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MIN-MAX AND MIN-MIN STACKELBERG STRATEGIES WITH CLOSED-LOOP INFORMATION STRUCTURE

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Abstract. This paper deals with the min-max and min-min Stackelberg strategies in the case of a closed-loop information structure. Two-player differential one-single stage games are considered with one leader and one follower. We first derive necessary conditions for the existence of the follower to characterize the best response set of the follower and to recast it, under weak assumptions, to an equivalent and more convenient form for expressing the constraints of the leader's optimization problem. Under a standard strict Legendre condition, we then derive optimality necessary conditions for the leader of both min-max and min-min Stackelberg strategies in the general case of nonlinear criteria for finite time horizon games. This leads to an expression of the optimal controls along the associated trajectory. Then, using focal point theory, the necessary conditions are also shown to be sufficient and lead to cheap control. The set of initial states allowing the existence of an optimal trajectory is emphasized. The linear-quadratic case is detailed to illustrate these results.

1. Introduction

A Stackelberg game, named after Heinrich von Stackelberg in recognition of his pioneering work on static games [56], designates a two-player noncooperative decision making problem formalized as a hierarchical combination of two optimization problems. The *lower level* decision maker, called the *follower*, selects a strategy optimizing his/her own objective function, depending on the strategy of the *upper level* decision maker, called the *leader*. The leader may decide his/her strategy, optimizing his/her objective function, relative to the decisions of both players by knowing the rational reaction of the follower. Such a problem may be viewed as a particular *bilevel optimization problem* [26, 55, 58]. When the rational reaction set of the follower is not reduced to a singleton, the situation is more complex and several formulations exist and have been introduced by Leitmann [36] (see also [4])

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and called *weak* and *strong* Stackelberg strategies by Breton et al. [17] or pessimistic and optimistic ones in [26, 27]. The term *strong* reflects the fact that the leader and the follower are seeking to minimize the criterion of the leader. Such a strategy leads to several motivating properties. In particular, it is stable to perturbations and could thus be called a *Stackelberg equilibrium* and in addition the resulting criterion of the leader will be equal or better than the one obtained via a Nash solution. The term *weak* is applied to Stackelberg strategies for which the latter two properties do not hold anymore. Nonetheless they are interesting and adapted to model performance guarantees (or for instance, robustness in control theory), without assuming additionally that the follower, after having minimized his/her own criterion, tries to maximize the criterion of the leader as proposed in [36]. The strong and weak Stackelberg strategies will be referred to respectively as the *min-min* and *min-max* ones in the whole paper. The class of strong-weak Stackelberg strategies, introduced in [3] generalizes and gathers together strong and weak ones. Their computational aspects in static games have been studied in [38, 39].

Game theory being a generic multi-objective optimization framework, the field of applications of Stackelberg strategies is large and includes, for example, economy [9], social behaviors, marketing [31], advertising in licensing contracts [18], network communications [14, 34], military intelligence [45]. The Stackelberg strategy for differential games was introduced in [21, 47, 48]. We consider here two-player nonzero sum differential games with one leader and one follower.

The information structure [12] in the game is the set of all available information for the players to make their decisions. The methods used to tackle such a Stackelberg optimization problem depend on the specific information structure.

When *open-loop* information structure is considered, no measurement of the state of the system is available and the players are committed to follow a predetermined strategy based on their knowledge of the initial state, the system's model and the cost functional to be minimized. Necessary conditions for obtaining a Stackelberg strategy with an open-loop information structure are well known [1, 29, 33, 42, 47–50, 57] and are derived from the standard Pontryagin minimum principle [35]. The obtained controls in this case are only functions of time.

The Stackelberg strategy is known to be inconsistent in time [25, 28], and dynamic programming cannot help to derive the optimal controls. Note however that the concept of *feedback* Stackelberg control, not considered in the present paper, is defined as the limit of controls obtained by dynamic programming on infinitesimally small time subintervals (see [10, 12, 32, 40, 43, 44]), that is in a multistage framework of repeated games. This concept differs from the concept of closed-loop control under consideration in this paper.

For the *closed-loop* information structure case (or, more precisely, the memoryless closed-loop information structure), each player has access to current state measurements and thus can adapt his strategy to the evolution of the system. For the closed-loop information structure case, determining the Stackelberg strategy for differential games is much harder than the other information structures and has been an open problem for a long time. The main difficulty comes from the presence, in the expression of the rational reaction set of the follower, of the partial derivative of the leader's control with respect to the measurement of the state. Several attempts have been proposed in the literature to overcome this difficulty [12]. Among such techniques, two main approaches could be distinguished.

The first method is dedicated to min-min Stackelberg strategies with a team-optimal approach introduced in [11, 13]. At the first step, the leader and the follower are aiming at optimizing the leader criterion as a team. Under some weak assumptions for linear-quadratic games [52], the optimal value of the leader criterion is attained for a parametrized family of controls for the leader and the follower. At the second step, the parameters of both controls are chosen such that the control of the follower lies in the rational reaction set in response to the control of the leader [51]. This could be interpreted as a threat formulated by the leader towards the follower [52], that is the leader punishes the follower, if he/she does not comply with the leader's policy, like (grim-) trigger for repeated games [8, 30].

The second approach consists in defining the whole rational reaction set of the follower for a given control of the leader. The resulting optimal control problem turns out to be nonclassical, not solvable *a priori* with the usual Pontryagin minimum principle. To solve this kind of nonclassical problem, a variational method is proposed in [46], assuming that this is a normal optimization problem (the possible occurence of an abnormal case is not mentioned). Moreover, in [41] it is emphasized that this technique does not lead to a solution for all initial states, and the difficulty is bypassed by assuming that the initial state of the system is uniformly distributed over the unit sphere and replacing the optimization criterion with its mean value over the initial state.

In this paper, we investigate both min-max and min-min Stackelberg strategies with closed-loop information structure. The best response set of the follower is characterized. This allows a convenient reformulation of the constraints of the leader's optimization problem under an assumption that is weaker than considering the best response set reduced to a singleton. Nevertheless it is shown that under a standard strict Legendre condition, it is possible to solve both min-max and min-min Stackelberg strategies. Note that min-max and min-min Stackelberg strategies coincide whenever the best response set is reduced to a singleton, and this happens in the important linear-quadratic case. The optimality necessary conditions for the leader are obtained, for both min-max and min-min Stackelberg strategies, *along the associated trajectory*, in the same spirit as in [46] for min-min Stackelberg strategy and by considering all cases. In addition, sufficient conditions of the optimization problem for linear-quadratic differential games are established using focal times theory. Based on these necessary and/or sufficient conditions, we then characterize all initial states from which there emanates an optimal trajectory. Also, an extension is proposed to associate with every initial state an optimal trajectory by introducing the Jacobian of the leader's control in his own criterion. Note that in [46], although the final result (for linear-quadratic games only) is correct, some of the arguments thereof used to derive the necessary conditions are either erroneous or missing.

The outline of the paper is as follows. In Sec. 2, the min-max and min-min Stackelberg strategies are mathematically formulated. Section 3 gathers the necessary conditions of existence for a strategy for the follower (Sec. 3.1) and for the leader (Sec. 3.2) for min-max and min-min Stackelberg strategies. A degeneration property of the Stackelberg strategy is emphasized in Sec. 3.3. These necessary conditions are detailed in the case of linear-quadratic two-player differential games in Sec. 3.4. The sufficient conditions are provided for the linear-quadratic case in Sec. 4. All these results lead to the two main results of this paper Theorem 3.1 and Theorem 4.1, which ensure the existence of optimal trajectories. Concluding remarks make up Sec. 5. The main proofs are gathered in Appendix A.

2. Preliminaries: Stackelberg strategy

A two-player differential game with finite horizon comprises

- a set of two players $\mathcal{K} = \{1, 2\}$, where Player 1 is called the leader and Player 2 the follower;
- a game duration $\mathcal{T} = [0, t_f]$, with $t_f > 0$;
- a state space $\mathcal{X} = \mathbb{R}^n$ which contains all the states $x(t)$ at time $t \in \mathcal{T}$;
- open sets $\mathcal{U} \subset \mathcal{L}^\infty(\mathcal{T}\times\mathcal{X}, \mathbb{R}^{m_1})$ and $\mathcal{V} \subset \mathcal{L}^\infty(\mathcal{T}\times\mathcal{X}, \mathbb{R}^{m_2})$ representing the action sets (or control sets) respectively of the leader and the follower (they will be specified below);
- a mapping $f : \mathcal{T} \times \mathcal{X} \times \mathcal{U} \times \mathcal{V} \to \mathcal{X}$ of class $\mathcal{C}^1$ defining the evolution law of the state as

$$\dot{x}(t) = f\left(t, x(t), u_{|_t}, v_{|_t}\right), \quad x(0) = x_0, \tag{2.1}$$

where $u_{|_t}$ and $v_{|_t}$ are the values at time t of the controls of both players $u \in \mathcal{U}$ and $v \in \mathcal{V}$;

- utility functions or criteria $J_i : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ of Player $i \in \mathcal{K}$, defined as
$$J_i(u, v) = g_i(x(t_f)) + \int_0^{t_f} L_i\big(t, x(t), u_{|t}, v_{|t}\big)\mathrm{d}t. \tag{2.2}$$
The functions L_1 and L_2 are $\mathcal{C}^1$ with respect to x, u and v are continuous with respect to time t;
- the information structure of Player $i \in \mathcal{K}$, which will be discussed below.

The leader (Player 1), who chooses the control u, aims at minimizing the criterion J_1 and the follower (Player 2), who chooses the control v, aims at minimizing the criterion J_2. We assume that the game is a single stage one, that is the choice of the controls should be done before the game begins at $t = 0$. It is assumed in this paper that there exists an information bias in the game which induces a hierarchy between the two players. We therefore have a Stackelberg differential game. The follower will rationally react to an observed control u of the leader. The leader is aware of this rational behavior and will use this bias of information to choose his/her control and to minimize his/her own criterion.

Definition 2.1. The rational reaction set or best response set of the follower is defined by
$$T : u \in \mathcal{U} \longmapsto Tu \subset \mathcal{V} \tag{2.3}$$
where $Tu = \{v \mid v \text{ minimizes } J_2(u, \bar{v}), \bar{v} \in \mathcal{V}\}$.

Definition 2.2. A min-min Stackelberg strategy (u^{**}, v^{**}) is defined by the minimization problem
$$\begin{cases} v^{**} \in Tu^{**}, \\ u^{**} \text{ minimizes } \min\limits_{v \in Tu} J_1(u, v). \end{cases} \tag{2.4}$$

The min-min Stackelberg strategy is stable to a deviation of the controls and allows the leader to reach a criterion value at least as small as that associated with a Nash equilibrium [36]. This definition, considered in [46] is suitable in *team optimization problem* [11], that is games where the follower selects in his/her rational reaction set the control which minimizes the utility function of the leader. As mentioned in [36], the follower could nonetheless be interested not only in minimizing his/her own cost (in response to the leader's control) but also in maximizing that of the leader. We thus consider the min-max Stackelberg strategy.

Definition 2.3. A min-max Stackelberg strategy (u^*, v^*) is defined by the minimization problem
$$\begin{cases} v^* \in Tu^*, \\ u^* \text{ minimizes } \max_{v \in Tu} J_1(u, v). \end{cases} \tag{2.5}$$

A min-max Stackelberg strategy can be interpreted as the minimization of the leader's criterion whenever the worst choice of the follower's control among the rational reaction set occurs. It can be viewed as a robust or risk-averse property with respect to the choice of the follower, which could be crucial in automatic control for robust control such as $\mathcal{H}_2/\mathcal{H}_\infty$-control (see [37, 59]).

Remark 2.1. These two definitions coincide whenever T is a single-valued mapping. Considering that T is a single-valued mapping is a widespread assumption in the literature simplifying the arguments. Here we do not assume that and we make a weaker assumption (Assumption 1 in Sec. 3.2) that is related to the criterion J_1 and the set T. Assumption 1 will be discussed in Sec. 3.2. Note moreover that, in the linear-quadratic case (investigated in Sec. 3.4 and more specifically in Sec. 4), T is a single-valued mapping and hence both definitions min-min and min-max coincide in that case.

The structure of the controls has to be formalized to make precise the induced optimization problems:

- Whenever the controls are only functions of time t, that is $u_{|_t} = u(t)$ and $v_{|_t} = v(t)$, the game has an *open-loop* information structure. Necessary conditions for obtaining an open-loop Stackelberg solution, derived from the usual Pontryagin Minimum Principle, are well known [1, 47–50].
- The case where the controls are functions of time t and of the current value of the state $x(t)$, $u_{|_t} = u(t, x(t))$ and $v_{|_t} = v(t, x(t))$, is called *closed-loop* Stackelberg strategy. This is the case we consider in the present paper. The controls are thus designed along the trajectory $x(t)$ associated with the Stackelberg solution.
- Considering $u_{|_t} = u(t, x)$ and $v_{|_t} = v(t, x)$, defined for every $x \in \mathbb{R}^n$, and *not only along the trajectory* $x(t)$ corresponds to the concept of *feedback* Stackelberg solution, in the spirit of the dynamic programming approach [10, 47, 48], even though dynamic programming does not apply rigorously to such a Stackelberg strategy, due to time inconsistency [25].

As was said above, in this paper we consider *closed-loop Stackelberg strategies*. Note that, in the linear-quadratic case (see Sec. 3.4), the values of the closed-loop Stackelberg controls and feedback Stackelberg controls coincide *along the associated trajectory*.

Within the framework of a closed-loop information structure, the evolution law of the game state given by (2.1) is written as

$$\dot{x}(t) = f\left(t, x(t), u(t, x(t)), v(t, x(t))\right), \quad x(0) = x_0. \tag{2.6}$$

Moreover, the sets $\mathcal{U}$ and $\mathcal{V}$ can be specified such that, for every couple $(u, v) \in \mathcal{U} \times \mathcal{V}$, the associated trajectory $x(\cdot)$, solution of (2.6), is well defined on $\mathcal{T}$. Throughout the paper, for the sake of clarity, we use the notation $u_x = \dfrac{\partial u}{\partial x}$ to denote the Jacobian of $u(t, x)$ with respect to the second variable x. We thus have

$$\mathcal{U} = \Big\{ u(\cdot,\cdot) \in \mathcal{L}^\infty(\mathcal{T} \times \mathcal{X}, \mathbb{R}^{m_1}), \text{ such that } \frac{\partial u}{\partial x}(t, x(t)) = u_x(t, x(t)) \text{ exists and } u(t, x(t)) \text{ as } u_x(t, x(t)) \text{ are continuous in } x(t) \text{ and piecewise continuous in } t \Big\},$$

$$\mathcal{V} = \Big\{ v(\cdot,\cdot) \in \mathcal{L}^\infty(\mathcal{T} \times \mathcal{X}, \mathbb{R}^{m_2}), \text{ such } v(t, x(t)) \text{ is continuous in } x(t) \text{ and piecewise continuous in } t \Big\}.$$

The main difficulty in a closed-loop Stackelberg strategy is the presence of the partial derivative $\dfrac{\partial u^*}{\partial x}$ in the necessary conditions for the follower. Different alternatives, surveyed for example in [12], have been proposed in the literature to overcome the difficulty raised by the presence of the partial derivative $\dfrac{\partial u^*}{\partial x}$ in the necessary conditions for the follower. The first approach consists in finding an equivalent team problem leading to a global minimization of the leader's cost and obtaining a particular representation of the leader's control [11]. The second approach consists in determining the follower's rational reaction set and the necessary conditions for the leader optimizing a dynamical problem over an infinite dimensional strategy space subject to dynamical constraints (evolution of the state vector and follower's rational reaction set). In [46], this problem is handled using a variational method, which however does not lead to all solutions. In this paper, based on the Pontryagin minimum principle, we derive necessary conditions for a min-max and min-min Stackelberg strategies, in the sense discussed formerly. Our study permits to compute the values of the controls $u^*(t, x(t))$ and $v^*(t, x(t))$ *along the optimal trajectories.* We do not provide an expression of the Stackelberg controls $u^*(t, x)$ and $v^*(t, x)$ for every x, except in the linear-quadratic case (see Sec. 3.4) where our main result can be made more precise and more explicit. Finally, using the theory of focal points, we provide sufficient conditions for local optimality (which are global in the linear-quadratic case).

3. Necessary conditions for min-max and min-min Stackelberg strategies

Due to the hierarchy between the two players, necessary conditions are first established for the follower, and then for the leader.

3.1. **For the follower.** The best response set or rational reaction set T of the follower defined by Definition 2.1 does not depend on the choice of min-max or min-min Stackelberg strategy. The best response set of the follower Tu^* involved in the definition of Stackelberg strategy (2.4) or (2.5) (implies that, for a fixed control u^*, the control v^* of the follower solves the following optimization problem.

Problem 3.1.

$$\min_{v \in \mathcal{V}} J_2(u^*, v),$$

subject to

$$\dot{x}(t) = f\big(t, x(t), u^*(t, x(t)), v(t, x(t))\big), \quad x(0) = x_0. \tag{3.1}$$

Necessary conditions for the existence of an optimal solution of Problem 3.1 for the follower are derived in the next proposition, proved in Appendix.

Proposition 3.1. *Consider a closed-loop min-max Stackelberg pair of controls* (u^*, v^*) (*or respectively min-min Stackelberg pair of controls* (u^{**}, v^{**})) *for system* (3.1)*, associated with the trajectory* $x(\cdot)$*, then there exists an absolutely continuous mapping* $p_2 : [0, t_f] \to \mathbb{R}^n$*, being a non trivial line vector, such that*

$$\begin{aligned}
0 &= \frac{\partial H_2}{\partial v}\Big(t, x(t), u^*(t, x(t)), v(t, x(t))\Big) \\
&= p_2(t)\frac{\partial f}{\partial v}\Big(t, x(t), u^*(t, x(t)), v(t, x(t))\Big) \\
&\quad + \frac{\partial L_2}{\partial v}\Big(t, x(t), u^*(t, x(t)), v(t, x(t))\Big),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\dot{p}_2(t) &= -\frac{\mathrm{d}H_2}{\mathrm{d}x}\Big(t, x(t), u^*(t, x(t)), v(t, x(t)), p_2(t)\Big) \\
&= -p_2(t)\frac{\partial f}{\partial x}\Big(t, x(t), u^*(t, x(t)), v(t, x(t))\Big) \\
&\quad - \frac{\partial L_2}{\partial x}\Big(t, x(t), u^*(t, x(t)), v(t, x(t))\Big) \\
&\quad - p_2(t)\frac{\partial f}{\partial u}\Big(t, x(t), u^*(t, x(t)), v(t, x(t))\Big)\frac{\partial u^*}{\partial x}(t, x(t)) \\
&\quad - \frac{\partial L_2}{\partial u}\Big(t, x(t), u^*(t, x(t)), v(t, x(t))\Big)\frac{\partial u^*}{\partial x}(t, x(t)),
\end{aligned} \tag{3.3}$$

$$p_2(t_f) = \frac{\partial g_2(x(t_f))}{\partial x}, \tag{3.4}$$

where H_2 *denotes the Hamiltonian of the follower,*

$$H_2(t, x, u, v, p_2) = p_2 f(t, x, u, v) + L_2(t, x, u, v).$$

All solutions v *of Eqs.* (3.2)–(3.4) *are gathered in the set valued mapping* $T' : \mathcal{U} \to \mathcal{V}$.

Remark 3.1. Note that condition (3.3) may seem akin to open-loop control since it does not involve terms in $\dfrac{\partial v}{\partial x}$. This dependency comes from condition (3.2) that implies that the open-loop and closed-loop control for the follower coincide.

The set valued mapping T', which is defined by Eqs. (3.2)–(3.4) and satisfies $Tu \subseteq T'u$, will be used in the next subsection to derive necessary conditions for the leader, under a weak assumption on T, T' and the criterion J_1, as explained next.

3.2. **For the leader.** From the leader's point of view, unlike the follower's one, the optimization problems related to min-max and min-min Stackelberg strategies differ. We first consider the min-max Stackelberg strategy and we make the following assumption, needed to derive Proposition 3.2.

Assumption 1.

$$J_1(u, v') \leq J_1(u, v) \quad \forall v' \in T'u,\ v \in Tu,\ u \in \mathcal{U}^*_{nb}, \tag{3.5}$$

where $\mathcal{U}^*_{nb}$ denotes a neighborhood of u^* in $\mathcal{U}$.

Proposition 3.2. *Consider a pair of controls* (u^*, v^*) *associated with a min-max Stackelberg solution. The control* u^* *is defined by Eq.* (2.5)*, i.e.,*

$$u^* \in \arg\min_{u\in\mathcal{U}} \max_{v\in Tu} J_1(u, v). \tag{3.6}$$

Under Assumption 1, we have

$$u^* \in \arg\min_{u\in\mathcal{U}} \max_{v\in T'u} J_1(u, v). \tag{3.7}$$

This proposition is proved in the Appendix.

In the same way, let us now consider the min-min Stackelberg strategy and make the following assumption necessary to derive Proposition 3.3.

Assumption 2.

$$J_1(u, v') \geq J_1(u, v) \quad \forall v' \in T'u,\ v \in Tu,\ u \in \mathcal{U}^{**}_{nb}, \tag{3.8}$$

where $\mathcal{U}^{**}_{nb}$ denotes a neighborhood of u^{**} in $\mathcal{U}$.

Proposition 3.3. *Consider a pair of controls* (u^{**}, v^{**}) *associated with a min-min Stackelberg solution. The control* u^{**} *is defined by Eq.* (2.4)*, i.e.,*

$$u^{**} \in \arg\min_{u\in\mathcal{U}} \min_{v\in Tu} J_1(u, v). \tag{3.9}$$

Under Assumption 2, we have

$$u^{**} \in \arg\min_{u\in\mathcal{U}} \min_{v\in T'u} J_1(u, v). \tag{3.10}$$

Remark 3.2. Assumptions 1 and 2 devoted respectively to min-max and min-min Stackelberg strategies, are weaker than assuming that T is reduced to a singleton and differ only by their sign. It preserves the diversity between these both frameworks. Note thus that Assumption 2 has been already assumed in [46] to solve the min-min Stackelberg strategy.

Remark 3.3. Note that, under standard assumptions such as the convexity of J_2 [35, Chap. 5], or the fact that the set $T'u^*$ be reduced to a singleton (as in the linear-quadratic case, see Sec. 3.4), or the fact that $Tu = T'u$ (as assumed in [10, 46] for example), the necessary conditions for the follower are also sufficient (see [35, Chap. 5]) and Assumptions 1 and 2 are fulfilled. In the linear-quadratic case in particular, Assumptions 1 and 2 are automatically satisfied.

Propositions 3.2 and 3.3 stress out that in the constraints of the leader optimization problem for min-max and min-min Stackelberg strategies, the set Tu could be replaced by the set $T'u$, without loss of generality. Assume that Eq. (3.2) is solvable, but admits several (local) solutions. Then, assume that f and L_2 are $\mathcal{C}^2$ with respect to the variable v, that is $\dfrac{\partial H_2}{\partial v}$ is $\mathcal{C}^1$ with respect to v. If the strict Legendre condition holds at every (local) solution, i.e., $\dfrac{\partial^2 H_2}{\partial v^2}$ is positive definite, then it follows from the implicit-function theorem that, locally, every solution v can be written as

$$v(t,x) = \mathcal{S}(t, x(t), p_2(t), u^*(t, x(t))) \in T'u^*, \tag{3.11}$$

with $\mathcal{S}$ continuous with respect to t and $\mathcal{C}^1$ with respect to x and p_2. These solutions being isolated, the set of all these (local) solutions is discrete. Then, our main results apply for min-max (respectively, for min-min) Stackelberg strategies by selecting among the discrete set of solutions (3.11) the one maximizing (respectively minimizing) the criterion J_1 of the leader. We stress again that, in the linear-quadratic case (see Sec. 3.4), there exists a unique global solution. This allows to obtain the same solution for both min-max and min-min Stackelberg strategies.

The leader, with his top hierarchical position with respect to the follower, can impose the control of the follower. The leader knows the reaction of the follower, i.e., he knows the function $\mathcal{S}$. Then the leader seeks to minimize his own criterion where v is replaced by the function $\mathcal{S}$. Using the notation $\tilde{L}_1(t, x, p_2, u) = L_1(t, x, u, \mathcal{S}(t, x, p_2, u))$ and

$$\tilde{J}_1(u) = \int_0^{t_f} \tilde{L}_1(t, x(t), p_2(t), u(t, x(t)))\mathrm{d}t + g_1(x(t_f)), \tag{3.12}$$

the following problem is considered:

$$\min_{u \in \mathcal{U}} \tilde{J}_1(u) \tag{3.13}$$

under the following two dynamical constraints:

$$\begin{aligned}\dot{x}(t) &= f\Big(t,\ x(t),\ u(t,x(t)),\ \mathcal{S}\big(t,x(t),p_2(t),u(t,x(t))\big)\Big),\\ &= F_1\Big(t,\ x(t),\ p_2(t),\ u(t,x(t))\Big),\end{aligned} \tag{3.14}$$

$$\begin{aligned}\dot{p}_2(t) &= -p_2(t)\frac{\partial f}{\partial x}\Big(t,\ x(t),\ u(t,x(t)),\ \mathcal{S}\big(t,x(t),p_2(t),u(t,x(t))\big)\Big)\\ &\quad -\frac{\partial L_2}{\partial x}\Big(t,\ x(t),\ u(t,x(t)),\ \mathcal{S}\big(t,x(t),p_2(t),u(t,x(t))\big)\Big)\\ &\quad - p_2\frac{\partial f}{\partial u}\Big(t,\ x(t),\ u(t,x(t)),\ \mathcal{S}\big(t,x(t),p_2(t),u(t,x(t))\big)\Big)\frac{\partial u}{\partial x}(t,x(t))\\ &\quad -\frac{\partial L_2}{\partial u}\Big(t,\ x(t),\ u(t,x(t)),\ \mathcal{S}\big(t,x(t),p_2(t),u(t,x(t))\big)\Big)\frac{\partial u}{\partial x}(t,x(t))\\ &= F_{21}\Big(t,x(t),p_2(t),u(t,x(t))\Big)\\ &\quad + F_{22}\Big(t,x(t),p_2(t),u(t,x(t))\Big)\frac{\partial u}{\partial x}(t,x(t)),\end{aligned} \tag{3.15}$$

and

$$x(0)=x_0, \quad p_2(t_f)=\frac{\partial g_2}{\partial x}(x(t_f)).$$

Denote

$$\begin{aligned}\tilde{L}_2(t,x,p_2,u) &= L_2(t,x,u,\mathcal{S}(t,x,p_2,u)),\\ F_{21}(t,x,p_2,u) &= -p_2\frac{\partial F_1}{\partial x}(t,x,p_2,u)-\frac{\partial \tilde{L}_2}{\partial x}(t,x,p_2,u),\\ F_{22}(t,x,p_2,u) &= -p_2\frac{\partial F_1}{\partial u}(t,x,p_2,u)-\frac{\partial \tilde{L}_2}{\partial u}(t,x,p_2,u).\end{aligned}$$

Due to the nonclassical term u_x, the usual Pontryagin minimum principle (see [35]) cannot be applied. However, it is possible to adapt its proof and derive a version of the Pontryagin minimum principle adapted to the system (3.14)–(3.15) (see the Appendix). The following proposition is proved in the Appendix.

Proposition 3.4. *If the trajectory $x(\cdot)$ associated with the pair (u^*, v^*) of closed-loop Stackelberg controls is a solution of the Stackelberg problem, then there exist absolutely continuous mappings $\lambda_1, \lambda_2 : [0, t_f] \to \mathbb{R}^n$, called costate vectors (written as line vectors by convention), and a scalar $\lambda^0 \geq 0$, such that*

$$0 = \lambda_2(t)\left(\frac{\partial F_{22}}{\partial u}\Big(t,x(t),p_2(t),u(t,x(t))\Big)u_x + \frac{\partial F_{21}}{\partial u}\Big(t,x(t),p_2(t),u(t,x(t))\Big)\right)^T$$

$$
\begin{aligned}
&+ \lambda_1(t)\frac{\partial F_1}{\partial u}\Big(t, x(t), p_2(t), u(t, x(t))\Big) \\
&+ \lambda^\circ \frac{\partial \tilde{L}_1}{\partial u}\Big(t, x(t), p_2(t), u(t, x(t))\Big), \qquad (3.16)\\
0 &= \lambda_2^T(t) F_{22}\Big(t, x(t), p_2(t), u(t, x(t))\Big) \\
&= \lambda_2^T(t)\Bigg(p_2(t)\frac{\partial F_1}{\partial u}\Big(t, x(t), p_2(t), u(t, x(t))\Big) \\
&+ \frac{\partial \tilde{L}_2}{\partial u}\Big(t, x(t), p_2(t), u(t, x(t))\Big)\Bigg), \qquad (3.17)\\
\dot{\lambda}_1(t) &= -\lambda_2(t)\Bigg(\frac{\partial F_{21}}{\partial x}\Big(t, x(t), p_2(t), u(t, x(t))\Big) \\
&+ \frac{\partial F_{22}}{\partial x}\Big(t, x(t), p_2(t), u(t, x(t))\Big)u_x\Bigg)^T \\
&- \lambda_1(t)\frac{\partial F_1}{\partial x}\Big(t, x(t), p_2(t), u(t, x(t))\Big) \\
&- \lambda^\circ \frac{\partial \tilde{L}_1}{\partial x}\Big(t, x(t), p_2(t), u(t, x(t))\Big), \qquad (3.18)\\
\dot{\lambda}_2(t) &= -\lambda_2(t)\Bigg(\frac{\partial F_{21}}{\partial p_2}\Big(t, x(t), p_2(t), u(t, x(t))\Big) \\
&+ \frac{\partial F_{22}}{\partial p_2}\Big(t, x(t), p_2(t), u(t, x(t))\Big)u_x\Bigg)^T \\
&- \lambda_1(t)\frac{\partial F_1}{\partial p_2}\Big(t, x(t), p_2(t), u(t, x(t))\Big) \qquad (3.19)\\
&- \lambda^\circ \frac{\partial \tilde{L}_1}{\partial p_2}\Big(t, x(t), p_2(t), u(t, x(t))\Big), \qquad (3.20)
\end{aligned}
$$

for almost every $t \in [0, t_f]$. Moreover, the following relations, called the transversality conditions, hold:

$$
\lambda_2(0) = 0, \qquad \lambda_1(t_f) - \lambda^\circ \frac{\partial g_1}{\partial x}(x(t_f)) + \lambda_2(t_f)\frac{\partial^2 g_2}{\partial x^2}(x(t_f)) = 0. \qquad (3.21)
$$

3.3. **Degeneration property.** Equation (3.17) implies either that $\lambda_2 \equiv 0$ or $F_{22} \equiv 0$ (or both) along the interval $[0, t_f]$, where $F_{22} = -p_2\frac{\partial F_1}{\partial u} - \frac{\partial \tilde{L}_2}{\partial u}$. In the general case, the relation $F_{22} \equiv 0$ is not obvious to analyze, however we will see that, in the linear-quadratic case, this relation does not hold under a weak additional assumption of the Kalman type (see Proposition 3.7

below). In the general nonlinear case, based on the genericity strategies developed in [5,22–24], we conjecture that the relation $F_{22} = 0$ does not hold generically. The strategy would consist in deriving an infinite number of times the latter relation, to infer an infinite number of independent relations, and to use Thom's transversality theorem. However, it is not obvious to turn this fact into a proper theorem and we let this question open in the general nonlinear case. The next proposition, proved in the Appendix, investigates the first case of that alternative, which is, in some sense, the generic one.

Proposition 3.5. *Under the additional assumption that the* $(m_1 \times m_1)$-*matrix*

$$\frac{\partial}{\partial u}\left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u}\right)^T \tag{3.22}$$

is invertible, we have

$$\lambda_2 \equiv 0. \tag{3.23}$$

Remark 3.4. The fact that $\lambda_2 \equiv 0$ means that the leader does not take into account the rational reaction set of the follower. It is actually not in contradiction with the hierarchical position between the leader and the follower; indeed, in this case the leader does not take into account the reaction of the follower, because he can impose his desired control to the follower. The leader is *omnipotent* with respect to the follower. The condition $\frac{\partial F_{22}}{\partial u}$ invertible formalizes this privileged position of the leader.

Proposition 3.5, under a weak assumption, emphasizes the omnipotence of the leader leading to a degeneration of the min-max and min-min Stackelberg strategies. The hierarchical roles of the players seem to disappear. An omnipotent leader is able to impose his/her control to the other player without taking into account the rational reaction set of the follower.

These conditions happen to be more explicit in the linear-quadratic case. In the next paragraph we focus on that case, and analyze more deeply the former necessary conditions. Our analysis finally leads to a more precise result on the Stackelberg controls in the linear-quadratic case.

3.4. **Linear-quadratic case.** In this section, we focus on the linear-quadratic case, due to its widespread presence in the literature [20], and reformulate and make more explicit our previous results. Consider a linear dynamic constraint

$$\dot{x} = Ax + B_1 u + B_2 v \tag{3.24}$$

and the quadratic criteria

$$J_1(u,v) = \int_0^{t_f} \frac{1}{2}\Big(x^T Q_1 x + u^T R_{11} u + v^T R_{12} v\Big)\,\mathrm{d}t$$

$$+ \frac{1}{2} x(t_f)^T K_{1f} x(t_f), \tag{3.25}$$

$$J_2(u,v) = \int_0^{t_f} \frac{1}{2}\Big(x^T Q_2 x + u^T R_{21} u + v^T R_{22} v \Big) \, \mathrm{d}t$$
$$+ \frac{1}{2} x(t_f)^T K_{2f} x(t_f), \tag{3.26}$$

where the matrices Q_i, R_{ij}, and K_{if} are symmetric for $i, j \in \mathcal{K}$, and $Q_i \geq 0$, $R_{ii} > 0$, $R_{12} > 0$, and R_{21} invertible. In what follows, denote

$$S_{ij} = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T, \quad S_i = B_i R_{ii}^{-1} B_i^T$$

for $i, j \in \mathcal{K}$.

3.4.1. *Necessary conditions for the follower.* The Hamiltonian associated with the follower (dynamic constraint (3.24) and criterion (3.26)) is

$$H_2(t, x, p_2, u, v) = p_2(Ax + B_1 u + B_2 v)$$
$$+ \frac{1}{2}(x^T Q_2 x + u^T R_{21} u + v^T R_{22} v). \tag{3.27}$$

Applying the relations (3.2) and (3.3), we obtain

$$\dot{p}_2(t) = -\frac{\mathrm{d}H_2}{\mathrm{d}x}\Big(t, x(t), p_2(t), u(t, x(t))\Big) \tag{3.28}$$
$$= -p_2(t)A - x^T(t) Q_2 - p_2(t) B_1 \frac{\partial u^*}{\partial x}(t, x(t))$$
$$- u^T(t, x(t)) R_{21} \frac{\partial u^*}{\partial x}(t, x(t)),$$
$$p_2(t_f) = x(t_f)^T K_{2f}, \tag{3.29}$$
$$\frac{\partial H_2}{\partial v} = 0 = p_2(t) B_2 + v^T(t, x(t)) R_{22}. \tag{3.30}$$

Since R_{22} is invertible by assumption, the optimal control is

$$v(t, x(t)) = -R_{22}^{-1} B_2^T p_2^T(t) = \mathcal{S}(t, x(t), p_2(t), u(t, x(t))). \tag{3.31}$$

3.4.2. *Necessary conditions for the leader.* In the case of quadratic criteria, we have

$$F_1(t, x, p_2, u) = Ax + B_1 u - S_2 p_2^T, \tag{3.32}$$
$$F_{21}(t, x, p_2, u) = -p_2 A - x^T Q_2, \tag{3.33}$$
$$F_{22}(t, x, p_2, u) = -p_2 B_1 - u^T R_{21}. \tag{3.34}$$

Using the expression of the optimal control of the follower (3.31), the instantaneous leader's criterion can be written as

$$\tilde{L}_1(t, x, p_2, u) = \frac{1}{2}\left(x^T Q_1 x + u^T R_{11} u + p_2 S_{12} p_2 \right).$$

The necessary conditions (3.16), (3.17), (3.18), and (3.20) lead to

$$\frac{\partial H}{\partial u} = 0 = \lambda_1(t)B_1 - \lambda_2(t)\left(\frac{\partial u}{\partial x}(t,x(t))\right)^T R_{21} + \lambda^\circ u^T(t,x(t))R_{11}, \tag{3.35}$$

$$\frac{\partial H}{\partial u_y} = 0 = -\lambda_2^T(t)\left(p_2(t)B_1 + u^T(t,x(t))R_{21}\right), \tag{3.36}$$

$$\dot{\lambda}_1(t) = -\lambda_1(t)A + \lambda_2(t)Q_2 - \lambda^\circ x^T(t)Q_1, \tag{3.37}$$

$$\dot{\lambda}_2(t) = \lambda_1(t)S_2 + \lambda_2(t)\left(A + B_1\left(\frac{\partial u}{\partial x}(t,x(t))\right)\right)^T - \lambda^\circ p_2(t)S_{12} \tag{3.38}$$

with the transversality conditions

$$\lambda_1(t_f) = \lambda^\circ x(t_f)^T K_{1f} - \lambda_2(t_f)K_{2f}, \qquad \lambda_2(0) = 0. \tag{3.39}$$

From (3.36), as discussed in Sec. 3.3, either $\lambda_2 \equiv 0$ or

$$p_2(t)B_1 + u^T(t,x(t))R_{21} \equiv 0$$

or both along the interval $[0,t_f]$. Without *a priori* consideration about $p_2(t)B_1 + u^T(t,x(t))R_{21}$, by assuming that

$$\frac{\partial}{\partial u}\left(p_2\frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u}\right) = R_{21}$$

is invertible and by Proposition 3.5, we can deduce that $\lambda_2 \equiv 0$.

We next prove by contradiction that $\lambda^0 \neq 0$. If λ^0 were equal to 0, then we would infer from (3.37)–(3.39) that λ_1, like λ_2 is identically equal to zero by Cauchy uniqueness; thus, $(\lambda_1, \lambda_2, \lambda^0)$ is trivial, and this is a contradiction with the Pontryagin minimum principle. From now on, we normalize the adjoint vector so that $\lambda^\circ = 1$.

From (3.35), we deduce with the invertibility of R_{11}, that

$$u(t,x(t)) = -R_{11}^{-1}B_1^T\lambda_1^T(t). \tag{3.40}$$

Moreover, Eq. (3.38) becomes, with $\lambda_2 \equiv 0$ along the interval $[0,t_f]$,

$$\lambda_1(t)S_2 - p_2(t)S_{12} \equiv 0.$$

Assuming that the rank of B_2 is maximal, that is, rank $B_2 = m_2$ (the number of the components of the control v), this relation yields

$$\lambda_1(t)B_2 = p_2(t)B_2R_{22}^{-1}R_{12}. \tag{3.41}$$

Substitution of v from (3.31) into (3.41) gives $R_{12}v(t,x(t)) = -B_2^T\lambda_1^T(t)$. If R_{12} is invertible, then the control v admits two expressions:

$$v(t,x(t)) = -R_{12}^{-1}B_2^T\lambda_1^T(t) = -R_{22}^{-1}B_2^Tp_2^T(t). \tag{3.42}$$

We gather the previous necessary conditions for optimality in the following proposition.

Proposition 3.6. *For $x_0 \neq 0$, if the matrices Q_i, R_{ij}, and K_{if} are symmetric, if $R_{11} > 0$, $R_{22} > 0$, $R_{12} > 0$, and R_{21} invertible and if rank $B_2 = m_2$ (B_2 is of full rank), then the controls issued from a min-max or min-min Stackelberg strategy with a closed-loop information structure are*

$$u(t, x(t)) = -R_{11}^{-1} B_1^T \lambda_1^T(t), \tag{3.43}$$
$$v(t, x(t)) = -R_{22}^{-1} B_2^T p_2^T(t) = -R_{12}^{-1} B_2^T \lambda_1^T(t), \tag{3.44}$$

with

$$\dot{x}(t) = Ax(t) + B_1 u(t, x(t)) + B_2 v(t, x(t)), \quad x(0) = x_0, \tag{3.45}$$
$$\dot{p}_2(t) = -p_2(t)A - x^T(t)Q_2 - \left(p_2(t)B_1 + u^T(t, x(t))R_{21}\right) \frac{\partial u}{\partial x}(t, x(t)), \tag{3.46}$$
$$\dot{\lambda}_1(t) = -\lambda_1(t)A - x^T(t)Q_1, \quad \lambda_1(t_f) = x_f^T K_{1f}, \quad p_2(t_f) = x_f^T K_{2f}, \tag{3.47}$$
$$\lambda_1(t)B_2 = p_2(t)B_2 R_{22}^{-1} R_{12}. \tag{3.48}$$

Remark 3.5. As will be justified below by Proposition 3.7, the case $x_0 = 0$ leads only to the trivial solution, which has only few interest. Thus in the sequel of the paper, x_0 is always considered non trivial to avoid the trivial optimal trajectory.

From (3.36), even if $\lambda_2 \equiv 0$, two cases must be yet considered to precise necessary conditions: either $p_2 B_1 + u^T R_{21} \equiv 0$ or $p_2 B_1 + u^T R_{21} \not\equiv 0$. We next prove that the first case is irrelevant under some additional weak assumptions.

3.4.3. *Case $p_2 B_1 + u^T R_{21} \equiv 0$.*

Proposition 3.7. *If the pair (A^T, Q_1) and one of the pairs (A, B_1) or (A, B_2) satisfy the Kalman condition, then*

$$x(t) \equiv \lambda_1^T(t) \equiv p_2^T(t) \equiv 0 \quad \forall t \in [0, t_f]. \tag{3.49}$$

This means that there exists a unique optimal trajectory, which is trivial.

The proof of Proposition 3.7 (see the Appendix) relies on the following lemma concerning the Kalman condition (also proved in the Appendix).

Lemma 3.1. *Assuming that the pair $\left(A^T, Q_1\right)$ and one of the pairs (A, B_1) or (A, B_2) satisfy the Kalman condition, then the pair $(\mathcal{A}, \mathcal{B})$ satisfies also the Kalman condition, where*

$$\mathcal{A} = \begin{bmatrix} A^T & -Q_1 & -Q_2 \\ -S_1 - B_2 R_{12}^{-1} B_2^T & -A & 0 \\ 0 & 0 & -A \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 0 & 0 \\ B_2 & B_1 R_{11}^{-1} R_{21} \\ -B_2 R_{22}^{-1} R_{12} & -B_1 \end{bmatrix}.$$

Remark 3.6. This means that the particular case $p_2 B_1 + u^T R_{21} \equiv 0$ can be discarded under weak assumptions on the system. The leader should be able to observe the system (pair (Q_1, A) observable) and at least one player should be able to control the system ((A, B_1) or (A, B_2) controllable). Once again, it is emphasized that the roles of the players are not symmetric.

3.4.4. *Case* $p_2 B_1 + u^T R_{21} \not\equiv 0$. Relation (3.48) is equivalent to the following two relations:

$$\lambda_1(t_f) B_2 = x^T(t_f) K_{1f} B_2 = p_2(t_f) B_2 R_{22}^{-1} R_{12} = x^T(t_f) K_{2f} B_2 R_{22}^{-1} R_{12}, \tag{3.50}$$

and

$$\begin{aligned} \dot\lambda_1(t) B_2 = \dot p_2(t) B_2 R_{22}^{-1} R_{12} &= -\left(\lambda_1(t) A + x^T(t) Q_1\right) B_2 \\ &= -\left(p_2(t) A + x^T(t) Q_2\right) B_2 R_{22}^{-1} R_{12} \\ &\quad - (p_2(t) B_1 + u^T(t, x(t)) R_{21}) \left(\frac{\partial u}{\partial x}(t, x(t))\right) B_2 R_{22}^{-1} R_{12}. \end{aligned}$$

Hence along the interval $\mathcal{T}$

$$\begin{aligned} &(p_2(t) B_1 + u^T(t, x(t)) R_{21}) \left(\frac{\partial u}{\partial x}(t, x(t))\right) B_2 \\ &\quad = \left(\lambda_1(t) A + x^T(t) Q_1\right) B_2 R_{12}^{-1} R_{22} - (p_2(t) A + x^T(t) Q_2) B_2. \end{aligned} \tag{3.51}$$

Therefore, (3.48) is equivalent to

$$\begin{cases} \left(B_2^T K_{1f} - R_{12} R_{22}^{-1} B_2^T K_{2f}\right) x(t_f) = 0, \\ \left(p_2(t) B_1 + u^T(t, x(t)) R_{21}\right) \left(\dfrac{\partial u}{\partial x}(t, x(t))\right) B_2 \\ \quad \equiv \left(\lambda_1(t) A + x^T(t) Q_1\right) B_2 R_{12}^{-1} R_{22} - (p_2(t) A + x^T(t) Q_2) B_2. \end{cases} \tag{3.52}$$

Equation (3.51) permits to derive an expression of $\dfrac{\partial u}{\partial x}$, since $p_2 B_1 + u^T R_{21} \not\equiv 0$

$$(p_2 B_1 + u^T R_{21}) \left(\frac{\partial u}{\partial x}\right) \equiv w_2 + w_2', \tag{3.53}$$

with

$$\begin{aligned} w_2 \equiv \Big(&\left(\lambda_1(t) A + x^T(t) Q_1\right) B_2 R_{12}^{-1} R_{22} \\ &- (p_2(t) A + x^T(t) Q_2) B_2 \Big) (B_2^T B_2)^{-1} B_2^T, \end{aligned} \tag{3.54}$$

and $(w_2')^T \in \operatorname{Ker}\left(B_2^T\right)$ (arbitrary).

Constraint (3.50) translates into a constraint on the set of initial points $x_0 \in \mathbb{R}^n$ from which a solution starts.

Lemma 3.2. *The optimal solutions must emanate from initial conditions x_0 lying in a subspace of $\mathbb{R}^n$ of codimension m_2 (at most).*

Note that given a starting point x_0 lying in the subspace of Lemma 3.2, there exists a unique trajectory starting from x_0, but it is achieved by all controls which satisfy relation (3.51). An optimal trajectory induces several possible $\dfrac{\partial u}{\partial x}$.

This fact appears in [41] where it is assumed that the initial state is uniformly distributed over the unit sphere and replacing the optimization criterion with its mean value over the initial state.

Remark 3.7. In the case of an optimization problem without terminal criteria, relation (3.50) does not reduce the set of initial states x_0 associated with optimal trajectories.

We gather all previous results in the following theorem.

Theorem 3.1. *Assume that the following assumptions hold*:

- $x_0 \neq 0$,
- Q_i, R_{ij}, *and* K_{if} *are symmetric, with* $Q_i \geq 0$,
- $R_{11} > 0$, $R_{22} > 0$, $R_{12} > 0$, *and* R_{21} *is invertible,*
- *the pair* (A^T, Q_1) *and one of the pairs* (A, B_1) *or* (A, B_2) *satisfy the Kalman condition,*
- *rank* $B_2 = m_2$ *(*B_2 *of full rank).*

Then the optimal trajectory satisfies the necessary conditions

$$u(t, x(t)) = -R_{11}^{-1} B_1^T K_1(t) x(t), \quad v(t, x(t)) = -R_{12}^{-1} B_2^T K_1(t) x(t), \tag{3.55}$$

where

$$\dot{x}(t) = \big(A - (B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T) K_1(t)\big) x(t), \quad x(0) = x_0, \tag{3.56}$$

where K_1 is the unique solution of the matrix differential equation

$$\begin{aligned} \dot{K}_1(t) &= -K_1(t)A - A^T K_1(t) - Q_1 \\ &\quad + K_1(t)\Big(B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T\Big) K_1(t), \\ K_1(t_f) &= K_{1f}. \end{aligned} \tag{3.57}$$

Furthermore, $\dfrac{\partial u(t, x(t))}{\partial x}$ *satisfies along the interval* $[0, t_f]$

$$\begin{aligned} &(p_2(t) B_1 + u^T(t, x(t) R_{21}) \frac{\partial u}{\partial x}(t, x(t)) B_2 \\ &\qquad = (\lambda_1(t) A + x^T(t) Q_1) B_2 R_{12}^{-1} R_{22} - (p_2(t) A + x^T(t) Q_2) B_2, \end{aligned} \tag{3.58}$$

where

$$\dot{p}_2(t) = -p_2(t)A - x^T(t)Q_2 - \Big(p_2(t)B_1 + u^T(t,x(t))R_{21}\Big)\frac{\partial u}{\partial x}(t,x(t)), \quad (3.59)$$

and

$$p_2(t_f) = x^T(t_f)K_{2f}, \quad (B_2^T K_{1f} - R_{12}R_{22}^{-1}B_2^T K_{2f})x(t_f) = 0.$$

Theorem 3.1 provides rigorous necessary conditions for closed-loop min-max and min-min Stackelberg solutions of generic linear-quadratic games. Up to now, this problem has remained open and was only partially solved in particular cases in [46]. It should be stressed again that the trajectory associated with closed-loop Stackelberg solution is unique, nevertheless it induces several possible u_x, which satisfy Eqs. (3.58)–(3.59) and are completely characterized by 3.53. This degree of freedom in the choice of u_x, leading to the same trajectory, requires an additional objective, e.g., arguments related to the robustness or the sensitivity of the min-max or min-min Stackelberg solution.

4. Sufficient conditions

In this section, using elements of focal point theory, we derive sufficient optimality conditions, first for the leader, and then for the follower in the case of linear-quadratic games.

4.1. Preliminary comments, focal times.

The optimization problem of the leader is $\min_u \hat{J}_1(u)$, where

$$\begin{cases} \hat{J}_1(u) = \dfrac{1}{2}\displaystyle\int_0^{t_f} \Big(x^T(t)Q_1x(t) + u^T(t,x(t))R_{11}u(t,x(t)) \\ \qquad\qquad + p_2(t)S_{12}p_2^T(t)\Big)\mathrm{d}t + \dfrac{1}{2}x^T(t_f)K_{1f}x(t_f), \\ \dot{x}(t) = Ax(t) - S_2p_2^T(t) + B_1u(t,x(t)), \\ \dot{p}_2^T(t) = -A^Tp_2^T(t) - Q_2x(t) - w^T\Big(p_2(t)B_1 + u^T(t,x(t))R_{21}\Big)^T, \end{cases} \quad (4.1)$$

with $x(0) = x_0$ and $p_2(t_f) = x^T(t_f)K_{1f}$. When $p_2(t)B_1 + u^T(t,x(t))R_{21} \not\equiv 0$, the control w is cheap (see [15] for the concept of cheap control), since it only appears in the dynamics of p_2, and nowhere else (it does also not appear in the cost); then, we rather consider p_2 as a control. Note that this is a particular case of the so-called Goh transformation (see [6, 7, 15, 53] for the definition and properties of the Goh transformation, related to singular trajectories or abnormal extremals). Actually, in what follows we consider $\xi = B_2^T p_2^T$ as a control. Then problem (4.1) can be rewritten as

$\min_{(u,\xi)} \hat{J}_1(u,\xi)$, where

$$\begin{cases} \dot{x}(t) = Ax(t) - B_2R_{22}^{-1}\xi(t) + B_1u(t,x(t)), \\ \hat{J}_1(u,\xi) = \dfrac{1}{2}x^T(t_f)K_{1f}x(t_f) + \dfrac{1}{2}\displaystyle\int_0^{t_f} \Big(x^T(t)Q_1x(t) \\ \qquad\qquad + u^T(t,x(t))R_{11}u(t,x(t)) + \xi^T(t)R_{22}^{-1}R_{12}R_{22}^{-1}\xi(t)\Big)\mathrm{d}t. \end{cases} \tag{4.2}$$

Remark 4.1. Note that this linear-quadratic problem with controls (u, ξ) is related to the Team optimal approach in [11]. In this reference, the first step in the research of min-max or min-min Stackelberg strategy is to obtain the minimum of the criterion of the leader, by a team cooperation between the leader and the follower. Then the follower control is modified to achieve the minimum of the criterion of the follower.

A necessary condition for the existence of an optimal control of problem (4.2) is

$$R_{22}^{-1}R_{12}R_{22}^{-1} \geq 0.$$

It is equivalent to $R_{12} \geq 0$, since R_{22} is positive definite. When t_f is small, $R_{12} > 0$ is a sufficient condition for the existence of an optimal control (see, e.g., [19, 35, 54]). In the following, it is assumed that $R_{12} > 0$.

Under this assumption, the optimal controls u and ξ are given by

$$u(t,x(t)) = -R_{11}^{-1}B_1^T\lambda_1^T(t), \quad \xi(t) = R_{22}R_{12}^{-1}B_2^T\lambda_1^T(t). \tag{4.3}$$

Recall that, in order to characterize focal points (for a definition and properties of focal points we refer the reader to [6, 15, 16]), we consider the variational system

$$\begin{cases} \delta\dot{x}(t) = A\delta x(t) - \Big(B_1R_{11}^{-1}B_1^T + B_2R_{12}^{-1}B_2^T\Big)\delta\lambda_1^T(t), \\ \delta\dot{\lambda}_1(t) = -\delta\lambda_1(t)A - \delta x^T(t)Q_1. \end{cases} \tag{4.4}$$

By definition, the first focal time $t_c > 0$ along the trajectory $x(t)$ associated with the controls (u, ξ) is the first positive time at which there exists a solution $(\delta x, \delta\lambda_1)$ satisfying (recall that $x(0) = x_0$ is fixed)

$$\begin{cases} \delta x(0) = 0, \\ \delta\lambda_1(t_c) = \delta x^T(t_c)K_{1f}. \end{cases} \tag{4.5}$$

It is well known that this condition is equivalent to

$$\|K(t)\| \xrightarrow{t\to t_c,\ t<t_c} +\infty, \tag{4.6}$$

where $K(t)$ is the solution of the Riccati differential equation

$$\begin{cases} \dot{K}(t) = K(t)A + A^T K(t) + Q_1 \\ \qquad\qquad - K(t)\Big(B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T\Big) K(t), \\ K(0) = K_{1f}. \end{cases} \tag{4.7}$$

The first focal time t_c is a finite escape time for the Riccati differential equation (4.7). Note that $K(t) = K_1(t_f - t)$, where $K_1(t)$ is defined by (3.57). Rigorously, since the first focal time is defined by an infimum, its existence must be proved (see the following lemma).

Lemma 4.1 (see [6, 15, 16])**.** *If $R_{11} > 0$ and $R_{12} > 0$, then the first focal time t_c is well defined, and t_c is either a positive real number, or is equal to $+\infty$.*

Remark 4.2. If $Q_1 \geq 0$, then Eq. (4.7) admits a solution on $[0, +\infty[$. There is no finite escape time for this equation. Thus, the first focal time is infinite ($t_c = +\infty$) (see [2, Corollary 3.6.7 and Example 3.6.8]).

The optimization problem for the follower is $\min \hat{J}_2$, where

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 u(t, x(t)) + B_2 v(t, x(t)), \quad x(0) = x_0, \\ \hat{J}_2 = \dfrac{1}{2} x^T(t_f) K_{2f} x(t_f) + \dfrac{1}{2} \displaystyle\int_0^{t_f} \Big(x^T(t) Q_2 x(t) \\ \qquad + u^T(t, x(t)) R_{21} u(t, x(t)) + v^T(t, x(t)) R_{22} v(t, x(t))\Big) \mathrm{d}t \end{cases} \tag{4.8}$$

with

$$v(t, x(t)) = -R_{22}^{-1} B_2^T p_2^T(t),$$

where $p_2(t_f) = x^T(t_f) K_{2f}$ and

$$\dot{p}_2(t) = -p_2(t)A - x^T(t) Q_2 - \Big(p_2(t) B_1 + u^T(t, x(t)) R_{21}\Big) \frac{\partial u}{\partial x}(t, x(t)). \tag{4.9}$$

The variational system along the trajectory $x(\cdot)$ is

$$\delta\dot{x}(t) = A\delta x(t) + B_1 \frac{\partial u}{\partial x} \delta x - S_2 p_2^T(t), \tag{4.10}$$

$$\begin{aligned} \delta\dot{p}_2(t) &= -\delta p_2(t) A - \delta x^T(t) Q_2 \\ &\quad - \big(p_2(t) B_1 + u^T(t, x(t)) R_{21}\big) \frac{\partial^2 u}{\partial x^2}(t, x(t)) \delta x \\ &\quad - \left(\delta p_2(t) B_1 + \left(\frac{\partial u}{\partial x}(t, x(t)) \delta x(t)\right)^T R_{21}\right) \frac{\partial u}{\partial x}(t, x(t)). \end{aligned} \tag{4.11}$$

Here, due to the freedom in the choice of w_2' to obtain u_x by relation (3.53), we choose $u(t,x(t))$ affine with respect to $x(t)$, thus $\dfrac{\partial^2 u}{\partial x^2} = 0$. Equation (4.11) takes the form

$$\delta \dot{p}_2(t) = -\delta p_2(t)A - \delta x^T(t)Q_2 - \left(\delta p_2(t)B_1 + \left(\frac{\partial u}{\partial x}(t,x(t))\delta x(t)\right)^T R_{21}\right)\frac{\partial u}{\partial x}(t,x(t)). \quad (4.12)$$

By definition, the first focal time t_c' along the trajectory $x(t)$ associated with the control v is the first time at which there exists a solution $(\delta x, \delta p_2)$ of (4.10)–(4.11) such that $\delta x(0) = 0$ and $\delta p_2(t_c') = \delta x^T(t_c')K_{2f}$. For each choice of admissible term u_x (that is choice of w_2') satisfying relation (3.58), there exists a first focal time t_c'.

4.2. **Sufficient conditions for min-max and min-min Stackelberg strategies.** We gather the previous remarks in the following result.

Theorem 4.1. *Under the assumptions of Theorem* 3.1*, let w_2' be a function of time t such that $w_2' \in (B_2^T)^\perp$. This choice of w_2' leads to design the Jacobian u_x satisfying* (3.53):

$$(p_2 B_1 + u^T R_{21})u_x = w_2 + w_2'.$$

Let

$$T^* = \min(t_c, t_c') > 0, \quad (4.13)$$

where t_c is the first focal time of the Riccati differential equation (3.57) *and t_c' is the first focal time of system* (4.10)*–*(4.12)*, induced by u_x, that is by w_2'.*

For every $t_f < T^*$, there exists a unique solution of the Riccati differential equation (3.57)*. Denoting $x(t,x_0)$ the obtained trajectory, let*

$$\mathcal{H} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \left(B_2^T K_{1f} - R_{12}R_{22}^{-1}B_2^T K_{2f}\right)x(t_f,x_0) = 0 \right\}. \quad (4.14)$$

Then, for every $x_0 \in \mathcal{H}$, there exists a unique optimal solution of the optimization problem (4.8) *on $[0,t_f]$ associated with w_2'. The optimal controls (u^*, v^*) associated with this unique optimal trajectory satisfy $(p_2 B_1 + u^T R_{21})u_x = w_2 + w_2'$ and, furthermore,*

$$u(t,x(t)) = -R_{11}^{-1}B_1^T K_1(t)x(t), \qquad v(t,x(t)) = -R_{12}^{-1}B_2^T K_1(t)x(t). \quad (4.15)$$

In addition, for every $x_0 \notin \mathcal{H}$, there exists no optimal trajectory starting from x_0.

Remark 4.3. Theorem 4.1 is a result of existence of closed-loop min-max and min-min Stackelberg strategies for linear-quadratic differential games, which is new, to the best of our knowledge.

Remark 4.4. The sufficient conditions for optimality are developed in the linear-quadratic case, and are global in that case. It is also, by the same argument, possible to express similar sufficient conditions in the general case of nonlinear criteria. However, they are not developed here, because their expressions are more technical and because they lead only to local optimality results (see [6] or [16, Chap. 9]).

Remark 4.5. The assumption $R_{12} > 0$ is required to derive Theorem 4.1. This assumption is used in a crucial way in order to derive Lemma 4.1 (more precisely, to derive inequality (A.53)). It is natural to make such an assumption when inspecting the minimization criterion $\hat{J}_1(u, \xi)$ defined by (4.2): indeed, as explained few lines above (4.2), the problem degenerates into a cheap control problem. In this sense, $\xi = B_2^T p_2^T$ may then be considered as a control, and therefore it is clear that one has to assume that $R_{12} > 0$ in order to ensure nice coercivity properties for the quadratic criterion $\hat{J}_1(u, \xi)$.

Remark 4.6. From Remark 4.2, the assumption $Q_1 \geq 0$ ensures that $t_c = +\infty$. A lower bound of $T^* = \min(t_c, t_c')$ corresponds to a lower bound of t_c' the first focal time of the nonlinear variational system (4.10)–(4.12). It depends implicitly on the choice of u_x, that is the choice of w_2'. The problem of determining a lower bound of t_c' is open.

4.3. **Extension: weighting of u_x in criteria.** The problem is degenerate since for each $x_0 \in \mathcal{H}$, there may exist an infinite choice of terms $\dfrac{\partial u^*}{\partial x}$. A way to yield a unique $\dfrac{\partial u^*}{\partial x}$ is to include a weight on the term $\dfrac{\partial u^*}{\partial x}$ in the criterion J_1 of the leader, as in [46]. Then the leader takes into account a restriction on the Jacobian of its control. The leader is no more omnipotent.

The new criterion of the leader is then

$$
\begin{aligned}
J_1(u, v) = \frac{1}{2} x(t_f)^T K_{1f} x(t_f) + \frac{1}{2} \int_0^{t_f} \Bigg[& x^T(t) Q_1 x(t) \\
& + u^T(t, x(t)) R_{11} u(t, x(t)) + v^T(t, x(t)) R_{12} v(t, x(t)) \\
& + \sum_{j=1}^{m_1} \left(\frac{\partial u_j}{\partial x}(t, x(t)) \right) R_j \left(\frac{\partial u_j}{\partial x}(t, x(t)) \right)^T \Bigg] \mathrm{d}t, \quad (4.16)
\end{aligned}
$$

where u_j are the m_1 components of the control u, and $R_j \in \mathbb{R}^{n \times n}$, $j = 1, \ldots, m_1$, are symmetric positive definite matrices.

Nothing changes for the follower. However the necessary conditions for the leader are modified as follows:

$$
\frac{\partial H}{\partial u} = 0 = \lambda_1(t) B_1 - \lambda_2(t) \left(\frac{\partial u}{\partial x}(t, x(t)) \right)^T R_{21}
$$

$$+ \lambda^\circ u^T(t, x(t)) R_{11}, \tag{4.17}$$

$$\frac{\partial H}{\partial u_y} = 0 = \Bigg(\lambda_2(t) \left(B_1^T p_2^T(t) + R_{21} u(t, x(t)) \right)_j + \frac{\partial u_j}{\partial x}(t, x(t)) R_j \Bigg)_{j=1,\dots,m_1}. \tag{4.18}$$

The other necessary conditions (3.18) and (3.20) are the same. Equations (4.17) and (4.18) are easily solvable, without considering different cases. In this framework, λ_2 cannot be trivial. As in [46], we obtain from (4.18) that

$$\frac{\partial u_j^*}{\partial x}(t, x(t)) = \Big(\left(B_1^T p_2^T(t) + R_{21} u(t, x(t)) \right)_j \lambda_2(t) R_j^{-1} \Big)_{j=1,\dots,m_1}. \tag{4.19}$$

For simplicity, we next assume, as in [46], that $R_j = R > 0$ for every $j = 1, \dots, m_1$. Then

$$\frac{\partial u^*}{\partial x}(t, x(t)) = \left(B_1^T p_2^T(t) + R_{21} u(t, x(t)) \right) \lambda_2(t) R^{-1}. \tag{4.20}$$

Substituting this expression in (4.17), we get

$$R_{11} u(t, x(t)) = -B_1^T \lambda_1^T(t) + R_{21} B_1^T p_2^T(t) \lambda_2(t) R^{-1} \lambda_2^T(t) + R_{21}^2 u \lambda_2(t) R^{-1} \lambda_2^T(t), \tag{4.21}$$

or

$$\left(R_{11} - \lambda_2(t) R^{-1} \lambda_2^T(t) R_{21}^2 \right) u(t, x(t)) = -B_1^T \lambda_1^T(t) + R_{21} B_1^T p_2^T(t) \lambda_2(t) R^{-1} \lambda_2^T(t). \tag{4.22}$$

Remark 4.7. For $t = 0$, $\lambda_2(0) = 0$, then

$$R_{11} - \lambda_2(0) R^{-1} \lambda_2^T(0) R_{21}^2 = R_{11} > 0$$

is invertible. For sufficiently small $t \geq 0$, the matrix $R_{11} - \lambda_2(t) R^{-1} \lambda_2^T(t) R_{21}^2$ is invertible.

As long as $R_{11} - \lambda_2(t) R^{-1} \lambda_2^T(t) R_{21}^2$ is invertible, the optimal control is

$$u(t, x(t)) = \left(R_{11} - \lambda_2(t) R^{-1} \lambda_2^T(t) R_{21}^2 \right)^{-1} \times \left(-B_1^T \lambda_1^T(t) + R_{21} B_1^T p_2^T(t) \lambda_2(t) R^{-1} \lambda_2^T(t) \right). \tag{4.23}$$

The nonlinear optimization problem becomes

$$\dot{x}(t) = Ax(t) - S_2 p_2^T(t) + B_1 \Big(R_{11} - \lambda_2(t) R^{-1} \lambda_2^T(t) R_{21}^2 \Big)^{-1} \times \Big(-B_1^T \lambda_1^T(t) + R_{21} B_1^T p_2^T(t) \lambda_2(t) R^{-1} \lambda_2^T(t) \Big), \tag{4.24}$$

$$\dot{p}_2(t) = -p_2(t) A - x^T(t) Q_2 - \Big\| p_2(t) B_1$$

$$+ u^T(t,x(t))R_{21}\Big\|^2 \lambda_2(t)R^{-1}, \tag{4.25}$$

$$\dot\lambda_1(t) = -\lambda_1(t)A + \lambda_2(t)Q_2 - x^T(t)Q_1, \tag{4.26}$$

$$\dot\lambda_2(t) = +\lambda_2(t)\Big(A^T + R^{-1}\lambda_2^T(t)\big(p_2(t)B_1 + u^T(t,x(t))R_{21}\big)B_1^T\Big) + \lambda_1(t)S_2 - p_2(t)S_{12}. \tag{4.27}$$

with boundary conditions

$$x(0) = x_0, \qquad p_2(t_f) = x^T(t_f)K_{2f}, \tag{4.28}$$

$$\lambda_2(0) = 0, \qquad \lambda_1(t_f) = x^T(t_f)K_{1f} - \lambda_2(t_f)K_{2f}. \tag{4.29}$$

Remark 4.8. For $R = \gamma\,\mathrm{Id}$, if we let γ tend to $+\infty$, then we recover the necessary conditions for the strategy of Stackelberg with an open-loop information structure. Note that this coincidence is obtained only by taking the limit $\gamma \to +\infty$ without modifying the criterion of the leader.

Remark 4.9. These conditions are necessary conditions. As previously, the theory of focal points leads to sufficient conditions associated with the min-max or min-min Stackelberg strategy with closed-loop information structure including a weight for u_x in the criterion of the leader, namely, given $x_0 \in \mathbb{R}^n$. For t_f less than the global focal time of the system, there exists only one trajectory starting from x_0 solution of (4.24)–(4.29) associated with the optimal control (u, u_x) (see (4.23)–(4.20).

5. Conclusion

In this paper, the min-max and min-min Stackelberg strategies with a closed-loop information structure are studied. The framework is restricted to two-player differential games. Necessary conditions for the existence of a closed-loop min-max and min-min Stackelberg strategies are derived by considering all cases. It is also shown that they may degenerate whenever the leader is *omnipotent* and can impose his control to the follower. The focal times theory provides sufficient conditions for the optimization problems of the two players. The linear-quadratic case is used to illustrate the obtained necessary and sufficient conditions. Moreover in this linear-quadratic case, the control $u(t, x)$ is obtained for each state x. An extension is proposed to allow an optimal trajectory starting from any initial state by including, in the criterion, the Jacobian of his/her control in the criterion of the leader.

Appendix A. Proofs of theorems

Preliminaries. We start with preliminaries essentially borrowed from [6, 15, 35, 54]. First, consider a usual optimal control problem:

$$\begin{cases} \min C(u) \\ \text{under } \dot{x}(t) = f(t, x(t), u(t)), \\ \text{with } C(u) = \int\limits_0^{t_f} f^0(t, x(t), u(t))\mathrm{d}t, \end{cases} \tag{A.1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. A usual way to derive the Pontryagin minimum principle for such an optimal control problem is to extend the control system with a new state variable representing the cost, in the following way. Define the extended state $z = (x, x^0)^T \in \mathbb{R}^{n+1}$, where $x^0(0) = 0$, and $\tilde{f} = (f, f^0)^T$. Consider the extended control system

$$\dot{z}(t) = \tilde{f}(t, z(t), u(t)). \tag{A.2}$$

The associated endpoint mapping e_{z_0,t_f} at time t_f is defined by

$$e_{z_0,t_f} : \mathcal{U} \longrightarrow \mathbb{R}^{n+1}, \quad v \longmapsto z_u(t_f), \tag{A.3}$$

with z_u the trajectory solution of (A.2) associated to the control u, and $\mathcal{U}$ is the set of admissible controls. The crucial remark which is at the basis of the proof of the minimum principle is the following: if a trajectory $x(\cdot)$, associated with a control u on $[0, t_f]$ is optimal, then the endpoint mapping e_{z_0,t_f} is not locally surjective at u, then it follows from an implicit function argument that the first differential of the endpoint mapping at u is not surjective (at least in the case where there is no constraint on the controls). This fact leads to a Lagrange multipliers type equation which finally leads to the well known Pontryagin minimum principle (see [54] for details).

In the present paper we are not dealing with such a classic optimal control problem, however the previous reasoning may be adapted, even though our controls now depend also on $x(t)$, and we first derive a proof of Proposition 3.1.

Proof of Proposition 3.1. Define the extended state $z = (x, \hat{x})^T \in \mathbb{R}^{n+1}$ (and the projector $q(z) = x$), where $\hat{x}$ is the instantaneous cost associated with the criterion of the follower such that

$$\dot{\hat{x}}(t) = L_2(t, x(t), u^*(t, x(t)), v(t, x(t)), \quad \hat{x}(0) = 0,$$

which leads to the dynamic of the extended state z:

$$\dot{z}(t) = \hat{f}(t, z(t), v(t, x(t))) = \begin{pmatrix} f \\ L_2 \end{pmatrix}. \tag{A.4}$$

It is pointed out that the function $t \mapsto u^*(t, x(t))$ is fixed and the control of the follower $v(t, x(t))$ is the optimization variable.

Definition A.1. The *end-point mapping* at time t_f of system (A.4) with the initial state $z_0 = (x_0, 0)^T$ is the mapping

$$e_{z_0,t_f} : \mathcal{V} \longrightarrow \mathbb{R}^{n+1}, \quad v \longmapsto z_v(t_f), \tag{A.5}$$

where z_v is the solution of (A.4) associated to v, starting from z_0, and $\mathcal{V}$ is the set of admissible controls.

To compute the Fréchet first derivative, consider a fixed control δv such that v and $v + \delta v$ belong to $\mathcal{V}$ and denote $z + \delta z$ the trajectory associated with the latter control [35, 54]. An expansion to the first order of $\hat{f}$ leads to

$$\begin{aligned}
&\frac{\mathrm{d}(z+\delta z)}{\mathrm{d}t} \\
&= \hat{f}\Big(t, z+\delta z, u^*(t, q(z+\delta z)), v(t, q(z+\delta z)) + \delta v(t, q(z+\delta z))\Big) + o(\delta z), \\
&\quad = \hat{f}\Big(t, z, u^*(t, q(z)), v(t, q(z))\Big) + \hat{f}_z\Big(t, z, u^*(t, q(z)), v(t, q(z))\Big)\delta z \\
&\qquad\qquad + \hat{f}_u\Big(t, z, u^*(t, q(z)), v(t, q(z))\Big)u^*_y + o(\delta z).
\end{aligned} \tag{A.6}$$

Furthermore, to the first order,

$$\begin{aligned}
v(t, q(z+\delta z)) &= v\Big(t, q(z) + q_z(z)\delta z + o(\delta z)\Big) \\
&= v(t, q(z)) + v_y(t, q(z))q_z(z)\delta z + o(\delta z), \\
u^*(t, q(z+\delta z)) &= u^*(t, q(z)) + u^*_y(t, q(z))q_z(z)\delta z + o(\delta z), \\
\delta v(t, q(z+\delta z)) &= \delta v(t, q(z)) + o(\delta z).
\end{aligned}$$

Substituting these Taylor series expansions into relation (A.6), we have, at the first order,

$$\begin{aligned}
\frac{\mathrm{d}(\delta z)}{\mathrm{d}t} &= \hat{f}_z\delta z + \hat{f}_u u^*_y q_z \delta z + \hat{f}_v \delta v + \hat{f}_v v_y q_z(z)\delta z \\
&= \underbrace{\Big(\hat{f}_z + \hat{f}_u u^*_y q_z + \hat{f}_v v_y q_z(z)\Big)}_{a(t)}\delta z + \underbrace{\hat{f}_v}_{b(t)}\delta v.
\end{aligned}$$

Using the transition matrix $\Phi(t)$ satisfying $\dot{\Phi}(t) = a\Phi(t)$ and $\Phi(0) = \mathrm{Id}$ (Id denoting the identity matrix), it follows that

$$\mathrm{d}e_{z_0,t_f}(v)\cdot\delta v = \delta z(t_f) = \Phi(t_f)\int_0^{t_f} \Phi^{-1}(s)b(s)\delta v \mathrm{d}s. \tag{A.7}$$

If a closed-loop Stackelberg control $v^* \in \mathcal{V}$ of the follower is optimal, then the first derivative of the end-point mapping, $\mathrm{d}e_{z_0,t_f}(v^*)$, is not surjective, and hence there exists a line vector $\tilde{\phi} \in \mathbb{R}^{n+1}$, $\tilde{\phi} \neq 0$ such that

$$\tilde{\phi}\cdot\mathrm{d}e_{z_0,t_f}(v^*)\delta v = 0 \quad \forall \delta v \in \mathcal{V}. \tag{A.8}$$

Set $\phi(t) = \tilde{\phi}\Phi(t_f)\Phi^{-1}(t)$; then relation (A.8) is satisfied for every control δv, and thus

$$\int_0^{t_f} \phi(t) b(t) \delta v \mathrm{d}t = 0.$$

This implies that almost everywhere on $[0, t_f]$

$$\phi(t)\, b(t) = 0. \tag{A.9}$$

Furthermore, deriving

$$\phi(t) = \tilde{\phi}\Phi(t_f)\Phi^{-1}(t)$$

with respect to t, we obtain

$$\dot{\phi}(t) = -\phi(t)a = -\phi(t)\Big(\hat{f}_z + \hat{f}_u u_y^* q_z + \hat{f}_v v_y q_z(z)\Big) = -\phi(t)\Big(\hat{f}_z + \hat{f}_u u_y^* q_z\Big), \tag{A.10}$$

the last equality holds due to relation (A.9). Denoting

$$\phi(t) = \begin{pmatrix} p_2(t) & p_2^\circ(t) \end{pmatrix},$$

we obtain that p_2° is a constant scalar. Finally, the initial condition z_0 being fixed and the final condition $z(t_f)$ being free, the standard transversality condition associated with system (A.4) implies that $p_2^\circ \neq 0$ since

$$\tilde{\phi} = \begin{pmatrix} p_2(t_f) & p_2^\circ \end{pmatrix} \neq 0.$$

We normalize the costate vector so that $p_2^\circ = 1$, since $(p_2(t_f), p_2^\circ)$ is defined up to a multiplicative scalar. The transversality condition leads to Eq. (3.4) and p_2 satisfies Eq. (3.3). In addition, relation (A.9) could be reformulated into Eq. (3.2)).

Proof of Proposition 3.2. For each $u \in \mathcal{U}$, the inclusion $Tu \subseteq T'u$ implies that

$$\max_{v \in Tu} J_1(u, v) \leq \max_{v' \in T'u} J_1(u, v'). \tag{A.11}$$

With Assumption 1, in a neighborhood $\mathcal{U}_{nb}^*$ of $u^* \in \mathcal{U}$, we have

$$\max_{v \in Tu} J_1(u, v) = \max_{v' \in T'u} J_1(u, v') \quad \forall u \in \mathcal{U}_{nb}^*. \tag{A.12}$$

Thus, u^* defined by (3.6) is also given by (3.7).

Proof of Proposition 3.4. As above, we define the extended state

$$Z = \begin{pmatrix} x^T & p_2 & x^\circ \end{pmatrix}^T \in \mathbb{R}^{2n+1}, \tag{A.13}$$

where x° is the instantaneous cost associated with the criterion of the leader satisfying

$$\dot{x}^\circ(t) = \tilde{L}_1(t, x, p_2, u), \quad x^\circ(0) = 0.$$

The extended system is subject to the dynamics

$$\dot{Z}(t) = \tilde{F}(t, Z, u, u_y^T) = \begin{pmatrix} F_1(t, x, p_2, u) \\ (F_{21}(t, x, p_2, u) + F_{22}(t, x, p_2, u)u_y)^T \\ \tilde{L}_1(t, x, p_2, u) \end{pmatrix}, \quad \text{(A.14)}$$

where $u = u(t, h(Z))$ is a function of time t and of the projection $h(Z) = x$.

The *end-point mapping* at time t_f of system (A.14) with initial state Z_0 is the mapping

$$E_{Z_0,t_f} : \mathcal{U} \longrightarrow \mathbb{R}^{2n+1}, \quad u \longmapsto Z_u(t_f), \quad \text{(A.15)}$$

where Z_u is the solution of (A.14) associated to u, starting from Z_0. Here $\mathcal{U}$ denotes the open set of controls $u \in \mathcal{L}^\infty([0, t_f] \times \mathbb{R}^n, \mathbb{R}^{m_1})$ such that the solution $Z_u(\cdot)$ of (A.14), associated with u and starting from Z_0, is well defined on $[0, t_f]$.

Note that, if $\tilde{F}$ is of class $\mathcal{C}^p$, $p \geq 1$, then E_{Z_0,t_f} is also of class $\mathcal{C}^p$.

To compute the Fréchet first derivative, we proceed as in [35,54], consider a fixed control δu on $\mathcal{U}$ and note $Z + \delta Z$ the trajectory associated with the control $u + \delta u$. An expansion to the first order of $\tilde{F}$ leads to

$$\begin{aligned} \frac{\mathrm{d}(Z + \delta Z)}{\mathrm{d}t} &= \tilde{F}\Big(t, Z + \delta Z, u(t, h(Z + \delta Z)) + \delta u(t, h(Z + \delta Z)), \\ &\qquad u_x(t, h(Z + \delta Z))^T + \delta u_x(t, h(Z + \delta Z))^T\Big) + o(\delta Z). \quad \text{(A.16)} \end{aligned}$$

Furthermore, an expansion to the first order of the control u gives

$$\begin{aligned} u(t, h(Z + \delta Z)) &= u(t, h(Z) + h_Z(Z)\delta Z + o(\delta Z)) \\ &= u(t, h(Z)) + u_x(t, h(Z))h_Z(Z)\delta Z + o(\delta Z). \end{aligned}$$

Therefore, at the first order,

$$\begin{aligned} \frac{\mathrm{d}(\delta Z)}{\mathrm{d}t} &= \tilde{F}_Z\delta Z + \tilde{F}_u u_x h_Z \delta Z + \tilde{F}_u \delta u + \tilde{F}_{u_x} u_{xx} h_Z \delta Z + \tilde{F}_{u_x}\delta u_x^T \\ &= \underbrace{\Big(\tilde{F}_Z + \tilde{F}_u u_x h_Z + \tilde{F}_{u_x} u_{xx} h_Z\Big)}_{A}\delta Z + \underbrace{\tilde{F}_u}_{B}\delta u + \underbrace{\tilde{F}_{u_x}}_{C}\delta u_x^T. \quad \text{(A.17)} \end{aligned}$$

Using the transition matrix M defined by $\dot{M}(t) = A(t)M(t)$ and $M(0) = \mathrm{Id}$ (Id denoting the identity matrix), it follows that

$$\mathrm{d}E_{Z_0,t_f}(u) \cdot \delta u = \delta Z(t_f) = M(t_f) \int_0^{t_f} M^{-1}(s)\Big(B(s)\delta u(s) + C(s)\delta u_y^T(s)\Big)\mathrm{d}s. \quad \text{(A.18)}$$

If u is the control of the leader in a closed-loop min-max or min-min Stackelberg solution, then there exists a vector $\tilde{\psi} \in \mathbb{R}^{2n+1}$, $\tilde{\psi} \neq 0$ such that

$$\tilde{\psi} \cdot \mathrm{d}E_{Z_0,t_f}(u)\delta u = 0, \quad \forall \delta u \in \mathcal{U}. \quad \text{(A.19)}$$

Set $\psi(t) = \tilde{\psi} M(t_f) M^{-1}(t)$; then

$$\dot{\psi}(t) = -\psi(t)A(t) = -\psi(t)\left(\tilde{F}_Z + \tilde{F}_u u_y h_Z + \tilde{F}_{u_y} u_{yy} h_Z\right). \qquad \text{(A.20)}$$

Furthermore, relation (A.19) holds for every control δu, and thus

$$\int_0^{t_f} \psi(t)\Big(B(t)\delta u(t,x) + C(t)\delta u_y^T(t,x)\Big)\mathrm{d}t = 0. \qquad \text{(A.21)}$$

This relation is satisfied for all controls u functions of t and x. In particular it is also satisfied for controls u functions of t only. For such a control, relation (A.21) becomes

$$\int_0^{t_f} \psi(t)\Big(B(t)\delta u(t)\Big)\mathrm{d}t = 0.$$

This implies that almost everywhere on $[0, t_f]$, $\psi(t)\,B(t) = 0$. Then (A.21) leads to

$$\int_0^{t_f} \psi(t)\Big(C(t)\delta u_y^T(t,x)\Big)\mathrm{d}t = 0.$$

Hence, almost everywhere on $[0, t_f]$, we have $\psi(t)\,C(t) = 0$.

Let $H(t, Z, u, u_y) = \psi(t)\,\tilde{F}(t, Z, u, u_y)$ be the Hamiltonian associated with this optimization problem. The last equations can be rewritten almost everywhere on $[0, t_f]$ as:

$$\begin{aligned}
\dot{Z}(t) &= \tilde{F}\Big(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))\Big) \\
&= \frac{\partial H}{\partial \psi}\Big(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))\Big), \\
\dot{\psi}(t) &= -\psi(t)\bigg(\tilde{F}_Z\Big(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))\Big) \\
&\quad + \tilde{F}_u\Big(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))\Big)u_y(t, h(Z(t)))h_Z(Z(t)) \\
&\quad + \tilde{F}_{u_y}\Big(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))\Big)u_{yy}(t, h(Z(t)))h_Z(Z(t))\bigg) \\
&= -\frac{\mathrm{d}H}{\mathrm{d}Z}\Big(t, Z(t), u(t, h(Z(t))), u_y(t, h(Z(t)))\Big), \\
\frac{\partial H}{\partial u} &= \psi(t)\,B(t) = 0, \qquad \frac{\partial H}{\partial u_y} = \psi(t)\,C(t) = 0.
\end{aligned}$$

Denoting $\psi = (\lambda_1, \lambda_2, \lambda^\circ)$, one obtains the necessary conditions (3.16)–(3.20) given by Proposition 3.4 for a closed-loop min-max or min-min Stackelberg strategy. Finally, some part of the initial and final values of the extended state Z, defined by (A.13) are imposed by the transversality condition for the follower optimization problem (3.4) and by the initial state $x(0) = x_0$. We can formalize these conditions by defining two sets M_0 and M_1

$$\begin{aligned} \begin{pmatrix} x(0) \\ p_2^T(0) \end{pmatrix} &= \begin{pmatrix} x_0 \\ p_2^T(0) \end{pmatrix} \in M_0, \\ \begin{pmatrix} x(t_f) \\ p_2^T(t_f) \end{pmatrix} &= \begin{pmatrix} x(t_f) \\ \dfrac{\partial g_2}{\partial x}(h(Z(t_f))) \end{pmatrix} \in M_1, \end{aligned} \tag{A.22}$$

where

$$M_0 = \{x_0\} \times \mathbb{R}^n = \left\{ \begin{pmatrix} x \\ p_2^T \end{pmatrix} \;\middle|\; \mathcal{F}_0 \begin{pmatrix} x \\ p_2^T \end{pmatrix} = x - x_0 = 0 \right\}, \tag{A.23}$$

$$M_1 = \left\{ \begin{pmatrix} x \\ p_2^T \end{pmatrix} \;\middle|\; \mathcal{F}_1 \begin{pmatrix} x \\ p_2^T \end{pmatrix} = \frac{\partial g_2}{\partial x}(h(Z(t_f))) - p_2 = 0 \right\}. \tag{A.24}$$

The tangent manifolds $T_{Z(0)}M_0$ and $T_{Z(t_f)}M_1$ are defined by

$$T_{Z(0)}M_0 = \left\{ (0, \alpha) \in \mathbb{R}^{2n} \;\middle|\; \alpha \in \mathbb{R}^n \right\}, \tag{A.25}$$

$$T_{Z(t_f)}M_1 = \left\{ \left(\beta, \beta \frac{\partial^2 g_2}{\partial x^2}\right) \;\middle|\; \beta \in \mathbb{R}^n \right\}. \tag{A.26}$$

The transversality conditions can be written as (see [6] or [54, p. 104])

$$\lambda(0) \perp T_{Z_0}M_0, \quad \lambda(t_f) - \lambda^\circ \frac{\partial g_1}{\partial Z}(h(Z(t_f))) \perp T_{Z(t_f)}M_1, \tag{A.27}$$

and lead to Eqs. (3.21).

Proof of Proposition 3.5. The proof goes by contradiction. The term $\lambda_2^T F_{22}$ in (3.17) is the product of a column vector (λ_2^T) and a row vector $F_{22} = p_2 \dfrac{\partial f}{\partial u} + \dfrac{\partial L_2}{\partial u}$, since λ_2 is a line costate vector. The triviality of this term induces that all components of λ_2 or all components of F_{22} are trivial (or both).

Assume that $\lambda_2 \not\equiv 0$, then

$$F_{22} = p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \equiv 0.$$

If, furthermore,

$$\frac{\partial F_{22}}{\partial u} = \frac{\partial}{\partial u}\left(p_2 \frac{\partial f}{\partial u} + \frac{\partial L_2}{\partial u} \right)$$

is invertible, then the implicit-function theorem applied to the function F_{22} with respect to the variable u permits to write locally along the trajectory the control $u = u(t, x, p_2)$.

The system in (x, p_2) is rewritten as

$$\begin{cases} \dot{x}(t) = F_1(t, x, p_2, u(t, x, p_2)), \\ \dot{p}_2(t) = F_{21}(t, x, p_2, u(t, x, p_2)), \end{cases} \tag{A.28}$$

since $F_{22} = 0$.

Since the dynamics and the criterion do not depend on u_y, we can deduce that any control u_y is extremal for the optimization problem. But the relation (3.16) is a constraint on u_y. Relation (3.23) follows.

Proof of Lemma 3.1. The proof uses the controllability Hautus test. The pair $(\mathcal{A}, \mathcal{B})$ satisfies the Kalman condition if and only if the matrix $\left[\ \mathcal{A} - \alpha I \ \vdots\ \mathcal{B}\ \right]$ is of full rank, for every $\alpha \in \mathbb{C}$. The proof consists in showing that all row vectors (z_1^T, z_2^T, z_3^T) satisfy

$$\begin{pmatrix} z_1^T & z_2^T & z_3^T \end{pmatrix} \left[\ \mathcal{A} - \alpha I \ \vdots\ \mathcal{B}\ \right] = 0, \tag{A.29}$$

are trivial. Developing Eq. (A.29), we have

$$-z_1^T Q_1 = z_2^T (A - \alpha I_n), \tag{A.30}$$

$$-z_1^T Q_2 = z_3^T (A - \alpha I_n), \tag{A.31}$$

$$z_1^T (A^T - \alpha I_n) = z_2^T (S_1 + B_2 R_{12}^{-1} B_2^T), \tag{A.32}$$

$$z_2^T B_2 = z_3^T B_2 R_{22}^{-1} R_{12}, \tag{A.33}$$

$$z_2^T B_1 R_{11}^{-1} R_{21} = z_3^T B_1. \tag{A.34}$$

Multiplying Eq. (A.30) by z_1 and Eq. (A.32) by z_2, we obtain

$$-z_1^T Q_1 z_1 = z_2^T (A - \alpha I_n) z_1 = z_2^T \left(S_1 + B_2 R_{12}^{-1} B_2^T \right) z_2. \tag{A.35}$$

The first term is nonpositive ($Q_1 \geq 0$) and the last term is nonnegative, hence both are zero. It follows that $z_1^T Q_1 = 0$, $z_2^T B_1 = 0$ and $z_2^T B_2 = 0$. Substituting these relations into (A.30) and (A.34) one gets

$$z_1^T (A^T - \alpha I_n) = 0, \quad z_1^T Q_1 = 0, \tag{A.36}$$

$$z_2^T (A - \alpha I_n) = 0, \quad z_2^T B_2 = 0, \quad z_2^T B_1 = 0, \tag{A.37}$$

$$z_3^T (A - \alpha I_n) = 0, \quad z_3^T B_2 = 0, \quad z_3^T B_1 = 0. \tag{A.38}$$

Relations (A.36) correspond to the observability Hautus test of the pair (Q_1, A), the relations (A.37) and (A.38) to the controllability Hautus test of the pair (A, B_1) or (A, B_2). The assumptions of controllability and observability lead to z_1, z_2 and z_3 trivial.

Proof of Proposition 3.7. With the condition

$$p_2 B_1 + u^T R_{21} \equiv 0, \tag{A.39}$$

the term $\dfrac{\partial u}{\partial x}$ does not appear anymore in the necessary conditions (3.45)–(3.48). Derivating with respect to time the relation (A.39) does not induce necessary conditions for $\dfrac{\partial u}{\partial x}$.

However assuming that R_{21} is invertible, the control u admits two representations

$$u(t, x(t)) = -R_{11}^{-1} B_1^T \lambda_1^T(t) = -R_{21}^{-1} B_1^T p_2^T(t). \tag{A.40}$$

From this relation and from (3.48), necessary conditions about $x(t_f)$ are developed by successive derivations with respect to time,

$$\begin{cases} \lambda_1(t) B_2 - p_2(t) B_2 R_{22}^{-1} R_{12} = 0, \\ \lambda_1(t) B_1 R_{11}^{-1} R_{21} - p_2(t) B_1 = 0. \end{cases} \tag{A.41}$$

These two relations can be rewritten for every $t \in [0, t_f]$ as follows:

$$\begin{pmatrix} x(t) & \lambda_1(t) & p_2(t) \end{pmatrix} \mathcal{B} = 0. \tag{A.42}$$

Substituting (A.39) into the dynamics of x, λ_1 and p_2, we obtain the autonomous differential system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x^T(t) & \lambda_1(t) & p_2(t) \end{pmatrix} = \begin{pmatrix} x^T(t) & \lambda_1(t) & p_2(t) \end{pmatrix} \mathcal{A} \tag{A.43}$$

The k-order derivation of (A.42) with respect to time, at time $t = t_f$, gives

$$\begin{bmatrix} x^T(t_f) & x^T(t_f) K_{1f} & x^T(t_f) K_{2f} \end{bmatrix} \mathcal{A}^k \mathcal{B} = 0 \quad \forall k \in \mathbb{N}. \tag{A.44}$$

The assumptions of Lemma 3.1 are verified. This leads to the controllability of the pair $(\mathcal{A}, \mathcal{B})$, which implies that $x(t_f) = 0$. Furthermore the autonomous linear system in x, λ_1 and p_2 with end value conditions $x(t_f) = \lambda_1^T(t_f) = p_2^T(t_f) = 0$ imposes, by a backward integration of (A.43)

$$x(t) \equiv \lambda_1^T(t) \equiv p_2^T(t) \equiv 0 \quad \forall t \in [0, t_f]. \tag{A.45}$$

The unique optimal trajectory in this case is the trivial one.

Proof of Lemma 3.2. Similarly as in the classic linear-quadratic problem, we seek a solution in the form $\lambda_1^T(t) = K_1(t)x(t)$. Then, the matrix $K_1(t) \in \mathbb{R}^{n \times n}$ must satisfy

$$\begin{aligned} \dot{K}_1(t)x(t) + K_1(t)\left(Ax(t) - \left(B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T\right) K_1(t)x(t)\right) \\ = -A^T K_1(t)x(t) - Q_1 x(t). \end{aligned} \tag{A.46}$$

This relation should hold for every x, which leads to define $K_1(t)$ as the solution of the following Riccati differential equation

$$\begin{cases} \dot{K}_1(t) = -K_1(t)A - A^T K_1(t) - Q_1 \\ \qquad\qquad + K_1(t)\left(B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T\right) K_1(t), \\ K_1(t_f) = K_{1f}. \end{cases} \tag{A.47}$$

The existence of a solution of the optimization problem is ensured in a standard way "a la Riccati" and by the uniqueness of an optimal trajectory. This is justified *a posteriori* in the following by using the theory of focal times.

Substituting $\lambda_1^T(t) = K_1(t)x(t)$ into (3.42) and (A.40), the state $x(t)$ has the dynamical constraint

$$\begin{cases} \dot{x}(t) = \left(A - \left(B_1 R_{11}^{-1} B_1^T + B_2 R_{12}^{-1} B_2^T\right) K_1(t)\right) x(t) = \tilde{A}x(t), \\ x(0) = x_0. \end{cases} \tag{A.48}$$

Let $\mathcal{M}(t)$ be the transition matrix associated with (A.48). Then $x(t) = \mathcal{M}(t)x_0$. Then the constraint (3.50) becomes

$$\left(B_2^T K_{1f} - R_{12} R_{22}^{-1} B_2^T K_{2f}\right) \mathcal{M}(t_f) x_0 = 0. \tag{A.49}$$

This is a m_2-codimension (at most) condition on the initial states x_0.

Proof of Lemma 4.1. From Eq. (4.1), one gets

$$x(t) = e^{tA}x_0 - \int_0^t e^{(t-s)A}\left(B_2 R_{22}^{-1}\xi(s) - B_1 u(s)\right) \mathrm{d}s.$$

There exist scalar constants $C_k \geq 0$ such that for a given $t_f > 0$, for every $t \in [0, t_f]$

$$\|x(t)\| \leq C_1\|x_0\| + C_2\sqrt{t_f}\left[\left(\int_0^{t_f} \|\xi(s)\|^2 \mathrm{d}s\right)^{1/2} + \left(\int_0^{t_f} \|u(s)\|^2 \mathrm{d}s\right)^{1/2}\right]. \tag{A.50}$$

Hence

$$\left\|\int_0^{t_f} x^T(s) Q_1 x(s) \mathrm{d}s\right\| \leq C_3 t_f^2 \left(\int_0^{t_f} \|\xi(s)\|^2 \mathrm{d}s + \int_0^{t_f} \|u(s)\|^2 \mathrm{d}s\right) + C_4\|x_0\|^2 + C_5 t_f + C_5 t_f^2 \int_0^{t_f} \|\xi(s)\|^2 \mathrm{d}s + C_5 t_f^2 \int_0^{t_f} \|u(s)\|^2 \mathrm{d}s. \tag{A.51}$$

In addition, assuming $R_{11} > 0$ and $R_{22}^{-1} R_{12} R_{22}^{-1} > 0$,

$$\left\| \int_0^{t_f} u^T(s) R_{11} u(s) \mathrm{d}s \right\| \geq C_6 \int_0^{t_f} \|u(s)\|^2 \mathrm{d}s, \tag{A.52}$$

$$\left\| \int_0^{t_f} \xi^T(s) R_{22}^{-1} R_{12} R_{22}^{-1} \xi(s) \mathrm{d}s \right\| \geq C_6 \int_0^{t_f} \|\xi(s)\|^2 \mathrm{d}s. \tag{A.53}$$

Using these inequalities and (4.2), we can compute a lower bound of the criterion $\hat{J}_1(u, \xi)$

$$2\hat{J}_1(u,\xi) \geq \left(C_6 - (C_3 + C_5)\, t_f^2\right) \left[\int_0^{t_f} \|u\|^2 \mathrm{d}s + \int_0^{t_f} \|\xi\|^2 \mathrm{d}s \right] + x^T(t_f) K_{1f} x(t_f) - C_4 \|x_0\|^2 - C_5 t_f. \tag{A.54}$$

For sufficiently small $t_f > 0$,

$$t_f \leq \sqrt{\frac{C_6}{C_3 + C_5}},$$

the criterion $\hat{J}_1(u, \xi)$ is finite, then $0 < t_f < t_c$.

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