_0^1 (1-s) f''\left(y_e(x) + s\left(w(t,x) + \frac{x(x-L)}{L} v(t)\right)\right) \mathrm{d}s.$$

However, this change of variable (2.4) induces the occurrence of a $\ddot{v}$ term in (2.5), while only a $\dot{v}$ term appears in (2.2a)–(2.2b). The control procedure presented in this paper could have also been applied in such a setting. Nevertheless, the consideration of the change of variable (2.1), instead of (2.4), allows a reduction of the complexity of the controller architecture by avoiding the introduction of an extra additional integral component.

Let $\mathcal{H} = \left\{ (w^1, w^2) \in H^1(0,L) \times L^2(0,L) \,:\, w^1(0) = 0 \right\}$ be the Hilbert space endowed with the inner product $\left\langle (w^1, w^2), (z^1, z^2) \right\rangle = \int_0^L (w^{1\prime} \overline{z^{1\prime}} + w^2 \overline{z^2})\, \mathrm{d}x$. Defining the state vector $W(t) = (w^1(t,\cdot), w^2(t,\cdot)) \in \mathcal{H}$, the wave equation with integral component (2.2) can be rewritten under the abstract form

$$\dot{W}(t) = \mathcal{A} W(t) + a v(t) + b \dot{v}(t) + R(t,\cdot), \qquad \dot{\zeta}(t) = \frac{\partial w^1}{\partial x}(t,0) - z_r(t), \tag{2.6a}$$

$$W(0,x) = \left(y_0(x) - y_e(x), y_1(x) - \frac{x}{\alpha L} v(0) \right), \qquad \zeta(0) = \zeta_0 \tag{2.6b}$$

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for $t > 0$ and $x \in (0, L)$, where $\mathcal{A}_0 = \Delta + f'(y_e)\,\mathrm{Id}$,

$$\mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathcal{A}_0 & 0 \end{pmatrix}, \tag{2.7}$$

defined on $D(\mathcal{A}) = \{(w^1, w^2) \in \mathcal{H} \,:\, w^1 \in H^2(0,L),\, w^2 \in H^1(0,L),\, w^2(0) = 0,\, (w^1)'(L) + \alpha w^2(L) = 0\}$, and $a, b, R(t,\cdot) \in \mathcal{H}$ are defined by

$$a(x) = (x/(\alpha L), 0), \quad b(x) = (0, -x/(\alpha L)), \quad R(t,x) = (0, r(t,x)). \tag{2.8}$$

We have $R(t,\cdot) \in \mathcal{H}$ because $r(t,\cdot) \in L^2(0,L)$, which follows from the facts that $w^1(t,\cdot) \in H^1(0,L) \subset L^\infty(0,L)$, f'' is continuous on $\mathbb{R}$, and y_e is continuous on $[0,L]$.

Remark 2.2. It is well-known that the operator $\mathcal{A}$ generates a C_0-semigroup [34]. Moreover, since the Neumann trace is $\mathcal{A}$-admissible, the application of [35, Lemma 1] shows that the augmentation of $\mathcal{A}$ with the integral component ζ still generates a C_0-semigroup. As $\dot{v}$ is seen as the control input, the state-space vector can further be augmented to include v, and the associated augmented operator also generates a C_0-semigroup. Now, noting that the residual term (2.3) can be rewritten under the form $r(t,x) = \int_{y_e(x)}^{w^1(t,x)+y_e(x)} (w^1(t,x) + y_e(x) - s) f''(s)\,\mathrm{d}s$, one can observe that $w^1 \mapsto \int_{y_e}^{w^1+y_e} (w^1 + y_e - s) f''(s)\,\mathrm{d}s$, when seen as a function from $\{w^1 \in H^1(0,L) \,:\, w^1(0) = 0\}$ to $L^2(0,L)$, is continuously differentiable. Consequently, the well-posedness of (2.6) follows from classical results [24]. In the subsequent developments, we will consider for initial conditions $W(0) \in D(\mathcal{A})$, continuously differentiable reference inputs z_r, and a control input $\dot{v}$ that will take the form of a state feedback, the concept of classical solution for (2.6) on its maximal interval of definition $[0, T_{\max})$ with $0 < T_{\max} \leq +\infty$, i.e., $W \in \mathcal{C}^0([0,T_{\max}); D(\mathcal{A})) \cap \mathcal{C}^1([0,T_{\max}); \mathcal{H})$.

2.2. Properties of the operator $\mathcal{A}$.

LEMMA 2.3. *The adjoint operator $\mathcal{A}^*$ is defined by $\mathcal{A}^*(z^1, z^2) = (-z^2 - g, -(z^1)'')$ on $D(\mathcal{A}^*) = \{(z^1, z^2) \in \mathcal{H} \,:\, z^1 \in H^2(0,L),\, z^2 \in H^1(0,L),\, z^2(0) = 0,\, (z^1)'(L) - \alpha z^2(L) = 0\}$ with $g \in \mathcal{C}^2([0,L])$ defined by $g'' = f'(y_e) z^2$ and $g(0) = g'(L) = 0$.*

Proof. We write $\mathcal{A} = \mathcal{A}_{tr} + \mathcal{A}_p$ with

$$\mathcal{A}_{tr} = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta & 0 \end{pmatrix}, \quad \mathcal{A}_p = \begin{pmatrix} 0 & 0 \\ f'(y_e)\,\mathrm{Id} & 0 \end{pmatrix},$$

where $\mathcal{A}_{tr}$ is an unbounded operator defined on the same domain as $\mathcal{A}$ while $\mathcal{A}_p$ in defined on $\mathcal{H}$. As $\mathcal{A}_p$ is bounded, straightforward computations show that $\mathcal{A}_p^* = (-g, 0)$, where g is defined as in the statement of the lemma. It remains to compute $\mathcal{A}_{tr}^*$. To do so, we observe that $0 \in \rho(\mathcal{A}_{tr})$ with $\mathcal{A}_{tr}^{-1} w = (-\alpha w^1(L) x + \int_0^x \int_L^\xi w^2(s)\,\mathrm{d}s\,\mathrm{d}\xi, w^1)$ for all $w = (w^1, w^2) \in \mathcal{H}$. Since $\mathcal{A}_{tr}^{-1}$ is bounded, straightforward computations show that $(\mathcal{A}_{tr}^{-1})^* w = (-\alpha w^1(L) x - \int_0^x \int_L^\xi w^2(s)\,\mathrm{d}s\,\mathrm{d}\xi, -w^1)$ for all $w = (w^1, w^2) \in \mathcal{H}$. We deduce the claimed result by computing the inverse of the latter operator. ∎

The strategy reported in this paper relies on the concept of Riesz bases. This concept is recalled in the following definition.

DEFINITION 2.4. *A family of vectors $(e_k)_{k \in \mathbb{Z}}$ of $\mathcal{H}$ is a Riesz basis if this family is maximal and there exist constants $m_r, M_R > 0$ such that*

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$$m_R \sum_{|k|\leq N} |c_k|^2 \leq \| \sum_{|k|\leq N} c_k e_k \|_{\mathcal{H}}^2$$
$$\leq M_R \sum_{|k|\leq N} |c_k|^2 \qquad \textit{for all } N \geq 0 \quad \textit{for all } c_{-N}, \ldots, c_N \in \mathbb{C}.$$

The dual Riesz basis of $(e_k)_{k\in\mathbb{Z}}$ *is the unique family of vectors* $(f_k)_{k\in\mathbb{Z}}$ *of* $\mathcal{H}$ *which is such that* $\langle e_k, f_l\rangle_{\mathcal{H}} = \delta_{k,l} \in \{0,1\}$ *with* $\delta_{k,l} = 1$ *if and only if* $k = l$.

We can now introduce the following properties of the operator $\mathcal{A}$. These properties, except the last item, are retrieved from [10, Lemmas 2 and 5].

LEMMA 2.5. *Let* $\alpha > 1$. *There exists a Riesz basis* $(e_k)_{k\in\mathbb{Z}}$ *of* $\mathcal{H}$ *consisting of generalized eigenfunctions of* $\mathcal{A}$*, associated to the eigenvalues* $(\lambda_k)_{k\in\mathbb{Z}}$ *and with dual Riesz basis* $(f_k)_{k\in\mathbb{Z}}$*, such that*

1. $e_k \in D(\mathcal{A})$ *and* $\|e_k\|_{\mathcal{H}} = 1$ *for every* $k \in \mathbb{Z}$*;*
2. *each eigenvalue* λ_k *is geometrically simple;*
3. *there exists* $n_0 \geq 0$ *such that, for any* $k \in \mathbb{Z}$ *with* $|k| \geq n_0 + 1$*,* λ_k *is algebraically simple and* $\lambda_k = \frac{1}{2L}\log(\frac{\alpha-1}{\alpha+1}) + i\frac{k\pi}{L} + O(\frac{1}{|k|})$ *as* $|k| \to +\infty$*.*
4. *if* $k \geq n_0 + 1$*, then* e_k *(resp.,* f_k*) is an eigenfunction of* $\mathcal{A}$ *(resp.,* $\mathcal{A}^*$*) associated with the algebraically simple eigenvalue* λ_k *(resp.,* $\overline{\lambda_k}$*);*
5. *for every* $k \geq n_0 + 1$*, one has* $e_k = \overline{e_{-k}}$ *and* $f_k = \overline{f_{-k}}$*;*
6. *if* $|k| \leq n_0$*, then* $\mathcal{A}e_k \in \operatorname{span}\{e_p : |p| \leq n_0\}$ *and* $\mathcal{A}^* f_k \in \operatorname{span}\{f_p : |p| \leq n_0\}$*;*
7. *introducing* $e_k = (e_k^1, e_k^2)$*, one has* $(e_k^1)'(0) = O(1)$ *as* $|k| \to +\infty$*.*

The proof of Lemma A, which is essentially extracted from [10], is placed in annex for self-completeness of the manuscript.

In what follows, we select the constant $\alpha > 1$ such that $\frac{1}{2L}\log(\frac{\alpha-1}{\alpha+1}) < -1$. Thanks to the third item, only a finite number of eigenvalues might have a nonnegative real part. Thus, without loss of generality, we also select the integer $n_0 \geq 0$ provided by Lemma 2.5 large enough such that $\operatorname{Re}\lambda_k < -1$ for all $|k| \geq n_0 + 1$.

Remark 2.6. The state space can be written as $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$ with the subspaces $\mathcal{H}_1 = \operatorname{span}\{e_p : |p| \leq n_0\}$ and $\mathcal{H}_2 = \overline{\operatorname{span}\{e_p : |p| \geq n_0 + 1\}}$. Introducing π_1, π_2 the projectors associated with this decomposition, Lemma 2.5 shows that $\mathcal{A}$ takes the form $\mathcal{A} = \mathcal{A}_1\pi_1 + \mathcal{A}_2\pi_2$, where $\mathcal{A}_1 \in \mathcal{L}(\mathcal{H}_1)$ and $\mathcal{A}_2 : D(\mathcal{A}_2) \subset \mathcal{H}_2 \to \mathcal{H}_2$ with $D(\mathcal{A}_2) = D(\mathcal{A}) \cap \mathcal{H}_2$. Moreover, Lemma 2.5 shows that $\mathcal{A}_2$ is a Riesz spectral operator. Then, as $D(\mathcal{A}) = \mathcal{H}_1 \bigoplus D(\mathcal{A}_2)$, we obtain that $D(\mathcal{A}) = \left\{\sum_{k\in\mathbb{Z}} w_k e_k : \sum_{k\in\mathbb{Z}} |\lambda_k w_k|^2 < \infty\right\}$ and, for any $w \in D(\mathcal{A})$, $\mathcal{A}w = \mathcal{A}_1\pi_1 w + \sum_{|k|\geq n_0+1} \lambda_k \langle w, f_k\rangle e_k$, where the equality holds in $\mathcal{H}$-norm. Now, for any $w \in D(\mathcal{A})$, consider the series expansion $w = (w^1, w^2) = \sum_{k\in\mathbb{Z}} \langle w, f_k\rangle e_k$. In particular, one has $w^1 = \sum_{k\in\mathbb{Z}} \langle w, f_k\rangle e_k^1$ in H^1-norm. Moreover, we have that $\mathcal{A}w = \sum_{k\in\mathbb{Z}} \langle w, f_k\rangle \mathcal{A}e_k$, and thus $\mathcal{A}_0 w^1 = \sum_{k\in\mathbb{Z}} \langle w, f_k\rangle \mathcal{A}_0 e_k^1$ in L^2-norm. Since $f'(y_e) \in L^\infty(0,L)$, the expansion of the latter identity shows that $(w^1)'' = \sum_{k\in\mathbb{Z}} \langle w, f_k\rangle (e_k^1)''$ in L^2-norm. Consequently, $w^1 = \sum_{k\in\mathbb{Z}} \langle w, f_k\rangle e_k^1$ in H^2-norm, and thus, by the continuous embedding $H^1(0,L) \subset L^\infty(0,L)$,

$$(w^1)'(0) = \sum_{k\in\mathbb{Z}} \langle w, f_k\rangle (e_k^1)'(0). \tag{2.9}$$

The latter series expansion will be intensively used in the remainder of this paper.

2.3. Spectral reduction and truncated model. Introducing, for every $k \in \mathbb{Z}$,

$$w_k(t) = \langle W(t), f_k\rangle_{\mathcal{H}}, \quad a_k = \langle a, f_k\rangle_{\mathcal{H}}, \quad b_k = \langle b, f_k\rangle_{\mathcal{H}}, \quad r_k(t) = \langle R(t,\cdot), f_k\rangle_{\mathcal{H}},$$

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we obtain from (2.6a) that

$$\dot{w}_k(t) = \langle \mathcal{A}W(t), f_k\rangle_{\mathcal{H}} + a_k v(t) + b_k \dot{v}(t) + r_k(t) = \langle W(t), \mathcal{A}^* f_k\rangle_{\mathcal{H}} + a_k v(t) + b_k \dot{v}(t) + r_k(t).$$

Recalling that $\mathcal{A}^* f_k = \overline{\lambda_k} f_k$ for $|k| \geq n_0 + 1$, we obtain that

$$\dot{w}_k(t) = \lambda_k w_k(t) + a_k v(t) + b_k \dot{v}(t) + r_k(t), \qquad |k| \geq n_0 + 1. \tag{2.10}$$

Moreover, after possibly linear recombination[2] of $(e_k)_{|k|\leq N_0}$ and $(f_k)_{|k|\leq N_0}$, still denoted by $(e_k)_{|k|\leq N_0}$ and $(f_k)_{|k|\leq N_0}$, to obtain matrices with real coefficients, we infer from Lemma 2.5, item 6, the existence of a matrix $A_0 \in \mathbb{R}^{(2n_0+1)\times(2n_0+1)}$ such that

$$\dot{X}_0(t) = A_0 X_0(t) + B_{0,1} v(t) + B_{0,2} \dot{v}(t) + R_0(t), \tag{2.11}$$

where $X_0(t), B_{0,1}, B_{0,2}, R_0(t) \in \mathbb{R}^{2n_0+1}$ are defined by

$$X_0(t) = \begin{bmatrix} w_{-n_0}(t) \\ \vdots \\ w_{n_0}(t) \end{bmatrix}, \quad B_{0,1} = \begin{bmatrix} a_{-n_0} \\ \vdots \\ a_{n_0} \end{bmatrix}, \quad B_{0,2} = \begin{bmatrix} b_{-n_0} \\ \vdots \\ b_{n_0} \end{bmatrix}, \quad R_0(t) = \begin{bmatrix} r_{-n_0}(t) \\ \vdots \\ r_{n_0}(t) \end{bmatrix}.$$

Introducing the auxiliary control input $v_d(t) = \dot{v}(t) \in \mathbb{R}$, we augment the state-space representation (2.11) with the actual control input v as follows:

$$\dot{X}_1(t) = A_1 X_1(t) + B_1 v_d(t) + R_1(t), \tag{2.12}$$

where $X_1(t), R_1(t), B_1 \in \mathbb{R}^{2n_0+2}$ and $A_1 \in \mathbb{R}^{(2n_0+2)\times(2n_0+2)}$ are defined by

$$X_1(t) = \begin{bmatrix} v(t) \\ X_0(t) \end{bmatrix}, \quad R_1(t) = \begin{bmatrix} 0 \\ R_0(t) \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ B_{0,1} & A_0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ B_{0,2} \end{bmatrix}. \tag{2.13}$$

Using v_d as the control input, we now apply a PI control design procedure to (2.12) similar to the one recalled in subsection 1.2. To do so we further augment the latter state-space representation to include an adequate integral component. First, we note from (2.9) that the dynamics of the integral component ζ satisfies $\dot{\zeta}(t) = \sum_{k\in\mathbb{Z}} w_k(t)(e_k^1)'(0) - z_r(t)$. We observe that the ζ-dynamics involves all the coefficients of projection $w_k(t)$, with $k \in \mathbb{Z}$, and hence cannot be used to augment (2.12). This motivates the change of variable

$$\xi(t) = \zeta(t) - \sum_{|k|\geq n_0+1} \frac{(e_k^1)'(0)}{\lambda_k} w_k(t) = \zeta(t) - 2 \sum_{k\geq n_0+1} \operatorname{Re}\left(\frac{(e_k^1)'(0)}{\lambda_k} w_k(t) \right), \tag{2.14}$$

where, by the Cauchy–Schwarz inequality, the series is convergent because $|\lambda_k| \sim \frac{|k|\pi}{l}$ and $(e_k^1)'(0) = O(1)$ as $|k| \to +\infty$. Moreover, the time derivative of ξ is given by

$$\dot{\xi}(t) = \sum_{|k|\leq n_0} w_k(t)(e_k^1)'(0) + \alpha_0 v(t) + \beta_0 \dot{v}(t) - \gamma(t) = L_1 X_1(t) + \beta_0 v_d(t) - \gamma(t),$$

[2] In that case, all the properties stated by Lemma 2.5 remain true except that $(e_k)_{|k|\leq N_0}$ might not be generalized eigenvectors of $\mathcal{A}$.

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where

$$\alpha_0 = -\sum_{|k|\geq n_0+1} \frac{(e_k^1)'(0)}{\lambda_k} a_k = -2 \sum_{k\geq n_0+1} \operatorname{Re}\Big(\frac{(e_k^1)'(0)}{\lambda_k} a_k\Big), \tag{2.15a}$$

$$\beta_0 = -\sum_{|k|\geq n_0+1} \frac{(e_k^1)'(0)}{\lambda_k} b_k = -2 \sum_{k\geq n_0+1} \operatorname{Re}\Big(\frac{(e_k^1)'(0)}{\lambda_k} b_k\Big), \tag{2.15b}$$

$$\gamma(t) = z_r(t) + \sum_{|k|\geq n_0+1} \frac{(e_k^1)'(0)}{\lambda_k} r_k(t) = z_r(t) + 2 \sum_{k\geq n_0+1} \operatorname{Re}\Big(\frac{(e_k^1)'(0)}{\lambda_k} r_k(t)\Big), \tag{2.15c}$$

and $L_1 = \begin{bmatrix}\alpha_0 & (e_{-n_0}^1)'(0) & \dots & (e_{n_0}^1)'(0)\end{bmatrix} \in \mathbb{R}^{1\times(2n_0+2)}$. Thus, with the introduction of $X(t), B, \Gamma(t) \in \mathbb{R}^{2n_0+3}$, and $A \in \mathbb{R}^{(2n_0+3)\times(2n_0+3)}$ defined by

$$X(t) = \begin{bmatrix} X_1(t) \\ \xi(t) \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ L_1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \beta_0 \end{bmatrix}, \quad \Gamma(t) = \begin{bmatrix} R_1(t) \\ -\gamma(t) \end{bmatrix}, \tag{2.16}$$

we obtain the truncated model $\dot{X}(t) = AX(t) + Bv_d(t) + \Gamma(t)$. Using the latter and (2.10), the wave equation with integral component (2.6) admits the following equivalent representation used for both control design and stability analysis:

$$\dot{X}(t) = AX(t) + Bv_d(t) + \Gamma(t), \tag{2.17a}$$

$$\dot{w}_k(t) = \lambda_k w_k(t) + a_k v(t) + b_k v_d(t) + r_k(t), \qquad |k| \geq n_0 + 1. \tag{2.17b}$$

Remark 2.7. The representation (2.17) shows that the dynamics of the wave equation with integral component (2.6) can be split into two parts. The first part (2.17a) consists of an ODE capturing the unstable dynamics plus a certain number of slow stable modes of the system. The second part (2.17b), referred to as the residual dynamics, captures the stable dynamics of the system which are such that $\operatorname{Re}\lambda_k < -1$. The control strategy consists now into the two following steps. First, a state feedback is designed to locally stabilize (2.17a). Then, a stability analysis is carried out to assess that such a control strategy achieves both the local stabilization of (2.17), as well as the fulfillment of the regulation of the left Neumann trace (1.3).

2.4. Control strategy and closed-loop dynamics.

The control design strategy consists in the design of a stabilizing state feedback for (2.17a). Such a pole shifting is allowed by the following result.

LEMMA 2.8. *The pair (A, B) satisfies the Kalman condition.*

Proof. From (2.16), the Hautus test easily shows that (A, B) satisfies the Kalman condition if and only if (A_1, B_1) satisfies the Kalman condition and the square matrix $T = \left[\begin{smallmatrix} A_1 & B_1 \\ L_1 & \beta_0 \end{smallmatrix}\right] \in \mathbb{R}^{(2n_0+3)\times(2n_0+3)}$ is invertible.

We first prove that, for any $\lambda \in \mathbb{C}$ and $z \in D(\mathcal{A}^*)$, $\langle a + \lambda b, z\rangle_{\mathcal{H}} = 0$ and $\mathcal{A}^* z = \overline{\lambda} z$ implies $z = 0$. Recall (based on Lemma 2.3) that $\mathcal{A}^* z = \overline{\lambda} z$ gives

$$z^2 + g = -\overline{\lambda} z^1, \qquad (z^1)'' = -\overline{\lambda} z^2, \qquad g'' = f'(y_e) z^2, \tag{2.18a}$$

$$z^1(0) = z^2(0) = g(0) = g'(L) = 0, \qquad (z^1)'(L) - \alpha z^2(L) = 0. \tag{2.18b}$$

From the definition (2.8) of $a, b \in \mathcal{H}$ one has

$$\begin{aligned} \langle a + \lambda b, z\rangle_{\mathcal{H}} &= \int_0^L \left(\left(\frac{x}{\alpha L}\right)' \overline{(z^1)'(x)} - \lambda \frac{x}{\alpha L} \overline{z^2(x)} \right) \mathrm{dx} \\ &= \left[\frac{x}{\alpha L} \overline{(z^1)'(x)} \right]_{x=0}^{x=L} - \int_0^L \frac{x}{\alpha L} \overline{\left((z^1)''(x) + \overline{\lambda} z^2(x)\right)} \,\mathrm{dx} = \frac{1}{\alpha} \overline{(z^1)'(L)}, \end{aligned}$$

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where we have used (2.18a). Thus we have $(z^1)'(L) = 0$. Then (2.18b) shows that $z^2(L) = 0$, and we infer from (2.18a) and (2.18b) that $(z^2)'(L) = 0$. Moreover, taking twice the derivative of the first equation of (2.18a), we obtain that $(z^2)'' + \left(f'(y_e) - (\overline{\lambda})^2\right) z^2 = 0$. By Cauchy uniqueness, we deduce that $z^2 = 0$. Using (2.18a), (2.18b), and $(z^1)'(L) = 0$, we reach the conclusion $z = 0$.

Assume now that (A_1, B_1) does not satisfy the Kalman condition. From (2.13), the Hautus test shows the existence of $\lambda \in \mathbb{C}$, $x_1 \in \mathbb{C}$, and $x_2 \in \mathbb{C}^{2n_0+1}$, with either $x_1 \neq 0$ or $x_2 \neq 0$, such that $x_2^* B_{0,1} = \lambda x_1^*$, $x_2^* A_0 = \lambda x_2^*$, and $x_1^* + x_2^* B_{0,2} = 0$. This implies the existence of $x_2 \neq 0$ such that $A_0^* x_2 = \overline{\lambda} x_2$ and $x_2^*(B_{0,1} + \lambda B_{0,2}) = 0$, where $B_{0,1} + \lambda B_{0,2} = (\langle a + \lambda b, f_k \rangle_{\mathcal{H}})_{-n_0 \leq k \leq n_0}$. Noting that A_0^* is the matrix of $\mathcal{A}^*$ in $(f_k)_{|k| \leq n_0}$, this shows the existence of a nonzero vector $z \in D(\mathcal{A}^*)$ such that $\langle a + \lambda b, z \rangle_{\mathcal{H}} = 0$ and $\mathcal{A}^* z = \overline{\lambda} z$. The result of the previous paragraph leads to the contraction $z = 0$. Hence (A_1, B_1) does satisfy the Kalman condition.

It remains to show that the matrix T is invertible. To do so, let a vector $X_e = \begin{bmatrix} v_e & w_{-n_0,e} & \dots & w_{-n_0,e} & v_{d,e} \end{bmatrix}^\top \in \mathbb{R}^{2n_0+3}$ be an element of the kernel of T. Then, by expanding $TX_e = 0$, we obtain that $v_{d,e} = 0$,

$$0 = A_0 \begin{bmatrix} w_{-n_0,e} \\ \vdots \\ w_{-n_0,e} \end{bmatrix} + B_{0,1} v_e, \qquad 0 = \sum_{|k| \leq n_0} w_{k,e} (e_k^1)'(0) - \left(\sum_{|k| \geq n_0+1} \frac{(e_k^1)'(0)}{\lambda_k} a_k \right) v_e.$$

We define, for $|k| \geq n_0 + 1$, $w_{k,e} = -\frac{a_k}{\lambda_k} v_e$. Then, as $(w_{k,e})_{k \in \mathbb{Z}}$ is square summable, we can introduce $w_e = \sum_{k \in \mathbb{Z}} w_{k,e} e_k \in \mathcal{H}$. We obtain that

$$0 = A_0 \begin{bmatrix} w_{-n_0,e} \\ \vdots \\ w_{-n_0,e} \end{bmatrix} + B_{0,1} v_e, \quad 0 = \lambda_k w_{k,e} + a_k v_e,\ |k| \geq n_0 + 1, \quad 0 = \sum_{k \in \mathbb{Z}} w_{k,e} (e_k^1)'(0).$$

In particular $(\lambda_k w_{k,e})_{k \in \mathbb{Z}}$ is square summable, and thus $w_e \in D(\mathcal{A})$. The developments of subsection 2.3 show that the above system is equivalent to $\mathcal{A} w_e + a v_e = 0$ and $(w_e^1)'(0) = 0$ with $w_e = (w_e^1, w_e^2)$. By expanding the former identity, we first have that $(w_e^1)'' + f'(y_e) w_e^1 = 0$ with $w_e^1(0) = (w_e^1)'(0) = 0$ and thus, by Cauchy uniqueness, $w_e^1 = 0$. We also have $w_e^2 = \frac{-x}{\alpha L} v_e$ with $v_e = -\alpha w_e^2(L) = (w_e^1)'(L) = 0$, and thus $w_e^2 = 0$. This yields $w_e = 0$, which shows that $w_{k,e} = 0$ for every $k \in \mathbb{Z}$. Hence the kernel of T is reduced to $\{0\}$, which concludes the proof. ∎

The result of Lemma 2.8 ensures the existence of a gain $K \in \mathbb{R}^{1 \times (2n_0+3)}$ such that $A_K = A + BK$ is Hurwitz. Then, we can take the state feedback

$$v_d(t) = \dot{v}(t) = K X(t), \tag{2.19}$$

which yields the closed-loop system dynamics

$$\dot{X}(t) = A_K X(t) + \Gamma(t), \tag{2.20a}$$
$$\dot{w}_k(t) = \lambda_k w_k(t) + a_k v(t) + b_k v_d(t) + r_k(t), \qquad |k| \geq n_0 + 1. \tag{2.20b}$$

Our main result establishes the local stability of the closed-loop system. We also study the tracking performance.

Remark 2.9. Ultimately, the controller takes the form of the preliminary feedback (1.5) which is combined with (2.19). It consists of a state feedback and two integral

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components. One is introduced due to the fact that the design is performed on the homogeneous wave equation composed of (2.2a)–(2.2b) and (2.2d), and one is due to the integral component (2.14) introduced to achieve the desired PI set point regulation control.

2.5. Dynamics of deviations. Consider now an arbitrary steady state $y_0 \in \mathcal{C}^2([0,L])$ of (1.1) with associated constant control input $u_0 \in \mathbb{R}$ and constant system output $z_0 \in \mathbb{R}$, i.e., $y_0''(x) + f(y_0(x)) = 0$ for all $x \in (0,L)$, $y_0(0) = 0$, $y_0'(L) = u_0$, and $z_0 = y_0'(0)$. Introducing $w_0^1(x) = y_{0,\delta}(x) = y_0(x) - y_e(x)$ and $w_0^2(x) = -\frac{x}{\alpha L} v_0$ with $v_0 = u_0 - u_e$, we infer that (w_0^1, w_0^2) is a steady state of (2.2a)–(2.2b) and (2.2d) because $w_0^2 + \frac{x}{\alpha L} v_0 = 0$ and $(w_0^1)'' + f'(y_e) w_0^1 + r_0 = 0$, where $r_0(x) = (w_0^1(x))^2 \int_0^1 (1-s) f''(y_e(x) + s w_0^1(x)) \,\mathrm{d}s$, with $w_0^1(0) = 0$, $\frac{\partial w_0^1}{\partial x}(L) + \alpha w_0^2(L) = 0$, and $(w_0^1)'(0) = z_0 - z_e$. Let $W_0 = (w_0^1, w_0^2) \in \mathcal{H}$, $R_0(x) = (0, r_0(x))$, $w_{0,k}(t) = \langle W_0, f_k \rangle_{\mathcal{H}}$, and $r_{0,k} = \langle R_0, f_k \rangle_{\mathcal{H}}$. Setting the constant reference signal $z_r(t) = z_0 - z_e$, we define Γ_0 similarly to $\Gamma(t)$ as in (2.16). Then, since A_K is Hurwitz, hence invertible, we can define the vector $X_0 = -A_K^{-1}\Gamma_0$. From the above reasoning, this implies the existence of $\xi_0 \in \mathbb{R}$ so that the triple (W_0, v_0, ξ_0) defines an equilibrium condition for (2.20), i.e, $0 = A_K X_0 + \Gamma_0$, $0 = \lambda_k w_{0,k} + a_k v_0 + r_{0,k}$, and $v_{d,0} = K X_0 = 0$. Defining the deviations of the system trajectories with respect to the steady state y_0 by $\Delta W = W - W_0$, $\Delta X = X - X_0$, $\Delta v = v - v_0$, $\Delta v_d = v_d - v_{d,0}$, $\Delta \xi = \xi - \xi_0$, $\Delta z_r = z_r - (z_0 - z_e)$, $\Delta\Gamma = \Gamma - \Gamma_0$, $\Delta r = r - r_0$, $\Delta R = R - R_0 = (0, \Delta r)$, and $\Delta r_k = r_k - r_{0,k}$ we infer that

$$\Delta v_d(t) = K \Delta X(t), \tag{2.21a}$$

$$\Delta \dot{X}(t) = A_K \Delta X(t) + \Delta\Gamma(t), \tag{2.21b}$$

$$\Delta \dot{w}_k(t) = \lambda_k \Delta w_k(t) + a_k \Delta v(t) + b_k \Delta v_d(t) + \Delta r_k(t), \qquad |k| \geq n_0 + 1. \tag{2.21c}$$

2.6. Main results. The two theorems below precisely formalize Theorem 1.4.

THEOREM 2.10. *There exist $\kappa \in (0,1)$ and $\overline{C}_1, \epsilon, \delta > 0$ such that, for any $\eta \in [0,1)$, there exists $\overline{C}_2 > 0$ such that, for any steady state $y_0 \in \mathcal{C}^2([0,L])$ of (1.1) so that $\|y_0 - y_e\|_{L^\infty(0,L)} \leq \epsilon$, for any initial condition satisfying $\|\Delta W(0)\|_{\mathcal{H}}^2 + |\Delta\xi(0)|^2 + |\Delta v(0)|^2 \leq \delta$, and any continuously differentiable z_r with $\|\Delta z_r\|_{L^\infty(\mathbb{R}_+)}^2 \leq \delta$, the solution of (2.2) with control law (2.19) is well defined on $\mathbb{R}_+$ and satisfies*

$$\|\Delta w^1(t,\cdot)\|_{L^\infty(0,L)} < 1 \qquad \textit{for all } t \geq 0, \tag{2.22}$$

(2.23)
$$\begin{aligned} \|\Delta W(t)\|_{\mathcal{H}}^2 + |\Delta\xi(t)|^2 + |\Delta v(t)|^2 \leq \overline{C}_1 e^{-2\kappa t} \left(\|\Delta W(0)\|_{\mathcal{H}}^2 + |\Delta\xi(0)|^2 + |\Delta v(0)|^2 \right) \\ + \overline{C}_2 \sup_{0 \leq s \leq t} e^{-2\eta\kappa(t-s)} |\Delta z_r(s)|^2 \qquad \textit{for all } t \geq 0. \end{aligned}$$

Remark 2.11. When f is linear, the closed-loop system is also linear (in particular the residual term (2.3) is identically zero). In that case, the local exponential stability result (2.23) stated by Theorem 2.10 becomes global.

Remark 2.12. Estimates (2.22) and (2.23) can be interpreted as a local input-to-state stability estimate with respect to Δz_r.

Remark 2.13. The result of Theorem 2.10 ensures the stability of the closed-loop system in w coordinates. This immediately induces the stability of the closed-loop

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system in its original coordinates because, from (2.1) and recalling that $w_0^1 = y_0 - y_e$ and $w_0^2 = -\frac{x}{\alpha L} v_0$, we have $\Delta W(t,x) = (y(t,x) - y_0(x), \frac{\partial y}{\partial t}(t,x) - \frac{x}{\alpha L}\Delta v(t))$ and hence

$$\left\| \left(y(t,\cdot) - y_0, \frac{\partial y}{\partial t}(t,\cdot) \right) \right\|_{\mathcal{H}} \leq \|\Delta W(t)\|_{\mathcal{H}} + \frac{1}{\alpha}\sqrt{\frac{L}{3}} |\Delta v(t)|.$$

THEOREM 2.14. *Let $\kappa \in (0,1)$, $\epsilon, \delta > 0$, and $\eta \in [0,1)$ be as provided by Theorem 2.10. There exist constants $\overline{C}_3, \overline{C}_4 > 0$ such that, for any steady state $y_0 \in \mathcal{C}^2([0,L])$ of (1.1) so that $\|y_0 - y_e\|_{L^\infty(0,L)} \leq \epsilon$, for any initial condition satisfying $\|\Delta W(0)\|_{\mathcal{H}}^2 + |\Delta\xi(0)|^2 + |\Delta v(0)|^2 \leq \delta$, and any continuously differentiable z_r with $\|\Delta z_r\|_{L^\infty(\mathbb{R}_+)}^2 \leq \delta$, the solution of (2.2) with control law (2.19) satisfies*

(2.24)
$$\begin{aligned} \left|(w^1)'(t,0) - z_r(t)\right| \leq \overline{C}_3 e^{-\kappa t} \left(\|\Delta W(0)\|_{\mathcal{H}} + |\Delta\xi(0)| + |\Delta v(0)| + \|\mathcal{A}\Delta W(0)\|_{\mathcal{H}} \right) \\ + \overline{C}_4 \sup_{0\leq s\leq t} e^{-\eta\kappa(t-s)} |\Delta z_r(s)| \qquad \textit{for all } t \geq 0. \end{aligned}$$

Remark 2.15. In particular, $z_r(t) \to z_0 - z_e$ implies $(w^1)'(t,0) \to z_0 - z_e$, i.e., $z(t) \to z_0$, which achieves the desired set point reference tracking.

3. Proof of the main results.

3.1. Proof of Theorem 2.10.

Let $M > 3(\|a\|_{\mathcal{H}}^2 + \|b\|_{\mathcal{H}}^2 \|K\|^2)/m_R$ be given, where we recall that $a, b \in \mathcal{H}$ are defined by (2.8) and the constant $m_R > 0$ is as provided by Definition 2.4. We set

$$V(t) = M\Delta X(t)^\top P \Delta X(t) + \frac{1}{2}\sum_{|k|\geq n_0+1} |\Delta w_k(t)|^2 \qquad \text{for all } t \geq 0, \tag{3.1}$$

where P is a symmetric definite positive matrix such that $A_K^\top P + PA_K = -I$. Then we obtain from (2.21) that

$$\begin{aligned} \dot{V}(t) &= M\Delta X(t)^\top \left(A_K^\top P + PA_K\right) \Delta X(t) \\ &\quad + M\left(\Delta\Gamma(t)^\top P \Delta X(t) + \Delta X(t)^\top P \Delta\Gamma(t)\right) + \sum_{|k|\geq n_0+1} \operatorname{Re}\lambda_k |\Delta w_k(t)|^2 \\ &\quad + \sum_{|k|\geq n_0+1} \operatorname{Re}\left(\overline{\Delta w_k(t)}\left(a_k \Delta v(t) + b_k \Delta v_d(t) + \Delta r_k(t)\right)\right) \\ &\leq -M\|\Delta X(t)\|^2 - \sum_{|k|\geq n_0+1} |\Delta w_k(t)|^2 + 2M\|P\|\|\Delta X(t)\|\|\Delta\Gamma(t)\| \\ &\quad + \sum_{|k|\geq n_0+1} |\Delta w_k(t)| \left(|a_k||\Delta v(t)| + |b_k||\Delta v_d(t)| + |\Delta r_k(t)|\right), \end{aligned}$$

where we have used that $\operatorname{Re}\lambda_k < -1$ for all $|k| \geq n_0+1$. Using now Young's inequality, we infer that $2M\|P\|\|\Delta X(t)\|\|\Delta\Gamma(t)\| \leq \frac{M}{2}\|\Delta X(t)\|^2 + 2M\|P\|^2\|\Delta\Gamma(t)\|^2$ and

$$\begin{aligned} &\sum_{|k|\geq n_0+1} |\Delta w_k(t)| \left(|a_k||\Delta v(t)| + |b_k||\Delta v_d(t)| + |\Delta r_k(t)|\right) \\ &\quad \leq \frac{1}{2}\sum_{|k|\geq n_0+1} |\Delta w_k(t)|^2 + \frac{3}{2}\sum_{|k|\geq n_0+1} \left(|a_k|^2|\Delta v(t)|^2 + |b_k|^2|\Delta v_d(t)|^2 + |\Delta r_k(t)|^2\right) \end{aligned}$$

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$$\leq \frac{1}{2} \sum_{|k| \geq n_0+1} |\Delta w_k(t)|^2 + \frac{3\|a\|_{\mathcal{H}}^2}{2m_R} |\Delta v(t)|^2 + \frac{3\|b\|_{\mathcal{H}}^2}{2m_R} |\Delta v_d(t)|^2 + \frac{3}{2m_R} \|\Delta R(t,\cdot)\|_{\mathcal{H}}^2$$

$$\leq \frac{1}{2} \sum_{|k| \geq n_0+1} |\Delta w_k(t)|^2 + \frac{3(\|a\|_{\mathcal{H}}^2 + \|b\|_{\mathcal{H}}^2 \|K\|^2)}{2m_R} \|\Delta X(t)\|^2 + \frac{3}{2m_R} \|\Delta R(t,\cdot)\|_{\mathcal{H}}^2,$$

where we have used (2.21) and the fact that $\Delta v(t)$ is the first component of $\Delta X(t)$. Thus, we obtain that

$$\dot{V}(t) \leq -\left(\frac{M}{2} - \frac{3(\|a\|_{\mathcal{H}}^2 + \|b\|_{\mathcal{H}}^2 \|K\|^2)}{2m_R} \right) \|\Delta X(t)\|^2 - \frac{1}{2} \sum_{|k| \geq n_0+1} |\Delta w_k(t)|^2 + 2M\|P\|^2 \|\Delta\Gamma(t)\|^2 + \frac{3}{2m_R} \|\Delta R(t,\cdot)\|_{\mathcal{H}}^2. \tag{3.2}$$

We evaluate the two last terms of the above inequality. Recalling that $R_1(t)$, $\gamma(t)$, and $\Gamma(t)$ are defined by (2.13), (2.15c), and (2.16), respectively, we have

$$\|\Delta\Gamma(t)\|^2 = \sum_{|k| \leq n_0} |\Delta r_k(t)|^2 + \left| \Delta z_r(t) + \sum_{|k| \geq n_0+1} \frac{(e_k^1)'(0)}{\lambda_k} \Delta r_k(t) \right|^2$$

$$\leq \sum_{|k| \leq n_0} |\Delta r_k(t)|^2 + 2|\Delta z_r(t)|^2 + 2 \sum_{|k| \geq n_0+1} \left| \frac{(e_k^1)'(0)}{\lambda_k} \right|^2 \sum_{|k| \geq n_0+1} |\Delta r_k(t)|^2$$

$$\leq C_0^2 \|\Delta R(t,\cdot)\|_{\mathcal{H}}^2 + 2|\Delta z_r(t)|^2, \tag{3.3}$$

with $C_0^2 = \frac{1}{m_R} \max(1, 2\sum_{|k| \geq n_0+1} |\frac{(e_k^1)'(0)}{\lambda_k}|^2) < +\infty$. We now need to evaluate $\|\Delta R(t,\cdot)\|_{\mathcal{H}}^2 = \int_0^L |\Delta r(t,x)|^2 \, \mathrm{d}x$. From (2.3) we have

$$\begin{aligned} \Delta r(t,x) &= r(t,x) - r_0(x) \\ &= \int_{y_e(x)}^{w^1(t,x)+y_e(x)} (w^1(t,x) + y_e(x) - s) f''(s) \, \mathrm{d}s \\ &\quad - \int_{y_e(x)}^{w_0^1(x)+y_e(x)} (w_0^1(x) + y_e(x) - s) f''(s) \, \mathrm{d}s \\ &= \Delta w^1(t,x) \int_{w_0^1(x)+y_e(x)}^{w^1(t,x)+y_e(x)} f''(s) \, \mathrm{d}s + \Delta w^1(t,x) \int_{y_e(x)}^{w_0^1(x)+y_e(x)} f''(s) \, \mathrm{d}s \\ &\quad - |\Delta w^1(t,x)|^2 \int_0^1 (1-s) f''(w^1(t,x) + y_e(x) + s(w_0^1(x) - w^1(t,x))) \, \mathrm{d}s. \end{aligned}$$

Let $\epsilon \in (0,1)$ to be chosen later. Let $C_I \geq 0$ be the maximum of $|f''|$ over $[\min y_e - 2, \max y_e + 2]$. For[3] $\|w_0^1\|_{L^\infty(0,L)} = \|y_0 - y_e\|_{L^\infty(0,L)} \leq \epsilon < 1$ and assuming that

$$\|\Delta w^1(t,\cdot)\|_{L^\infty(0,L)} \leq \epsilon < 1, \tag{3.4}$$

we obtain that $|\Delta r(t,x)| \leq 2C_I |\Delta w^1(t,x)|^2 + C_I |w_0^1(x)| |\Delta w^1(t,x)| \leq 3\epsilon C_I |\Delta w^1(t,x)|$, and thus

[3]The evaluation of the distance between y_0 and y_e in L^∞-norm, rather than in L^2-norm, is required to obtain the upcoming estimate for $|\Delta r(t,x)|$.

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$$\|\Delta R(t,\cdot)\|_{\mathcal{H}}^2$$
$$\leq 9\epsilon^2 C_I^2 \|\Delta w^1(t,\cdot)\|_{L^2(0,L)}^2 \leq 9\epsilon^2 L^2 C_I^2 \|\Delta W(t)\|_{\mathcal{H}}^2 \tag{3.5}$$
$$\leq 9\epsilon^2 M_R L^2 C_I^2 \sum_{k\in\mathbb{Z}} |\Delta w_k(t)|^2 \leq \epsilon^2 C_1^2 \left(\|\Delta X(t)\|^2 + \sum_{|k|\geq n_0+1} |\Delta w_k(t)|^2 \right), \tag{3.6}$$

where the second inequality follows from Poincaré inequality, the fourth inequality follows from Definition 2.4, and with the constant $C_1 \geq 0$ defined by $C_1^2 = 9M_R L^2 C_I^2$. We deduce from (3.2)–(3.3) and, under the a priori estimate (3.4) and (3.6) that

$$\dot{V}(t) \leq -\left(\frac{M}{2} - \frac{3(\|a\|_{\mathcal{H}}^2 + \|b\|_{\mathcal{H}}^2\|K\|^2)}{2m_R} - \epsilon^2 C_1^2 \left(2M\|P\|^2 C_0^2 + \frac{3}{2m_R} \right) \right) \|\Delta X(t)\|^2$$
$$- \frac{1}{2}\left(1 - 2\epsilon^2 C_1^2 \left(2M\|P\|^2 C_0^2 + \frac{3}{2m_R} \right) \right) \sum_{|k|\geq n_0+1} |\Delta w_k(t)|^2 + 4M\|P\|^2 |\Delta z_r(t)|^2.$$

With $M > 3(\|a\|_{\mathcal{H}}^2 + \|b\|_{\mathcal{H}}^2\|K\|^2)/m_R$ and by selecting $\epsilon \in (0,1)$ small enough (independently of the initial conditions and the reference signal z_r), we obtain the existence of constants $\kappa, C_2 > 0$ such that, under the a priori estimate (3.4), $\dot{V}(t) \leq -2\kappa V(t) + C_2 |\Delta z_r(t)|^2$. Let $\eta \in [0,1)$ be arbitrary. Assuming that the a priori estimate (3.4) holds on $[0,t]$ for some $t > 0$, we obtain that

$$V(t) \leq e^{-2\kappa t} V(0) + C_2 \int_0^t e^{-2\kappa(t-s)} |\Delta z_r(s)|^2 \,\mathrm{d}s$$
$$\leq e^{-2\kappa t} V(0) + \frac{C_2}{2\kappa(1-\eta)} \sup_{0\leq s\leq t} e^{-2\eta\kappa(t-s)} |\Delta z_r(s)|^2.$$

Since P is symmetric positive definite, we obtain from (3.1) the existence of constants $C_3, C_4 > 0$ such that $V(0) \leq C_3 \left(\|\Delta W(0)\|_{\mathcal{H}}^2 + |\Delta\xi(0)|^2 + |\Delta v(0)|^2 \right)$ and $V(t) \geq C_4 \left(\|\Delta W(t)\|_{\mathcal{H}}^2 + |\Delta\xi(t)|^2 + |\Delta v(t)|^2 \right)$. Thus, assuming that the a priori estimate (3.4) holds over $[0,t]$ for some $t > 0$, we deduce from the three above estimates the existence of constants $\overline{C}_1, \overline{C}_2 > 0$ such that

$$\|\Delta W(t)\|_{\mathcal{H}}^2 + |\Delta\xi(t)|^2 + |\Delta v(t)|^2 \leq \overline{C}_1 e^{-2\kappa t} \left(\|\Delta W(0)\|_{\mathcal{H}}^2 + |\Delta\xi(0)|^2 + |\Delta v(0)|^2 \right)$$
$$+ \overline{C}_2 \sup_{0\leq s\leq t} e^{-2\eta\kappa(t-s)} |\Delta z_r(s)|^2. \tag{3.7}$$

We now note that $\|\Delta w^1(t,\cdot)\|_{L^\infty(0,L)} \leq \sqrt{L}\|(\Delta w^1)'(t,\cdot)\|_{L^2(0,L)} \leq \sqrt{L}\|\Delta W(t)\|_{\mathcal{H}}$. Hence, for a steady state y_0 so that $\|w_0^1\|_{L^\infty(0,L)} = \|y_0 - y_e\|_{L^\infty(0,L)} \leq \epsilon$, if the initial condition is selected such that $\|\Delta W(0)\|_{\mathcal{H}}^2 + |\Delta\xi(0)|^2 + |\Delta v(0)|^2 \leq \frac{\epsilon^2}{2L}\min(1, \frac{1}{2\overline{C}_1})$, which in particular ensures that $\|\Delta w^1(0,\cdot)\|_{L^\infty(0,L)} \leq \epsilon/\sqrt{2} < \epsilon$, and the reference input is chosen such that $\|\Delta z_r\|_{L^\infty(0,L)}^2 \leq \frac{\epsilon^2}{4L\overline{C}_2}$, it is readily checked based on (3.7) that the a priori estimate (3.4) holds for all $t \geq 0$. In this case, the stability estimate (3.7) holds for all $t \geq 0$.

3.2. Proof of Theorem 2.14. We first derive an estimate of $\|\frac{\mathrm{d}(\Delta R)}{\mathrm{d}t}(t,\cdot)\|_{\mathcal{H}}$. To do so, we assume that the assumptions and conclusions of Theorem 2.10 apply. Following up with Remark 2.2, we have that

$$\frac{\mathrm{d}(\Delta R)}{\mathrm{d}t} = \left(0, \frac{\mathrm{d}r}{\mathrm{d}t}\right), \qquad \frac{\mathrm{d}r}{\mathrm{d}t}(t,\cdot) = \left(w^2(t,\cdot) + \frac{(\cdot)}{\alpha L} v(t) \right) \int_{y_e}^{w^1(t,\cdot)+y_e} f''(s)\,\mathrm{d}s.$$

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Let $C_J \geq 0$ be the maximum of $|f''|$ over $[\min y_e - (\epsilon+1), \max y_e + (\epsilon+1)]$. Recalling that $w^1 = \Delta w^1 + w^1_0$ with $w^1_0 = y_0 - y_e$, the use of $\|y_0 - y_e\|_{L^\infty(0,L)} \leq \epsilon$ and (2.22) imply that

$$\begin{aligned}
\left\| \frac{\mathrm{d}(\Delta R)}{\mathrm{d}t}(t,\cdot) \right\|_{\mathcal{H}}^2 &\leq \int_0^L \left| w^2(t,x) + \frac{x}{\alpha L} v(t) \right|^2 \left| \int_{y_e(x)}^{w^1(t,x)+y_e(x)} |f''(s)| \,\mathrm{d}s \right|^2 \mathrm{d}x \\
&\leq 2C_J^2(1+\epsilon)^2 \left(\|\Delta w^2(t)\|^2_{L^2(0,L)} + \frac{L}{3\alpha^2} |\Delta v(t)|^2 \right) \\
&\leq 2C_J^2(1+\epsilon)^2 \left(\|\Delta W(t)\|^2_{\mathcal{H}} + \frac{L}{3\alpha^2} |\Delta v(t)|^2 \right), \qquad (3.8)
\end{aligned}$$

where we have used that $0 = w^2_0 + \frac{x}{\alpha L} v_0$. We are now in position to complete the proof of Theorem 2.14. We fix an integer $N \geq n_0$ and a constant $\gamma > 0$ such that[4] $\mathrm{Re}\lambda_k \leq -\gamma < -\kappa < 0$ for all $|k| \geq N+1$. Recalling that $(w^1_0)'(0) = z_0 - z_e$, we have from (2.9) that

$$\begin{aligned}
&\left|(w^1)'(t,0) - z_r(t)\right| = \left|(\Delta w^1)'(t,0) - \Delta z_r(t)\right| \\
&\quad \leq \sum_{k\in\mathbb{Z}} |\Delta w_k(t)| |(e^1_k)'(0)| + |\Delta z_r(t)| \\
&\quad \leq \sum_{|k|\leq N} |\Delta w_k(t)| |(e^1_k)'(0)| + \sum_{|k|\geq N+1} |\lambda_k \Delta w_k(t)| \left| \frac{(e^1_k)'(0)}{\lambda_k} \right| + |\Delta z_r(t)| \\
&\quad \leq C_5 \|\Delta W(t)\|_{\mathcal{H}} + C_5 \sqrt{\sum_{|k|\geq N+1} |\lambda_k \Delta w_k(t)|^2} + |\Delta z_r(t)|, \qquad (3.9)
\end{aligned}$$

with $C_5 = \max(\frac{1}{m_R} \sum_{|k|\leq N} |(e^1_k)'(0)|^2, \sum_{|k|\geq N+1} |\frac{(e^1_k)'(0)}{\lambda_k}|^2)^{1/2}$. Based on (2.23), we only need to evaluate the term $\sum_{|k|\leq N} |\lambda_k \Delta w_k(t)|^2$. We have from (2.21) that, for all $|k| \geq N+1 \geq n_0+1$,

$$\begin{aligned}
\lambda_k \Delta w_k(t) &= e^{\lambda_k t} \lambda_k \Delta w_k(0) + \int_0^t \lambda_k e^{\lambda_k (t-s)} \left(a_k \Delta v(s) + b_k \Delta v_d(s) + \Delta r_k(s) \right) \mathrm{d}s \\
&= e^{\lambda_k t} \lambda_k \Delta w_k(0) - a_k \Delta v(t) - b_k \Delta v_d(t) - \Delta r_k(t) \qquad (3.10) \\
&\quad + e^{\lambda_k t} \left(a_k \Delta v(0) + b_k \Delta v_d(0) + \Delta r_k(0) \right) \\
&\quad + \int_0^t e^{\lambda_k (t-s)} \left(a_k \Delta v_d(s) + b_k \Delta \dot{v}_d(s) + \Delta \dot{r}_k(s) \right) \mathrm{d}s.
\end{aligned}$$

We have from (2.21) that $\Delta v_d(t) = K \Delta X(t)$ and $\Delta \dot{v}_d(t) = K A_K \Delta X(t) + K \Delta \Gamma(t)$ with

$$\|\Delta X(t)\|^2 \leq \frac{1}{m_R} \|\Delta W(t)\|^2_{\mathcal{H}} + |\Delta \xi(t)|^2 + |\Delta v(t)|^2, \qquad (3.11a)$$

$$\|\Delta \Gamma(t)\|^2 \leq 9L^2 C_0^2 C_I^2 \|\Delta W(t)\|^2_{\mathcal{H}} + 2|\Delta z_r(t)|^2, \qquad (3.11b)$$

where the second inequality follows from (3.3) and (3.5). Using $\gamma > \kappa > \eta\kappa \geq 0$ and denoting $\mathrm{CI} = \sqrt{\|\Delta W(0)\|^2_{\mathcal{H}} + |\Delta \xi(0)|^2 + |\Delta v(0)|^2}$, we have from (2.23) that

[4] We recall that $\mathrm{Re}\,\lambda_k < -1$ for all $|k| \geq n_0 + 1$ and that $\kappa \in (0,1)$.

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$$\left|\int_0^t e^{\lambda_k(t-s)}\Delta v_d(s)\,\mathrm{d}s\right| \leq \|K\| \int_0^t e^{-\gamma(t-s)}\|\Delta X(s)\|\,\mathrm{d}s$$
$$\leq \|K\| \max(1, m_R^{-1/2}) e^{-\gamma t} \int_0^t e^{\gamma s}\left(\sqrt{\overline{C}_1}e^{-\kappa s}\mathrm{CI} + \sqrt{\overline{C}_2}\sup_{0\leq\tau\leq s} e^{-\eta\kappa(s-\tau)}|\Delta z_r(\tau)|\right)\mathrm{d}s$$
$$\leq C_6 e^{-\kappa t}\mathrm{CI} + C_6 \sup_{0\leq\tau\leq t} e^{-\eta\kappa(t-\tau)}|\Delta z_r(\tau)| \tag{3.12}$$

for some constant $C_6 > 0$, and, similarly,

$$\left|\int_0^t e^{\lambda_k(t-s)}\Delta \dot{v}_d(s)\,\mathrm{d}s\right| \leq C_7 e^{-\kappa t}\mathrm{CI} + C_7 \sup_{0\leq\tau\leq t} e^{-\eta\kappa(t-\tau)}|\Delta z_r(\tau)| \tag{3.13}$$

for some constant $C_7 > 0$. Finally, we also have, for $-\gamma < -\tilde{\kappa} < -\kappa < 0$,

$$\left|\int_0^t e^{\lambda_k(t-s)}\Delta \dot{r}_k(s)\,\mathrm{d}s\right| \leq \sqrt{\int_0^t e^{-2(\gamma-\tilde{\kappa})(t-s)}\,\mathrm{d}s}\sqrt{\int_0^t e^{-2\tilde{\kappa}(t-s)}|\Delta\dot{r}_k(s)|^2\,\mathrm{d}s}$$
$$\leq \sqrt{\frac{1}{2(\gamma-\tilde{\kappa})}}\sqrt{\int_0^t e^{-2\tilde{\kappa}(t-s)}|\Delta\dot{r}_k(s)|^2\,\mathrm{d}s}. \tag{3.14}$$

Taking the square on both sides of (3.10), using Young's inequality, substituting estimates (2.23), (3.5), and (3.11), and using the fact that $\sum_{|k|\geq N+1} |\langle z, f_k\rangle|^2 \leq \|z\|_{\mathcal{H}}^2/m_R$ for all $z \in \mathcal{H}$, we infer the existence of a constant $C_8 > 0$ such that

$$\sum_{|k|\geq N+1} |\lambda_k \Delta w_k(t)|^2 \leq C_8 e^{-2\kappa t} \sum_{|k|\geq N+1} |\lambda_k \Delta w_k(0)|^2$$
$$+ C_8 e^{-2\kappa t}\mathrm{CI}^2 + C_8 \sup_{0\leq\tau\leq t} e^{-2\eta\kappa(t-\tau)}|\Delta z_r(\tau)|^2 + C_8 \int_0^t e^{-2\tilde{\kappa}(t-s)}\left\|\frac{\mathrm{d}(\Delta R)}{\mathrm{d}t}(s,\cdot)\right\|_{\mathcal{H}}^2 \mathrm{d}s,$$

where we recall that $\mathrm{Re}\lambda_k \leq -\gamma < -\kappa < 0$ for all $|k| \geq N+1$. Noting that, for $|k| \geq N+1 \geq n_0+1$, $\langle \mathcal{A}\Delta W(0), f_k\rangle_{\mathcal{H}} = \lambda_k \Delta w_k(0)$, we infer that $\sum_{|k|\geq N+1} |\lambda_k \Delta w_k(0)|^2 \leq \sum_{k\in\mathbb{Z}} |\langle \mathcal{A}\Delta W(0), f_k\rangle_{\mathcal{H}}|^2 \leq \|\mathcal{A}\Delta W(0)\|_{\mathcal{H}}^2/m_R$. Then, based on (2.23) and (3.8), and as $\tilde{\kappa} > \kappa > 0$, estimations similar to the ones reported in (3.12)–(3.13) show the existence of a constant $C_9 > 0$ such that

$$\sum_{|k|\geq N+1} |\lambda_k \Delta w_k(t)|^2 \leq C_9 e^{-2\kappa t}\left(\mathrm{CI}^2 + \|\mathcal{A}\Delta W(0)\|_{\mathcal{H}}^2\right) + C_9 \sup_{0\leq\tau\leq t} e^{-2\eta\kappa(t-\tau)}|\Delta z_r(\tau)|^2.$$

Substituting this latter estimate and (2.23) into (3.9), we obtain the existence of constants $\overline{C}_3, \overline{C}_4 > 0$ such that (2.24) holds.

4. Numerical illustration.

For numerical illustration, we set $L = 1$, $\alpha = 1.1$, and we consider the nonlinear function $f(y) = y^3$. Since $F(y) = \int_0^y f(s)\,\mathrm{d}s = y^4/4 \to +\infty$ when $|y| \to +\infty$, it follows from Remark 1.2 the existence of a steady state $y_e \in \mathcal{C}^2([0,L])$ associated with any given value of the system output $z_e \in \mathbb{R} = y_e'(0)$. We set $z_e = 1.5$ and numerically compute the associated steady-state trajectory y_e, giving in particular the equilibrium control input $u_e = y_e'(L) \approx 0.781$. In the absence

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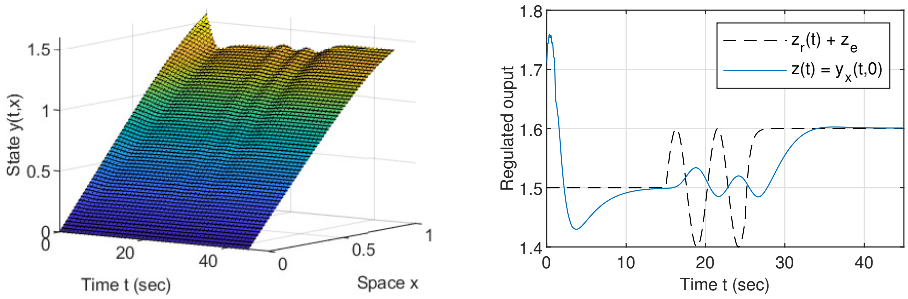

(a) State $y(t,x)$ (b) Regulated output $z(t) = \frac{\partial y}{\partial x}(t,0)$

FIG. 1. *Time domain evolution of the closed-loop system.*

of nonlinearity, i.e., $f = 0$, it is known (see, e.g., [28, section 4]) that the eigenvalues of the operator $\mathcal{A}$ defined by (2.7) are given by $\lambda_k = \frac{1}{2L}\log(\frac{\alpha-1}{\alpha+1}) + i\frac{k\pi}{L}$. These values are used as initial guesses to determine the eigenvalues λ_k and the associated eigenvectors e_k of the operator $\mathcal{A}$ in the presence of the nonlinearity $f(y) = y^3$ by using a shooting method. We obtain one unstable eigenvalue $\lambda_0 \approx 0.326$ while all other eigenvalues are stable with a real part less than 1. The feedback gain is computed to place the poles of the truncated model at -0.5, -1, and -1.5.

For numerical simulations, we take $W(0,x) = (\frac{\alpha}{5}x, -\frac{1}{5L}x) \in D(\mathcal{A})$. The adopted numerical scheme is the modal approximation of the infinite-dimensional system using its first 10 modes. The time domain evolution of the closed-loop system is depicted in Figure 1. The simulation results are compliant with the theoretical predictions.

5. Conclusion. In this paper we have investigated the boundary regulation control of the left Neumann trace of a one-dimensional semilinear wave equation using a PI control design procedure. Our control strategy combines a traditional velocity feedback and the design of an auxiliary control law performed on a finite-dimensional (spectrally) truncated model.

The PI control design procedure carried out in this paper has been focused on the ability for the system output to achieve local set point reference tracking of the reference signal. However, it is well known for finite-dimensional systems that PI control design schemes carry on certain robustness properties, e.g., w.r.t. external perturbations. We conjecture that such robustness properties also hold for the control design problem addressed in this paper. For instance, the addition of an additive perturbation $d(t,x)$ to the semilinear wave equation (1.1), sufficiently small in L^2-norm and evolving around a steady state $d_0(x)$, should conduct to a stability estimate similar to (2.22) from Theorem 2.10 but augmented on the right-hand side with a term of the type $\sup_{0\leq s\leq t} e^{-2\eta\kappa(t-s)}\|\Delta d(s,\cdot)\|_{L^2}^2$. A similar remark holds for Theorem 2.14 with estimate (2.24) also augmented by the above mentioned term, as well as a term of the type $\sup_{0\leq s\leq t} e^{-2\eta\kappa(t-s)}\|(\partial d/\partial t)(s,\cdot)\|_{L^2}^2$.

It was shown in [21] for a 1-D reaction-diffusion equation that the PI controller design procedure can be augmented with a predictor component [1, 7, 19, 22] to handle constant control input delays. Such a strategy fails in the context of the wave equation studied in this paper. The reason is that, even for arbitrarily small $h > 0$, the preliminary feedback $u_\delta(t) = -\alpha\frac{\partial y_\delta}{\partial t}(t-h, L)$ might fail to stabilize the linear wave equation, yielding an infinite number of unstable modes. To illustrate this remark, let us consider the following linear wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - \beta y, \qquad y(t,0) = 0, \qquad \frac{\partial y}{\partial x}(t,L) = -\alpha\frac{\partial y}{\partial t}(t-h,L), \tag{5.1}$$

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where $h > 0$ is a constant delay, $\alpha > 0$, and $\beta \in \mathbb{R}$. It was shown in [11] in the case $\alpha = 1$ and $\beta = 0$ that (5.1) admits an infinite number of unstable modes for some arbitrarily small values of the delay $h > 0$. We elaborate here this remark by extending it to the case $\alpha > 0$ and $\beta \in \mathbb{R}$. We proceed similarly to [11] by looking for solutions of (5.1) under the form $w(t,x) = e^{\lambda t}\sinh(\sqrt{\lambda^2+\beta}x)$ for some $\lambda \in \mathbb{C}$. It is easy to see that such a solution exists if and only if $\lambda \in \mathbb{C}$ satisfies $\sqrt{\lambda^2+\beta}\cosh(\sqrt{\lambda^2+\beta}L) = -\alpha\lambda e^{-\lambda h}\sinh(\sqrt{\lambda^2+\beta}L)$. We start by studying the special case $\beta = 0$. Then the above identity becomes equivalent to

$$e^{\lambda h} = -\alpha\tanh(\lambda L). \tag{5.2}$$

Let $k \in \mathbb{N}$ be arbitrarily fixed, and select the input delay $h = L/(k+1/2) > 0$. Let $\gamma > 0$ be the only positive number such that $e^{\gamma h} = \alpha\coth(\gamma L)$. It is easily seen that $\lambda_n^0 = \gamma + \frac{i}{L}(k+\frac{1}{2})(4n+1)\pi$, $n \in \mathbb{Z}$, are distinct solutions of (5.2) with $\operatorname{Re}\lambda_n = \gamma > 0$. We now turn out attention onto the case $\beta \neq 0$. Let $h, \gamma > 0$ be still defined as above. We introduce the open set $A = (\lambda \in \mathbb{C} : 0 < \operatorname{Re} z < 2\gamma\ , |\lambda| > \sqrt{\beta})$. We define for $\lambda \in A$ the holomorphic functions $f(\lambda) = \lambda\cosh(\lambda L) + \alpha\lambda e^{-\lambda h}\sinh(\lambda L)$ and $g(\lambda) = \sqrt{\lambda^2+\beta}\cosh(\sqrt{\lambda^2+\beta}L) + \alpha\lambda e^{-\lambda h}\sinh(\sqrt{\lambda^2+\beta}L)$ with $\sqrt{\lambda^2+\beta} = \lambda\exp(\frac{1}{2}\operatorname{Log}(1+\frac{\beta}{\lambda^2}))$, where Log denotes the principal determination of logarithm. We have $f(\lambda_n^0) = 0$ for all $n \in \mathbb{Z}$ and

$$g(\lambda) = f(\lambda) + O(1), \qquad \lambda \in A, \quad |\lambda| \to +\infty. \tag{5.3}$$

For a given constant $R > 0$ to be defined later, we consider the simple loop $\lambda_n : [0,2\pi] \to \mathbb{C}$ defined by $\lambda_n(\theta) = \lambda_n^0 + \frac{Re^{i\theta}}{n}$, $\theta \in [0,2\pi]$. We consider an integer $N_0 \geq 1$ such that $\lambda_n(\theta) \in A$ for all $|n| \geq N_0$ and all $\theta \in [0,2\pi]$. Standard computations yield $f(\lambda_n(\theta)) = (-1)^k i\lambda_n^0\frac{Re^{i\theta}}{n}\zeta\cosh(\gamma L) + O(\frac{1}{n})$ when $|n| \to +\infty$, uniformly with respect to $\theta \in [0,2\pi]$, where $\zeta = L + h\tanh(\gamma L) - L\tanh^2(\gamma L)$. We note that $\zeta \neq 0$. Indeed, the condition $\zeta = 0$ would imply that $\tanh(\gamma L) > 0$ is the positive root of $L + hX - LX^2$, yielding the contradiction $\tanh(\gamma L) = (h+\sqrt{h^2+4L^2})/(2L) > 1 + h/(2L) > 1$. We deduce that

$$|f(\lambda_n(\theta))| \sim CR \tag{5.4}$$

when $|n| \to +\infty$, uniformly with respect to $\theta \in [0,2\pi]$, where the constant $C = 2(2k+1)\pi|\zeta|\cosh(\gamma L)/L \neq 0$ is independent of $R > 0$. Hence, in view of (5.3) and (5.4) and by selecting $R > 0$ large enough, we can apply Rouché's theorem for $|n| \geq N_1$ with $N_1 > 0$ large enough. This shows for any $|n| \geq N_1$ the existence of $\lambda_n^\beta = \lambda_n^0 + O(\frac{1}{n})$ such that $g(\lambda_n^\beta) = 0$. Since $h = L/(k+1/2) > 0$, with $k \in \mathbb{N}$ arbitrarily fixed, $h > 0$ can be made arbitrarily small by selecting k large. Hence we have shown the existence of arbitrarily small delays $h > 0$ such that (5.1) admits an infinite number of unstable modes. Such an observation implies that the strategy reported in this paper cannot be successfully applied to the case of a delayed control input. Hence, the PI regulation control of (1.1) in the presence of a delay in the control input remains open. This research direction may be the topic of future works.

Appendix A. Annex—Proof of Lemma 2.5. From the definition of $\mathcal{A}$ given by (2.7), $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ associated with the nonzero eigenvector $w = (w^1, w^2) \in D(\mathcal{A})$ if and only if $w^2 = \lambda w^1$ and

$$(w^1)'' + f'(y_e)w^1 = \lambda^2 w^1, \qquad w^1(0) = 0, \quad (w^1)'(L) + \alpha\lambda w^1(L) = 0$$

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for $x \in (0,L)$. Then, for $|\lambda| \to +\infty$, we obtain for $x \in [0,L]$ that

$$w^1(x) = \sinh\left(\sqrt{\lambda^2 + O(1)}\,x\right), \quad (w^1)'(x) = \sqrt{\lambda^2 + O(1)}\cosh\left(\sqrt{\lambda^2 + O(1)}\,x\right),$$

uniformly with respect to $x \in [0,L]$. Using now the right boundary condition, we obtain the item 3. Then

$$e_k = \frac{1}{A_k}\left(\sinh\left(\sqrt{\lambda_k^2 + O(1)}\,x\right), \lambda_k \sinh\left(\sqrt{\lambda_k^2 + O(1)}\,x\right)\right)$$

is an associated unit eigenvector where, recalling that $\alpha > 1$ and introducing $\beta = -\frac{1}{2L}\log(\frac{\alpha-1}{\alpha+1}) > 0$, we have $A_k = |\lambda_k|\sqrt{\frac{\sinh(2\beta L)}{2\beta} + O(\frac{1}{|k|})}$. In particular, $(e_k^1)'(0) = O(1)$ as $|k| \to +\infty$, showing that item 7 holds true.

We show that the eigenvalues of $\mathcal{A}$ are geometrically simple (item 2). Assume that $w_i = (w_i^1, w_i^2) \in D(\mathcal{A})$, with $i \in \{1,2\}$, are two eigenvectors of $\mathcal{A}$ associated with the same eigenvalue $\lambda \in \mathbb{C}$. We note that $w_i(L) \neq 0$ because otherwise $(w_i^1)'(L) = -\alpha\lambda w_i^1(L) = 0$ hence, by Cauchy uniqueness, $w_i^1 = 0$ and thus $w_i^2 = \lambda w_i^1 = 0$, giving the contradiction $w = 0$. Then the function g defined by $g = w_2^1(L)w_1^1 - w_1^1(L)w_2^1 \neq 0$ satisfies $g'' + f'(y_e)g = \lambda^2 g$ and $g(L) = g'(L) = 0$, implying $g = 0$. Recalling that $w_i^2 = \lambda w_i^1$, this shows that w_1 and w_2 are not linearly independent.

Recalling that $\mathcal{A}$ has compact resolvent, we denote by $(e_k)_{k\in\mathbb{Z}}$ a complete set of unit generalized eigenfunctions of $\mathcal{A}$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{Z}}$ [14]. We are going to apply Bari's theorem [14] to show that $(e_k)_{k\in\mathbb{Z}}$ is a Riesz basis. To do so, we need a Riesz basis of reference. Based on the definition (2.7) of $\mathcal{A}$, we consider the operator $\mathcal{A}_{tr} = \left(\begin{smallmatrix} 0 & \mathrm{Id} \\ \Delta & 0 \end{smallmatrix}\right)$, obtained by removing $f'(y_e)\,\mathrm{Id}$, defined on the same domain as $\mathcal{A}$. We know from [28, section 4] that $\mathcal{A}_{tr}$ admits a Riesz basis of eigenvectors $(\phi_k)_{k\in\mathbb{Z}}$ associated with the eigenvalues $(\mu_k)_{k\in\mathbb{Z}}$ given for any $k \in \mathbb{Z}$ by

$$\mu_k = \frac{1}{2L}\log\left(\frac{\alpha-1}{\alpha+1}\right) + i\frac{k\pi}{L}, \quad \phi_k = \frac{1}{B_k}\left(\sinh(\mu_k x), \mu_k \sinh(\mu_k x)\right)$$

where $B_k = \frac{1}{L\sqrt{2\beta}}\sqrt{(\beta^2 L^2 + k^2\pi^2)\sinh(2\beta L)}$ with $\beta = -\frac{1}{2L}\log(\frac{\alpha-1}{\alpha+1}) > 0$. We deduce that $e_k = \phi_k + O(\frac{1}{|k|})$, in $\mathcal{H}$-norm as $|k| \to +\infty$. Hence $(e_k)_{k\in\mathbb{Z}}$ is quadratically close to the Riesz basis $(\phi_k)_{k\in\mathbb{Z}}$. Then, with the results of [15, Lemma 6.2 and Theorem 6.3] relying on Bari's theorem, we obtain that $(e_k)_{k\in\mathbb{Z}}$ is a Riesz basis. Introducing $(f_k)_{k\in\mathbb{Z}}$ as the dual Riesz basis of $(e_k)_{k\in\mathbb{Z}}$, items 1, 3, 4, and 6 hold true. Finally, an homotopy argument using the operator $\mathcal{A}_{tr}$ shows that the algebraic multiplicity of the real eigenvalues of $\mathcal{A}$ is odd, yielding item 5.

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