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GLOBAL STEADY-STATE STABILIZATION AND CONTROLLABILITY OF 1D SEMILINEAR WAVE EQUATIONS

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This paper is concerned with the exact boundary controllability of semilinear wave equations in one space dimension. We prove that it is possible to move from any steady-state to any other one by means of a boundary control, provided that they are in the same connected component of the set of steady-states. The proof is based on an expansion of the solution in a one-parameter Riesz basis of generalized eigenvectors, and on an effective feedback stabilization procedure which is implemented.

Keywords: Wave equation; stabilization; Riesz basis; pole shifting; Lyapunov functional.

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1. Introduction

Let $L > 0$ fixed and $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2. We are concerned with the exact controllability of the semilinear wave equation

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} + f(y), \\ y(t,0) = 0, \quad y_x(t,L) = u(t), \\ y(0,\cdot) = a_0(\cdot), \quad y_t(0,\cdot) = a_1(\cdot), \end{cases} \tag{1}$$

where the state is $(y(t,\cdot), y_t(t,\cdot)) : [0,L] \to \mathbb{R}^2$ and the control is $u(t) \in \mathbb{R}$.

The question we investigate in this paper is the following. For $T > 0$ large enough, given initial data (a_0, a_1) and final data (b_0, b_1) in a suitable Hilbert space,

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is it possible to construct a control u steering the control system (1) from the initial state (y_0, y_1) to the target (z_0, z_1) within time T? Moreover, is it possible to achieve this by an explicit and efficient numerical implementation?

If f is linear, the situation is well known (see, for instance, [18, 22]). In the general semilinear case, the main results as to the global controllability problem, using a variant of the Hilbert Uniqueness Method and a fixed point argument, assert that if f is asymptotically linear (see [26]), and more generally if f is globally Lipschitzian (see [27]), then the control system (1) is globally controllable within time $T > 2L$, in the space $H^1_{(0)}(0,L) \times L^2(0,L)$, with controls in $L^2(0,T)$. The situation extends to slightly super linear functions, or functions sharing a good sign growth condition, see [5, 17, 19, 26, 28]. Here, and throughout the paper, $H^1_{(0)}(0,L)$ denotes the Banach space

$$H^1_{(0)}(0,L) := \{y \in H^1(0,L) \mid y(0) = 0\},$$

endowed with the norm

$$\|y\|_{H^1_{(0)}(0,L)} = \left\| \frac{dy}{dx} \right\|_{L^2(0,L)}.$$

When f is highly super linear, the situation is far more intricate, in particular because of possible blowing up. It is proved in [28] that if there exists k large enough so that

$$\int_k^{+\infty} \frac{ds}{|F(s)|^{1/2}} < +\infty,$$

where $F(s) = \int_0^s f(t)\,dt$, then the system (1) is not exactly controllable in any time $T > 0$. More precisely, for every $T > 0$, there exist initial data $(a_0, a_1) \in H^1_{(0)}(0,L) \times L^2(0,L)$ for which the solution of (1) so that $y(0,\cdot) = a_0(\cdot)$ and $y_t(0,\cdot) = a_1(\cdot)$ blow up in time $t < T$, for every control $u \in C^0([0,T])$. Hence, there is no hope to get a general result on global controllability. The result of this paper is intermediate.

Definition 1. A function $y \in C^2([0,L])$ is a steady-state of the control system (1) if

$$\frac{d^2y}{dx^2}(x) + f(y(x)) = 0, \quad y(0) = 0.$$

Let $\mathcal{S}$ denote the set of steady-states, endowed with the C^2 topology.

Introduce the Banach space

$$Y_T := C^0([0,T], H^1_{(0)}(0,L)) \cap C^1([0,T], L^2(0,L)), \tag{2}$$

endowed with the norm

$$\|y\|_{Y_T} = \max_{t\in[0,T]} \left(\|y\|_{H^1_{(0)}(0,L)} + \left\| \frac{\partial y}{\partial t} \right\|_{L^2(0,L)} \right).$$

Note that, for every $u \in L^2(0,T)$, and for all initial data $(a_0, a_1) \in H^1_{(0)}(0,L) \times L^2(0,L)$, there exists at most one solution of (1) in Y_T.

The main results of the paper are the following.

Theorem 1. *Let y_0 and y_1 be two steady-states belonging to a same connected component of $\mathcal{S}$. For every $\delta > 0$, there exists $\varepsilon_1 > 0$ so that, for every $\varepsilon \in (0, \varepsilon_1]$, there exists a control $u \in H^2(0, 1/\varepsilon)$ such that the solution y in $Y_{1/\varepsilon}$ of the Cauchy–Dirichlet problem*

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} + f(y), \\ y(t,0) = 0, \quad y_x(t,L) = u(t), \\ y(0,x) = y_0(x), \quad y_t(0,x) = 0, \end{cases} \tag{3}$$

satisfies

$$\|y(1/\varepsilon, \cdot) - y_1(\cdot)\|_{H^1_{(0)}(0,L)} + \|y_t(1/\varepsilon)\|_{L^2(0,L)} \leq \delta.$$

Remark 1. The proof of this result, which represents the main part of the paper, relies on an explicit construction of the control u in a *feedback form*, and of a *Lyapunov functional*. We stress that the procedure is effective and consists actually in solving a stabilization problem in finite dimension. Indeed, in order to construct u, one only needs to compute a finite number of quantities related to a one-parameter dependent Riesz expansion of the solution. The numerical procedure is implemented, and simulations are presented in the last section of the paper.

Coupling Theorem 1 with a local controllability result yields the following corollary.

Corollary 1. *Let y_0 and y_1 be two steady-states belonging to a same connected component of $\mathcal{S}$. There exist a time $T > 0$ and a control function $u \in L^2(0,T)$ such that the solution $y(t,x)$ in Y_T of the Cauchy–Dirichlet problem* (3) *satisfies $y(T, \cdot) = y_1(\cdot)$, $y_t(T, \cdot) = 0$.*

Remark 2. The time T of controllability required in this result may be large. However, on the other part, due to the finite speed of propagation for the wave equation, the time T cannot be arbitrarily small. The question of a minimal time to reach a given target, using for instance *a priori* estimates, is open.

Remark 3. Similar results have been obtained in [8] in the context of the heat equation. The idea is to stabilize a finite dimensional part of the system using pole shifting. The problem investigated here is, however, much more challenging; on the one part, because of conservation properties of the wave equation, and on the other, because of the necessity of using Riesz basis expansions. This latter point is the key technical development of this paper, and is investigated in Sec. 2.3. There is a large body of literature dealing with Riesz basis analysis applied to the boundary controlled wave equation (see, for instance, [1, 24] and references therein). However, the analysis of this article requires a *one-parameter* Riesz expansion of the solution, so as to obtain a Riesz basis depending smoothly on the parameter (Lemma 5). This reduction procedure constitutes the main contribution of this work.

Remark 4. As proved in [8], the set of steady-states $\mathcal{S}$ is connected if one of the following situations occur:

- $F(y) = \int_0^y f(s)\, ds \xrightarrow[|y|\to+\infty]{} +\infty$;
- for every $\alpha > 0$, the indefinite integral
$$\int \frac{dy}{\sqrt{\alpha - F(y)}}$$
(if it makes sense) diverges in $-\infty$ and in $+\infty$;
- the function f is odd, i.e. $f(-y) = -f(y)$, for every $y \in \mathbb{R}$.

Remark 5. The result of Corollary 1 may be achieved directly by using repeatedly a local exact controllability theorem (see [26, 28], and Sec. 2.6 of this paper), but contrarily to our strategy, the control function is not constructed explicitly. Note also that our approach does not necessarily require controllability of the linearized system around an equilibrium (see [7]).

Remark 6. In the case of the heat equation [8], it was proved that, if the steady-states y_0 and y_1 belong to distinct connected components of the set $\mathcal{S}$ of steady-states, then it is impossible, either to move from y_0 to y_1, or the converse. Here, in the case of the wave equation, the question is open.

The idea of the proof of Theorem 1 is as follows. Linearizing the system (3) along a path of steady-states joining y_0 to y_1, we obtain a system of the form $w_{tt} = w_{xx} + cw$, where $c \in L^\infty(0,T)$, with boundary conditions $w(t,0) = 0$ and $w_x(t,L) = v(t)$. At the first glance, if we suppose that $c = 0$, then it is possible to choose a control $v(\cdot)$ stabilizing this equation; namely, if we set $v(t) = -\alpha w_t(t,L)$, with $\alpha > 0$, the *energy function*
$$t \mapsto \int_0^L (w_t(t,x)^2 + w_x(t,x)^2)\, dt$$
is exponentially decreasing (see, for instance, [14, 15] for some results in that direction). Moreover, an obvious spectral computation shows that the eigenvalues of the corresponding operator have their real part tending to $-\infty$ as α tends to 1. This result only holds asymptotically if $c \neq 0$. Therefore, in the general case, if α is close enough to 1, then only a finite number of eigenvalues may be positive. The system corresponding to these unstable modes can be written (using an expansion of the solution in a one-parameter dependent Riesz basis of generalized eigenvectors), at the first order, as a nonautonomous linear control system. It is then possible, by a pole shifting procedure together with a time reparametrization, to stabilize this subsystem using a control in a feedback form.

Remark 7. The method consisting in stabilizing a quasi-static deformation has already been used in [8] in the context of nonlinear heat equations, in [23] for Navier–Stokes equations, in [7] for shallow water equations, and in [3, 4] for a Schrödinger equation. However, in both latter cases, the deformation was naturally stable and a feedback procedure was not necessary.

2. Proof of the Main Results

2.1. *Construction of a path of steady-states*

The following lemma is obvious.

Lemma 1. *Let $\phi_0, \phi_1 \in \mathcal{S}$. Then, ϕ_0 and ϕ_1 belong to the same connected component of $\mathcal{S}$ if and only if, for every real number α between $\phi_0'(0)$ and $\phi_1'(0)$, the maximal solution of*

$$\frac{d^2y}{dx^2} + f(y) = 0, \quad y(0) = 0, \quad y'(0) = \alpha,$$

denoted by $y^\alpha(\cdot)$, is defined on $[0, L]$.

Now let y_0 and y_1 in the same connected component of $\mathcal{S}$. Let us construct in $\mathcal{S}$ a C^2 path $(\hat{y}(\tau, \cdot), \hat{u}(\tau))$, $0 \leq \tau \leq 1$, joining y_0 to y_1. For $i = 0, 1$, set $\alpha_i := y_i'(0)$. Then, with our previous notations, one has $y_i(\cdot) = y^{\alpha_i}(\cdot), i = 0, 1$. Now, set

$$\hat{y}(\tau, x) := y^{(1-\tau)\alpha_0 + \tau\alpha_1}(x) \quad \text{and} \quad \hat{u}(\tau) := \hat{y}_x(\tau, L),$$

where $\tau \in [0, 1]$ and $x \in [0, L]$, so that $\hat{y}(\tau, \cdot)$ satisfies

$$\begin{cases} \dfrac{\partial^2 \hat{y}}{\partial x^2}(\tau, x) + f(\hat{y}(\tau, x)) = 0, \quad x \in (0, L), \\ \hat{y}(\tau, 0) = 0, \quad \dfrac{\partial \hat{y}}{\partial x}(\tau, 0) = (1-\tau)\alpha_0 + \tau\alpha_1, \end{cases}$$

for all $\tau \in [0, 1]$. By construction, we have

$$\hat{y}(0, \cdot) = y_0(\cdot) \quad \text{and} \quad \hat{y}(1, \cdot) = y_1(\cdot),$$

and thus $(\hat{y}(\tau, \cdot), \hat{u}(\tau))$ is a C^2 path in $\mathcal{S}$ connecting y_0 to y_1.

2.2. *Reduction of the problem*

Let $\varepsilon > 0$, and let y denote the solution of (3) in $Y_{1/\varepsilon}$, associated to a control $u \in H^2(0, 1/\varepsilon)$. We set, for all $t \in [0, 1/\varepsilon]$ and $x \in [0, L]$,

$$\begin{aligned} z(t, x) &:= y(t, x) - \hat{y}(\varepsilon t, x), \\ u_1(t) &:= u(t) - \hat{u}(\varepsilon t). \end{aligned} \tag{4}$$

This time reparametrization will happen to be useful in order to perform a pole shifting procedure on the linear finite dimensional system representing the unstable part of the equation.

From the definition of $(\hat{y}, \hat{u})$, we infer that z satisfies the initial-boundary problem

$$\begin{cases} z_{tt} = z_{xx} + f'(\hat{y})z + z^2 \displaystyle\int_0^1 (1-s) f''(\hat{y} + sz)\, ds - \varepsilon^2 \hat{y}_{\tau\tau}, \\ z(t,0) = 0, \quad z_x(t,L) = u_1(t), \\ z(0,x) = 0, \quad z_t(0,x) = -\varepsilon \hat{y}_\tau(0,x). \end{cases} \tag{5}$$

Notice that, if the nonlinearity f and the residual term r were equal to zero, then, as explained previously, setting $u_1(t) = -\alpha z_t(t,L)$, the energy function

$$t \mapsto \int_0^L (z_t(t,x)^2 + z_x(t,x)^2)\, dt$$

would be exponentially decreasing. This suggests to seek the control function $u_1(t)$ in the form

$$u_1(t) = -\alpha z_t(t,L) + v(t),$$

where $\alpha > 0$ will be chosen later. Set

$$w(t,x) := z(t,x) - \frac{x(x-L)}{L} v(t). \tag{6}$$

This leads to the system

$$\begin{cases} w_{tt} = w_{xx} + f'(\hat{y})w - \dfrac{x(x-L)}{L} v'' + \left(\dfrac{x(x-L)}{L} f'(\hat{y}) + \dfrac{2}{L} \right) v + r(\varepsilon, t, x), \\ w(t,0) = 0, \quad w_x(t,L) = -\alpha w_t(t,L), \\ w(0,x) = -\dfrac{x(x-L)}{L} v(0), \quad w_t(0,x) = -\varepsilon \hat{y}_\tau(0,x) - \dfrac{x(x-L)}{L} v'(0), \end{cases} \tag{7}$$

where

$$r(\varepsilon,t,x) = \left(w + \frac{x(x-L)}{L} v \right)^2 \int_0^1 (1-s) f'' \left(\hat{y} + s \left(w + \frac{x(x-L)}{L} v \right) \right) ds - \varepsilon^2 \hat{y}_{\tau\tau}. \tag{8}$$

The aim is to prove that, given a neighborhood $\mathcal{V}$ of $(0,0,0,0)$ in $\mathbb{R} \times \mathbb{R} \times H^1_{(0)}(0,L) \times L^2(0,L)$, for $\varepsilon > 0$ small enough, there exists a pair (v, w) solution of (7), satisfying $v(0) = v'(0) = 0$, such that

$$(v(1/\varepsilon), v'(1/\varepsilon), w(1/\varepsilon, \cdot), w_t(1/\varepsilon, \cdot)) \in \mathcal{V}.$$

To achieve this, we shall construct an appropriate control function and a Lyapunov functional which stabilizes system (7) to 0.

Remark 8. Let us set an upper bound to the residual term r. First, it is not difficult to check that there exists a constant C_1 such that, if $|v(t)| + \|w(t,\cdot)\|_{L^\infty(0,L)} \leq 1$, then the inequality

$$\|r(\varepsilon,t,\cdot)\|_{L^\infty(0,L)} \leq C_1 \big(\varepsilon^2 + v(t)^2 + \|w(t,\cdot)\|^2_{L^\infty(0,L)} \big)$$

holds. Moreover, since $w(t,0) = 0$, we can assert that there exists a constant C_2 such that, if $|v(t)| + \|w(t,\cdot)\|_{L^\infty(0,L)} \leq 1$, then

$$\|r(\varepsilon,t,\cdot)\|_{L^\infty(0,L)} \leq C_2\big(\varepsilon^2 + v(t)^2 + \|w_x(t,\cdot)\|^2_{L^2(0,L)}\big). \tag{9}$$

This *a priori* estimate shall be used later.

2.3. *Asymptotic Riesz spectral analysis of the operator*

The proof is based on a spectral analysis of the operator representing the system (7). In what follows, we set

$$H := \left\{ \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \in H^1((0,L),\mathbb{C}) \times L^2((0,L),\mathbb{C}) \mid w^1(0) = 0 \right\}. \tag{10}$$

Endowed with the scalar product

$$\left\langle \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}, \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \right\rangle_H := \int_0^L \left(\bar{w}^1_x z^1_x + \bar{w}^2 z^2\right) dx, \tag{11}$$

where the overbar denotes the complex conjugate, H is a complex Hilbertian space.

It is relevant to write (7) in the form

$$\begin{cases} w^1_t = w^2, \\ w^2_t = w^1_{xx} + f'(\hat{y})w^1 - \dfrac{x(x-L)}{L}v'' + \left(\dfrac{x(x-L)}{L}f'(\hat{y}) + \dfrac{2}{L}\right)v + r(\varepsilon,t,x), \\ w^1(t,0) = 0, \quad w^1_x(t,L) = -\alpha w^2(t,L), \\ w^1(0,x) = -\dfrac{x}{L}v(0), \quad w^2(0,x) = -\varepsilon\hat{y}_\tau(0,x) - \dfrac{x(x-L)}{L}v'(0), \end{cases} \tag{12}$$

and to introduce the *one-parameter family of linear operators*

$$\widetilde{A}(\tau) := \begin{pmatrix} 0 & 1 \\ A(\tau) & 0 \end{pmatrix}, \tag{13}$$

where $A(\tau) := \triangle + f'(\hat{y}(\tau,\cdot))\mathrm{Id}$, $\tau \in [0,1]$, on the domain

$$D(\widetilde{A}(\tau)) := \left\{ \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \in H \mid w^1 \in H^2((0,L),\mathbb{C}),\ w^2 \in H^1((0,L),\mathbb{C}), \right.$$
$$\left. w^2(0) = 0,\ w^1_x(L) = -\alpha w^2(L) \right\}, \tag{14}$$

so that

$$W_t(t,x) = \widetilde{A}(\varepsilon t)W(t,x) + v(t)a(\varepsilon t,x) + v''(t)b(x) + R(\varepsilon,t,x), \tag{15}$$

where

$$a(\tau,x) := \begin{pmatrix} 0 \\ \dfrac{x(x-L)}{L} f'(\hat{y}(\tau,x)) + \dfrac{2}{L} \end{pmatrix} =: \begin{pmatrix} a^1(\tau,x) \\ a^2(\tau,x) \end{pmatrix},$$

$$b(x) := \begin{pmatrix} 0 \\ -\dfrac{x(x-L)}{L} \end{pmatrix} =: \begin{pmatrix} b^1(x) \\ b^2(x) \end{pmatrix}, \tag{16}$$

$$W(t,x) := \begin{pmatrix} w^1(t,x) \\ w^2(t,x) \end{pmatrix}, \quad R(\varepsilon,t,x) := \begin{pmatrix} 0 \\ r(\varepsilon,t,x) \end{pmatrix}.$$

Recall that, by definition, the sequence $(\psi_j)_{j\in\mathbb{Z}}$ is a *Riesz basis* of the Hilbert space H if and only if there exists an equivalent scalar product on H for which $(\psi_j)_{j\in\mathbb{Z}}$ is orthonormal (see [10]); this is equivalent to the existence of positive constants A, B such that, for every sequence of complex scalars $(c_j)_{j\in\mathbb{Z}}$, there holds

$$A\sum_{j\in\mathbb{Z}} |c_j|^2 \leq \left\| \sum_{j\in\mathbb{Z}} c_j\psi_j \right\|_H^2 \leq B\sum_{j\in\mathbb{Z}} |c_j|^2. \tag{17}$$

An operator A on H is said to have compact resolvent whenever there exists a real α in the resolvent of A so that $(\alpha\,\mathrm{Id} - A)^{-1}$ is compact in H.

A nontrivial element $v \in H$ is called a *generalized eigenvector* of A (respectively, an *eigenvector* of A), associated to the eigenvalue λ, if there exists a positive integer n so that $(\lambda\,\mathrm{Id} - A)^n v = 0$ (respectively, if $(\lambda\,\mathrm{Id} - A)v = 0$). The *algebraic multiplicity* (respectively, the *geometric multiplicity*) of λ is defined as the number of linearly independent generalized eigenvectors (respectively, eigenvectors) associated to λ.

Recall that the spectrum of operator A on H having compact resolvent consists of isolated eigenvalues only, and each eigenvalue has finite algebraic multiplicity.

Lemma 2. *For every $\tau \in [0,1]$, the operator $\widetilde{A}(\tau)$ in H has compact resolvent, and thus its spectrum consists of isolated eigenvalues. There exists a Riesz basis $(\tilde{e}_k(\tau,\cdot))_{k\in\mathbb{Z}}$ of H, consisting of generalized eigenfunctions of $\widetilde{A}(\tau)$, associated to the eigenvalues $(\lambda_k(\tau))_{k\in\mathbb{Z}}$, such that*:

(i) $\tilde{e}_k(\tau,\cdot) \in D(\widetilde{A}(\tau))$, *and* $\|\tilde{e}_k(\tau,\cdot)\|_H = 1$, *for every* $k \in \mathbb{Z}$ *and every* $\tau \in [0,1]$;
(ii) *each eigenvalue* $\lambda_k(\tau)$ *is geometrically simple*;
(iii) *there exists an integer* $n_0 \geq 0$ *so that, for every integer* k *satisfying* $|k| > n_0$, *the eigenvalue* $\lambda_k(\tau)$ *is algebraically simple, and satisfies*

$$\lambda_k(\tau) = \frac{1}{2L}\ln\frac{\alpha-1}{\alpha+1} + i\frac{k\pi}{L} + \mathrm{O}\left(\frac{1}{|k|}\right), \tag{18}$$

as $|k| \to +\infty$, *uniformly for* $\tau \in [0,1]$;

(iv) *if* $|k| > n_0$, *then the generalized eigenfunction* $\tilde{e}_k(\tau,\cdot)$ *is an eigenfunction of* $\widetilde{A}(\tau)$, *associated to the (algebraically simple) eigenvalue* $\lambda_k(\tau)$, *and the functions*

$$\begin{aligned}[0,1] &\to \mathbb{C}\\ \tau &\mapsto \lambda_k(\tau),\end{aligned}$$

and

$$\begin{aligned}[0,1] &\to H\\ \tau &\mapsto \tilde{e}_k(\tau,\cdot),\end{aligned}$$

are of class C^1.

(v) *for every integer* $k > n_0$ *and every* $\tau \in [0,1]$,

$$\lambda_k(\tau) = \overline{\lambda_{-k}(\tau)} \quad \textit{and} \quad \tilde{e}_k(\tau,\cdot) = \overline{\tilde{e}_{-k}(\tau,\cdot)}. \tag{19}$$

Moreover, the Riesz basis $(\tilde{e}_k(\tau,\cdot))_{k\in\mathbb{Z}}$ *of* H *is uniform with respect to* $\tau \in [0,1]$, *in the sense that there exist positive real numbers* A *and* B *such that, for every sequence of complex scalars* $(c_j)_{j\in\mathbb{Z}}$, *there holds*

$$A\sum_{j\in\mathbb{Z}} |c_j|^2 \leq \left\| \sum_{j\in\mathbb{Z}} c_j \tilde{e}_j(\tau,\cdot) \right\|_H^2 \leq B\sum_{j\in\mathbb{Z}} |c_j|^2. \tag{20}$$

Remark 9. Uniform Riesz property (20) would be obvious if all eigenfunctions $\tilde{e}_k$ were continuous with respect to τ. However, the function $\tau \mapsto e_k(\tau,\cdot)$ may fail to be continuous whenever $|k| \leq n_0$, due to the fact that the eigenvalue $\lambda_k(\tau)$ is not necessarily algebraically simple.

Proof of Lemma 2. The fact that the operator $\widetilde{A}(\tau)$ has compact resolvent on H is obvious. For $\tau \in [0,1]$, let $\binom{w^1}{w^2}$ be an eigenfunction of $\widetilde{A}(\tau)$ associated to the eigenvalue λ. Then,

$$\begin{gathered} w^2 = \lambda w^1, \quad A(\tau)w^1 = \lambda w^2,\\ w^1(0) = w^2(0) = 0, \quad w^1_x(L) = -\alpha w^2(L). \end{gathered}$$

Therefore, w^1 satisfies the boundary value problem

$$\begin{cases} w^1_{xx} + f'(\hat{y})w^1 = \lambda^2 w^1,\\ w^1(0) = 0, \quad w^1_x(L) = -\lambda\alpha w^1(L). \end{cases}$$

If we assume that $|\lambda|$ tends to $+\infty$, then it is not difficult to show that, for every $x \in [0,L]$,

$$w^1(x) = \sinh\sqrt{\lambda^2 + \mathrm{O}(1)}\,x \quad \text{and} \quad w^1_x(x) = \sqrt{\lambda^2 + \mathrm{O}(1)}\,\cosh\sqrt{\lambda^2 + \mathrm{O}(1)}\,x, \tag{21}$$

as $|\lambda| \to +\infty$, uniformly with respect to $\tau \in [0,1]$ and $x \in [0,L]$. If we seek λ in the form $\lambda = -\theta + i\nu$, with ν large enough, then easy computations show that there exists an integer k so that, as $|k| \to +\infty$,

$$\nu = \frac{k\pi}{L} + \mathrm{O}\left(\frac{1}{|k|}\right) \quad \text{and} \quad \frac{\alpha - 1}{\alpha + 1}\mathrm{e}^{2\theta L} = 1,$$

and thus,

$$\lambda_k(\tau) = \frac{1}{2L}\ln\frac{\alpha - 1}{\alpha + 1} + i\frac{k\pi}{L} + \mathrm{O}\left(\frac{1}{|k|}\right). \tag{22}$$

Let us prove that each eigenvalue $\lambda_k(\tau)$ is geometrically simple. If not, let $\binom{w_1^1}{w_1^2}$ and $\binom{w_2^1}{w_2^2}$ be two independent eigenfunctions associated to the eigenvalue λ. Let us first point out that $w_1^1(L) \neq 0$. Indeed, if $w_1^1(L) = 0$, then w_1^1 satisfies

$$\begin{cases} {w_1^1}_{xx} + f'(\hat{y})w_1^1 = \lambda^2 w_1^1, \\ w_1^1(L) = {w_1^1}_x(L) = 0, \end{cases}$$

and thus $w_1^1 \equiv 0$. Since $w_1^2 = \lambda w_1^1$, one gets $\left(w_1^1, w_1^2\right) \equiv (0,0)$, which is a contradiction. If we set $w = w_2^1(L)w_1^1 - w_1^1(L)w_2^1$, then w satisfies

$$\begin{cases} w_{xx} + f'(\hat{y})w = \lambda^2 w, \\ w(L) = w_x(L) = 0, \end{cases}$$

and thus $w \equiv 0$, whence $\binom{w_1^1}{w_1^2}$ and $\binom{w_2^1}{w_2^2}$ are not linearly independent, which is a contradiction. Hence, the item (ii) of the lemma follows.

Let $(\tilde{e}_k(\tau,\cdot))_{k\in\mathbb{Z}}$ denote a complete set of generalized eigenfunctions of $\widetilde{A}(\tau)$, associated to the eigenvalues $(\lambda_k(\tau))_{k\in\mathbb{Z}}$, and such that the item (i) of the lemma holds. In order to prove that $(\tilde{e}_k(\tau,\cdot))_{k\in\mathbb{Z}}$ is a Riesz basis of H, we use Bari's Theorem (see for instance [10, Theorem 2.3, p. 317], see also [11, Theorem 6.3]). From this result, if we are able to exhibit a Riesz basis $(\phi_k)_{k\in\mathbb{Z}}$ of H which is quadratically close to $(\tilde{e}_k(\tau,\cdot))_{k\in\mathbb{Z}}$, that is,

$$\sum_k \|\phi_k(\cdot) - \tilde{e}_k(\tau,\cdot)\|_H^2 < \infty,$$

then the sequence $(\tilde{e}_k(\tau,\cdot))_{k\in\mathbb{Z}}$ is a Riesz basis of H.

To this aim, we introduce in H the operator

$$\tilde{A}_0 := \begin{pmatrix} 0 & 1 \\ \triangle & 0 \end{pmatrix}$$

on the same domain (14) than $\widetilde{A}(\tau)$. Intuitively, this operator corresponds to a truncation of $\widetilde{A}(\tau)$, up to the compact part $f'(\hat{y}(\tau,\cdot))\mathrm{Id}$. Bari's theorem, and simple computations, all of them detailed in [22, Sec. 4, p. 667] show that the operator

$\tilde{A}_0$ admits a Riesz basis of eigenfunctions $(\phi_k)_{k\in\mathbb{Z}}$, associated to the eigenvalues $(\mu_k)_{k\in\mathbb{Z}}$, so that there holds, for every integer k,

$$\mu_k = \frac{1}{2L}\ln\frac{\alpha-1}{\alpha+1} + i\frac{k\pi}{L},$$

and

$$\phi_k = \begin{pmatrix} \phi_k^1 \\ \phi_k^2 \end{pmatrix},$$

where

$$\phi_k^1(x) = \frac{1}{A_k}\sinh\,\mu_k x, \quad \phi_k^2(x) = \frac{\mu_k}{A_k}\cosh\,\mu_k x,$$

with

$$A_k = \frac{1}{2L\sqrt{-\mathrm{Re}(\mu_k)}}\sqrt{(\mathrm{e}^{-2\mathrm{Re}(\mu_k)L} - \mathrm{e}^{2\mathrm{Re}(\mu_k)L})(k^2\pi^2 + (\mathrm{Re}(\mu_k))^2L^2)}.$$

Moreover, the eigenvalues μ_k are algebraically simple as $|k| \to +\infty$. From expansions (18) and (21), we get easily, in H,

$$\tilde{e}_k(\tau,\cdot) = \phi_k(\cdot) + \mathrm{O}(1/k),$$

uniformly for $\tau \in [0,1]$. Hence, the family $(\phi_k)_{k\in\mathbb{Z}}$ is quadratically close to $(\tilde{e}_k(\tau,\cdot))_{k\in\mathbb{Z}}$, uniformly for $\tau \in [0,1]$. The proof of Bari's Theorem in [10, Theorem 2.3, p. 317] readily extends to our case, and the uniform Riesz property (20) follows. Moreover, the eigenvalues $\lambda_k(\tau)$ are algebraically simple as $|k| \to +\infty$.

In particular, with the formula (22), the item (iii) follows.

Moreover, it is a standard fact that, if $|k| > n_0$, then the eigenfunction $\tilde{e}_k(\tau,\cdot)$ and the eigenvalue $\lambda_k(\tau)$ are C^1 functions of τ (see for instance [12, 21]). The item (iv) is proved.

Finally, note that it is possible to choose the eigenelements so that item (v) holds. Indeed, one just has to show that the operator $\widetilde{A}(\tau)$ admits (at least) a real eigenvalue. But this follows obviously from an homotopy argument using the operator $\tilde{A}_0$. □

Let $\widetilde{A}(\tau)^*$ denote the adjoint operator of $\widetilde{A}(\tau)$ on H. The following lemma is obvious.

Lemma 3. *For every $\tau \in [0,1]$, the domain of $\widetilde{A}(\tau)^*$ is given by*

$$D(\widetilde{A}(\tau)^*) = \Big\{ \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in H \mid z^1 \in H^2((0,L),\mathbb{C}),\ z^2 \in H^1((0,L),\mathbb{C}),$$
$$z^2(0) = 0,\ z_x^1(L) = \alpha z^2(L) \Big\}. \tag{23}$$

For every $\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in D(\widetilde{A}(\tau)^*)$*, there holds*

$$\widetilde{A}(\tau)^* \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = - \begin{pmatrix} z^2 + g \\ z^1_{xx} \end{pmatrix}, \tag{24}$$

where the function $g \in C^2([0, L], \mathbb{C})$ *is defined by*

$$\begin{cases} g_{xx} = f'(\hat{y}(\tau, \cdot))z^2, \\ g(0) = g_x(L) = 0. \end{cases}$$

We next introduce the dual Riesz basis $(\tilde{f}_j(\tau, \cdot))_{j\in\mathbb{Z}}$ of $(\tilde{e}_j(\tau, \cdot))_{j\in\mathbb{Z}}$. Recall that, by definition, there holds

$$\langle \tilde{f}_j(\tau, \cdot), \tilde{e}_k(\tau, \cdot)\rangle_H = \delta_{kj} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases}$$

and, moreover, $(\tilde{f}_j(\tau, \cdot))_{j\in\mathbb{Z}}$ is a Riesz basis on H of generalized eigenvectors of $\widetilde{A}(\tau)^*$ with associated eigenvalues $(\overline{\lambda_j(\tau)})_{j\in\mathbb{Z}}$.

Remark 10. Reasoning as in the proof of Lemma 2, we state that $(\tilde{f}_j(\tau, \cdot))_{j\in\mathbb{Z}}$ is a *uniform* Riesz basis on H, in the sense that there exist positive real numbers A' and B' such that, for every sequence of complex scalars $(c_j)_{j\in\mathbb{Z}}$, there holds

$$A' \sum_{j\in\mathbb{Z}} |c_j|^2 \leq \left\| \sum_{j\in\mathbb{Z}} c_j \tilde{f}_j(\tau, \cdot) \right\|_H^2 \leq B' \sum_{j\in\mathbb{Z}} |c_j|^2.$$

In particular, the sequence of real numbers $(\|\tilde{f}_j(\tau, \cdot)\|_H)_{j\in\mathbb{Z}}$ is bounded, uniformly for $\tau \in [0, 1]$.

In the sequel, we will need the following technical lemma.

Lemma 4. *There exists a constant* $C > 0$ *so that*

$$\|\tilde{e}_{k\tau}(\tau, \cdot)\|_H \leq \frac{C}{|k|} \quad \text{and} \quad \|\tilde{f}_{k\tau}(\tau, \cdot)\|_H \leq \frac{C}{|k|}, \tag{25}$$

for every integer k *satisfying* $|k| > n_0$*, and every* $\tau \in [0, 1]$.

Proof. Let k be an integer such that $|k| > n_0$. From Lemma 2, the eigenfunction $\tilde{e}_k(\tau, \cdot)$ is a C^1 function of τ. We consider the expansion of $\tilde{e}_{k\tau}(\tau, \cdot)$ as series in the Riesz basis $(\tilde{e}_j(\tau, \cdot))_{j\in\mathbb{Z}}$, convergent in H,

$$\tilde{e}_{k\tau}(\tau, \cdot) = \sum_{j\in\mathbb{Z}} \alpha_j^k(\tau)\tilde{e}_j(\tau, \cdot), \tag{26}$$

where $\alpha_j^k(\tau)$ is defined by

$$\alpha_j^k(\tau) := \langle \tilde{f}_j(\tau, \cdot), \tilde{e}_{k\tau}(\tau, \cdot)\rangle_H,$$

for every $\tau \in [0, 1]$, and all $j, k \in \mathbb{Z}$, with $|k| > n_0$.

Let us estimate $\alpha_j^k(\tau)$, for large values of $|j|$ and $|k|$. By definition of $\tilde{e}_k(\tau,\cdot)$, and from Lemma 2, we have, whenever $|k| > n_0$,

$$\widetilde{A}(\tau)\tilde{e}_k(\tau,\cdot) = \lambda_k(\tau)\tilde{e}_k(\tau,\cdot).$$

Since the domain of $\widetilde{A}(\tau)$ does not depend on τ, it is clear that $\tilde{e}_{k\tau}(\tau,\cdot) \in D(\widetilde{A}(\tau))$. Differentiating with respect to τ, we get

$$\widetilde{A}(\tau)\tilde{e}_{k\tau}(\tau,\cdot) = \lambda_k'(\tau)\tilde{e}_k(\tau,\cdot) + \lambda_k(\tau)\tilde{e}_{k\tau}(\tau,\cdot) - \widetilde{A}'(\tau)\tilde{e}_k(\tau,\cdot),$$

and thus, taking the scalar product with $f_j(\tau,\cdot)$, $j \in \mathbb{Z}$, we get

$$\begin{aligned}\langle \widetilde{A}(\tau)^* \tilde{f}_j(\tau,\cdot), \tilde{e}_{k\tau}(\tau,\cdot)\rangle_H &= \langle \tilde{f}_j(\tau,\cdot), \widetilde{A}(\tau)\tilde{e}_{k\tau}(\tau,\cdot)\rangle_H \\ &= \lambda_k'(\tau)\delta_{kj} + \lambda_k(\tau)\alpha_j^k(\tau) \\ &\quad - \langle \tilde{f}_j(\tau,\cdot), \widetilde{A}'(\tau)\tilde{e}_k(\tau,\cdot)\rangle_H. \end{aligned} \tag{27}$$

We distinguish between two cases.

First case. If $|j| > n_0$, then $\widetilde{A}(\tau)^* \tilde{f}_j(\tau,\cdot) = \overline{\lambda_j(\tau)}\tilde{f}_j(\tau,\cdot)$, and thus (27) yields, for $j \neq k$,

$$\alpha_j^k(\tau)\lambda_j(\tau) = \lambda_k(\tau)\alpha_j^k(\tau) - \langle \tilde{f}_j(\tau,\cdot), \widetilde{A}'(\tau)\tilde{e}_k(\tau,\cdot)\rangle_H,$$

and for $j = k$,

$$\lambda_k'(\tau) = \langle \tilde{f}_k(\tau,\cdot), \widetilde{A}'(\tau)\tilde{e}_k(\tau,\cdot)\rangle_H.$$

Since $\lambda_j(\tau) \neq \lambda_k(\tau)$ whenever $j \neq k$, $|j| > n_0$, $|k| > n_0$, there holds

$$\alpha_j^k(\tau) = \frac{1}{\lambda_k(\tau) - \lambda_j(\tau)} \langle \tilde{f}_j(\tau,\cdot), \widetilde{A}'(\tau)\tilde{e}_k(\tau,\cdot)\rangle_H. \tag{28}$$

Clearly,

$$\widetilde{A}'(\tau) = \begin{pmatrix} 0 & 0 \\ f''(\hat{y}(\tau,\cdot))\hat{y}_\tau(\tau,\cdot) & 0 \end{pmatrix},$$

and thus, denoting

$$\tilde{e}_k(\tau,\cdot) = \begin{pmatrix} \tilde{e}_k^1(\tau,\cdot) \\ \tilde{e}_k^2(\tau,\cdot) \end{pmatrix} \quad \text{and} \quad \tilde{f}_j(\tau,\cdot) = \begin{pmatrix} \tilde{f}_j^1(\tau,\cdot) \\ \tilde{f}_j^2(\tau,\cdot) \end{pmatrix},$$

we get

$$\langle \tilde{f}_j(\tau,\cdot), \widetilde{A}'(\tau)\tilde{e}_k(\tau,\cdot)\rangle_H = \int_0^L f''(\hat{y}(\tau,x))\hat{y}_\tau(\tau,x)\tilde{e}_k^1(\tau,x)\overline{\tilde{f}_j^2(\tau,x)}\,dx.$$

Since $\tilde{e}_k(\tau,\cdot)$ is an eigenfunction of $\widetilde{A}(\tau)$, associated to the eigenvalue $\lambda_k(\tau)$, there holds $\tilde{e}_k^2(\tau,\cdot) = \lambda_k(\tau)\tilde{e}_k^1(\tau,\cdot)$. Moreover, from the estimate (18), $\lambda_k(\tau) \sim ik\pi/L$ as k tends to $+\infty$, uniformly for $\tau \in [0,1]$. Hence, there exists a constant C_1 so that, if $j \neq k$, $|j| > n_0$, $|k| > n_0$, then

$$|\alpha_j^k(\tau)| \leq \frac{C_1}{|k(j-k)|}, \tag{29}$$

for every $\tau \in [0,1]$.

Second case. If $|j| \leq n_0$, then it follows from Lemma 2 that

$$\widetilde{A}(\tau)^* \tilde{f}_j(\tau,\cdot) \in \operatorname{Span}\{\tilde{f}_p(\tau,\cdot) \mid -n_0 \leq p \leq n_0\},$$

for every $\tau \in [0,1]$. Thus,

$$\widetilde{A}(\tau)^* \tilde{f}_j(\tau,\cdot) = \sum_{p=-n_0}^{n_0} \beta_p^j(\tau) \tilde{f}_p(\tau,\cdot),$$

where

$$\beta_p^j(\tau) = \langle \widetilde{A}(\tau)^* \tilde{f}_j(\tau,\cdot), \tilde{e}_p(\tau,\cdot)\rangle_H = \langle \tilde{f}_j(\tau,\cdot), \widetilde{A}(\tau)\tilde{e}_p(\tau,\cdot)\rangle_H.$$

It is not difficult to see that all coefficients $\beta_p^j(\tau)$, with $p,j \in \{-n_0,\ldots,n_0\}$, are bounded, uniformly for $\tau \in [0,1]$.

Then, (27) yields

$$\sum_{p=-n_0}^{n_0} \beta_p^j(\tau)\alpha_p^k(\tau) = \lambda_k(\tau)\alpha_j^k(\tau) - \langle \tilde{f}_j(\tau,\cdot), \widetilde{A}'(\tau)\tilde{e}_k(\tau,\cdot)\rangle_H.$$

Setting

$$X(\tau) := \begin{pmatrix} \alpha_{-n_0}^k(\tau) \\ \vdots \\ \alpha_{n_0}^k(\tau) \end{pmatrix} \quad \text{and} \quad Y(\tau) := \begin{pmatrix} \langle \tilde{f}_{-n_0}(\tau,\cdot), \widetilde{A}'(\tau)\tilde{e}_k(\tau,\cdot)\rangle_H \\ \vdots \\ \langle \tilde{f}_{n_0}(\tau,\cdot), \widetilde{A}'(\tau)\tilde{e}_k(\tau,\cdot)\rangle_H \end{pmatrix},$$

the latter equations can be written as

$$(\lambda_k(\tau)I + M(\tau))X(\tau) = Y(\tau),$$

where the matrix $M(\tau)$ is bounded, uniformly for $\tau \in [0,1]$. If $|k|$ is large enough, then $|\lambda_k(\tau)| \sim |k|\pi/L$, thus the matrix $(\lambda_k(\tau)I + M(\tau))$ is invertible, and this yields readily the estimate

$$|\alpha_j^k(\tau)| \leq \frac{C_2}{k^2}, \tag{30}$$

for every $\tau \in [0,1]$, and for all integers j,k so that $|k| > n_0$ and $|j| \leq n_0$, where C_2 is a constant.

Finally, let us estimate $\alpha_k^k(\tau)$, for $|k| > n_0$. From Lemma 2, $\|\tilde{e}_k(\tau,\cdot)\|_H = 1$, and hence, if $|k| > n_0$, one gets, by differentiation with respect to τ,

$$\langle \tilde{e}_{k\tau}(\tau,\cdot), \tilde{e}_k(\tau,\cdot)\rangle_H = 0.$$

From (26), we infer that

$$\alpha_k^k(\tau) = -\sum_{\substack{j\in\mathbb{Z}\\ j\neq k}} \alpha_j^k(\tau)\langle \tilde{e}_k(\tau,\cdot), \tilde{e}_j(\tau,\cdot)\rangle_H. \tag{31}$$

Therefore, there exist constants C_3 and C_4 such that

$$|\alpha_k^k(\tau)| = \left|\langle \tilde{e}_k(\tau,\cdot), \sum_{\substack{j\in\mathbb{Z}\\ j\neq k}} \alpha_j^k(\tau)\tilde{e}_j(\tau,\cdot)\rangle_H\right| \leq \left\|\sum_{\substack{j\in\mathbb{Z}\\ j\neq k}} \alpha_j^k(\tau)\tilde{e}_j(\tau,\cdot)\right\|_H$$
$$\leq C_3\left(\sum_{\substack{j\in\mathbb{Z}\\ j\neq k}} |\alpha_j^k(\tau)|^2\right)^{1/2} \leq \frac{C_4}{|k|}, \tag{32}$$

for every $\tau \in [0,1]$.

It then follows from (26), and from the estimates (29), (30), and (32), that

$$\|\tilde{e}_{k\tau}(\tau,\cdot)\|_H = \left\|\sum_{j\in\mathbb{Z}} \alpha_j^k(\tau)\tilde{e}_j(\tau,\cdot)\right\|_H \leq C_3\left(\sum_{j\in\mathbb{Z}} |\alpha_j^k(\tau)|^2\right)^{1/2} \leq \frac{C}{|k|},$$

where C is a constant.

A similar reasoning is done for $\|\tilde{f}_{k\tau}(\tau,\cdot)\|_H$. The lemma is proved. □

Lemma 2 states the existence of a Riesz basis of H, consisting of generalized eigenfunctions $(\tilde{e}_k(\tau,\cdot))_{k\in\mathbb{Z}}$ of $\widetilde{A}(\tau)$, associated to the eigenvalues $(\lambda_k(\tau))_{k\in\mathbb{Z}}$. Note that, if $|k| \leq n_0$, then the function

$$\begin{aligned}[0,1] &\to H\\ \tau &\mapsto \tilde{e}_k(\tau,\cdot)\end{aligned}$$

may fail to be of class C^1, since the corresponding eigenvalue $\lambda_k(\tau)$ is not necessarily algebraically simple.

However, our proof of Theorem 1 requires the existence of a Riesz basis $(e_k(\tau,\cdot))_{k\in\mathbb{Z}}$, satisfying the conclusions of Lemma 2, and such that, *for every integer k, the function $\tau \mapsto e_k(\tau,\cdot)$ is of class C^1*.

Hence, we next modify the generalized eigenfunctions $\tilde{e}_k(\tau,\cdot)$, for $|k| \leq n_0$, so as to obtain new vectors $e_k(\tau,\cdot)$, $|k| \leq n_0$, that are C^1 functions of τ, but are not necessarily generalized eigenfunctions of $\widetilde{A}(\tau)$. The same is done for the dual Riesz basis $(\tilde{f}_k(\tau,\cdot))_{k\in\mathbb{Z}}$. More precisely, we prove the following lemma.

Lemma 5. *There exist a Riesz basis $(e_k(\tau,\cdot))_{k\in\mathbb{Z}}$ of H, having a dual Riesz basis $(f_k(\tau,\cdot))_{k\in\mathbb{Z}}$, such that*:

(i) $e_k(\tau,\cdot) \in D(\widetilde{A}(\tau))$, *and* $\|e_k(\tau,\cdot)\|_H = 1$, *for every* $k \in I$ *and every* $\tau \in [0,1]$;
(ii) *for every integer k, the functions $\tau \mapsto e_k(\tau,\cdot)$ and $\tau \mapsto f_k(\tau,\cdot)$ are of class C^1 on* $[0,1]$;
(iii) *if $|k| > n_0$, then $e_k(\tau,\cdot)$ is an eigenfunction of $\widetilde{A}(\tau)$, associated to the* (*algebraically simple*) *eigenvalue $\lambda_k(\tau)$, and $f_k(\tau,\cdot)$ is an eigenfunction of $\widetilde{A}(\tau)^*$, associated to the* (*algebraically simple*) *eigenvalue* $\overline{\lambda_k(\tau)}$;

(iv) *for every integer* $k > n_0$ *and every* $\tau \in [0,1]$, *one has* $e_k(\tau,\cdot) = \overline{e_{-k}(\tau,\cdot)}$ *and* $f_k(\tau,\cdot) = \overline{f_{-k}(\tau,\cdot)}$;

(v) *for every integer* k *so that* $|k| \leq n_0$, *there holds*

$$\widetilde{A}(\tau)e_k(\tau,\cdot) \in \mathrm{Span}\{e_p(\tau,\cdot) \mid |p| \leq n_0\},$$

and

$$\widetilde{A}(\tau)^* f_k(\tau,\cdot) \in \mathrm{Span}\{f_p(\tau,\cdot) \mid |p| \leq n_0\}.$$

Proof. For every integer k so that $|k| > n_0$, set

$$e_k(\tau,\cdot) := \tilde{e}_k(\tau,\cdot) \quad \text{and} \quad f_k(\tau,\cdot) := \tilde{f}_k(\tau,\cdot).$$

Then, items (i)–(iii) hold. Moreover, for $|k| > n_0$, the functions $\tau \mapsto e_k(\tau,\cdot)$ and $\tau \mapsto f_k(\tau,\cdot)$ are of class C^1. We proceed with an induction argument. Assume that, for every $\tau \in [0,1]$, the subspace of H

$$E(\tau) := \overline{\mathrm{Span}\{e_k(\tau,\cdot) \mid |k| > n_0\}}$$

is of codimension $2n_0 + 1$. Let us construct $e_{n_0} \in C^1([0,1], H)$.

We first prove that there exists $x \in C^1([0,1], H)$ such that $x(\tau) \notin E(\tau)$, for every $\tau \in [0,1]$. Since there does not exist necessarily an element $a \in H$ such that $a \notin E(\tau)$ for every $\tau \in [0,1]$, we deal with a subdivision of $[0,1]$, and construct x using piecewise constant functions. By a compactness argument, it is clear that there exists an integer m, and elements $a_1, \ldots, a_m$ of H, such that $a_i \notin E(\tau)$, for every $\tau \in [\frac{i-1}{m}, \frac{i+1}{m}]$, and every $i \in \{1, \ldots, m-1\}$.

Assume that there exists $t_1 \in [0,1]$ such that $t_1 a_1 + (1-t_1)a_2 \in E\big(\frac{3}{2m}\big)$. Clearly, $0 < t_1 < 1$. Then, $ta_1 - (1-t)a_2 \notin E\big(\frac{3}{2m}\big)$, for every $t \in [0,1]$. Indeed, by contradiction, assume that there exists $t_2 \in [0,1]$ such that $t_2 a_1 - (1-t_2)a_2 \in E\big(\frac{3}{2m}\big)$. Necessarily, $0 < t_2 < 1$. Then, the linear combination

$$\frac{1-t_2}{1-t_1}(t_1 a_1 + (1-t_1)a_2) + t_2 a_1 - (1-t_2)a_2 = \left(t_1\frac{1-t_2}{1-t_1} + t_2\right)a_1$$

is an element of $E\big(\frac{3}{2m}\big)$. This yields a contradiction, since $a_1 \notin E\big(\frac{3}{2m}\big)$.

Finally, replacing if necessary a_2 by $-a_2$, we proved that

$$ta_1 + (1-t)a_2 \notin E\left(\frac{3}{2m}\right),$$

for every $t \in [0,1]$.

Then, using an easy induction argument, we may assume that

$$ta_i + (1-t)a_{i+1} \notin E\left(\frac{2i+1}{2m}\right), \quad i = 1, \ldots, m-2,$$

for every $t \in [0,1]$ (see Fig. 1).

Therefore, by continuity, there exists $\varepsilon > 0$ such that

$$ta_i + (1-t)a_{i+1} \notin E(\tau),$$

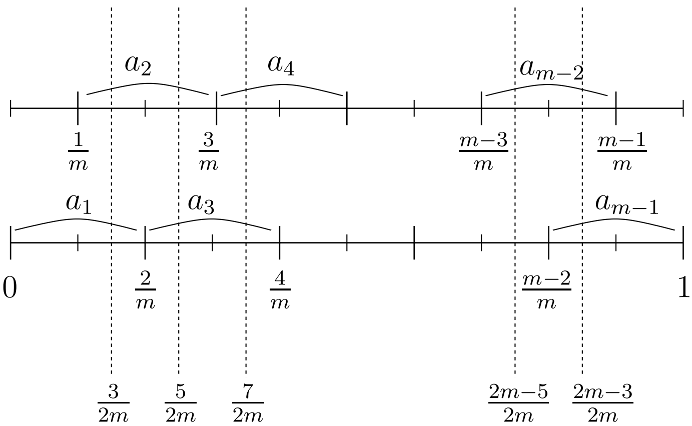

Fig. 1. Construction.

for every $\tau \in \left[\frac{2i+1}{2m} - \varepsilon, \frac{2i+1}{2m} + \varepsilon\right]$, and every $i \in \{1, \ldots, m-2\}$. As a consequence, the function y, defined as a piecewise linear continuous function, by

$$y(\tau) := \begin{cases} a_1 & \text{if } 0 \leq \tau \leq \dfrac{3}{2m} - \varepsilon, \\ a_i & \text{if } \dfrac{i-1}{m} + \varepsilon \leq \tau \leq \dfrac{i+1}{m} - \varepsilon, \\ a_{m-1} & \text{if } \dfrac{2m-3}{2m} + \varepsilon \leq \tau \leq 1, \\ \dfrac{a_{i+1} - a_i}{2\varepsilon}\tau + a_i & \\ \qquad - \dfrac{a_{i+1} - a_i}{2\varepsilon}\left(\dfrac{2i+1}{2m} - \varepsilon\right) & \text{if } \dfrac{2i+1}{2m} - \varepsilon \leq \tau \leq \dfrac{2i+1}{2m} + \varepsilon, \end{cases}$$

satisfies $y(\tau) \notin E(\tau)$, for every $\tau \in [0,1]$.

Using a convolution argument, we easily deduce the existence of $x \in C^1([0,1], H)$ such that $x(\tau) \notin E(\tau)$, for every $\tau \in [0,1]$.

Define $e_{n_0} : [0,1] \times [0,L] \to H$ by

$$e_{n_0}(\tau, \cdot) := \frac{x(\tau) - \displaystyle\sum_{|k|>n_0} \left\langle f_k(\tau, \cdot), x(\tau) \right\rangle_H e_k(\tau, \cdot)}{\left\| x(\tau) - \displaystyle\sum_{|k|>n_0} \left\langle f_k(\tau, \cdot), x(\tau) \right\rangle_H e_k(\tau, \cdot) \right\|_H}.$$

Using the estimate (25) of Lemma 4, the function $\tau \mapsto e_{n_0}(\tau, \cdot)$ is well defined, is of class C^1, and, by construction,

$$\|e_{n_0}(\tau, \cdot)\|_H = 1,$$

and

$$\langle f_k(\tau,\cdot), e_{n_0}(\tau,\cdot)\rangle_H = 0,$$

for $|k| > n_0$, and for every $\tau \in [0,1]$.

Define $f_{n_0} : [0,1]\times[0,L] \to H$ by

$$f_{n_0}(\tau,\cdot) := e_{n_0}(\tau,\cdot) - \sum_{|k|>n_0} \big\langle e_{n_0}(\tau,\cdot), e_k(\tau,\cdot)\big\rangle_H f_k(\tau,\cdot).$$

Using the estimate (25) of Lemma 4, the function $\tau \mapsto f_{n_0}(\tau,\cdot)$ is well defined, is of class C^1, and one has

$$\langle f_{n_0}(\tau,\cdot), e_{n_0}(\tau,\cdot)\rangle_H = 1,$$

and

$$\langle f_{n_0}(\tau,\cdot), e_k(\tau,\cdot)\rangle_H = 0,$$

for $|k| > n_0$, and for every $\tau \in [0,1]$.

For all integers k, l so that $|k| \leq n_0$ and $|l| > n_0$, there holds, by construction,

$$\langle f_l(\tau,\cdot), \widetilde{A}(\tau)e_k(\tau,\cdot)\rangle_H = \langle \widetilde{A}(\tau)^* f_l(\tau,\cdot), e_k(\tau,\cdot)\rangle_H = \lambda_l(\tau)\langle f_l(\tau,\cdot), e_k(\tau,\cdot)\rangle_H = 0,$$

and the item (v) follows easily. □

Remark 11. Denote $e_k(\tau,\cdot) = \binom{e_k^1(\tau,\cdot)}{e_k^2(\tau,\cdot)}$, for every integer k. If $|k| > n_0$, then, from (iii) in Lemma 5, there holds

$$e_k^2(\tau,\cdot) = \lambda_k(\tau) e_k^1(\tau,\cdot),$$

and

$$\begin{aligned} e^1_{k\,xx}(\tau,\cdot) + f'(\hat{y}(\tau,\cdot))e_k^1(\tau,\cdot) &= \lambda_k(\tau)^2 e_k^1(\tau,\cdot), \\ e_k^1(\tau,0) = 0, \quad e^1_{k\,x}(\tau,L) &= -\alpha\lambda_k(\tau)e_k^1(\tau,L). \end{aligned}$$

Similarly, denote $f_j(\tau,\cdot) = \binom{f_j^1(\tau,\cdot)}{f_j^2(\tau,\cdot)}$, for every integer j. If $|j| > n_0$, then

$$\begin{cases} f^2_{j\,xx}(\tau,\cdot) + f'(\hat{y}(\tau,\cdot))f_j^2(\tau,\cdot) = \overline{\lambda_j(\tau)}^2 f_j^2(\tau,\cdot), \\ f_j^2(\tau,0) = 0, \quad f^2_{j\,x}(\tau,L) = -\alpha\overline{\lambda_j(\tau)} f_j^2(\tau,L), \end{cases} \tag{33}$$

and

$$\begin{cases} f^1_{j\,xx}(\tau,\cdot) = -\overline{\lambda_j(\tau)} f_j^2(\tau,\cdot), \\ f_j^1(\tau,0) = 0, \quad f^1_{j\,x}(\tau,L) = \alpha f_j^2(\tau,L), \end{cases} \tag{34}$$

for every $\tau \in [0,1]$.

2.4. *The finite dimensional unstable part of the system*

Let $\alpha > 1$ so that

$$\frac{1}{2L}\ln\frac{\alpha-1}{\alpha+1} < -1.$$

Using (18), only a finite number of eigenvalues may have a nonnegative real part as $\tau \in [0,1]$. More precisely, there exists an integer n so that

$$\forall\,\tau \in [0,1], \quad \forall\,k \in \mathbb{Z}, \quad (|k| > n) \Rightarrow (\mathrm{Re}(\lambda_k(\tau)) < -1). \tag{35}$$

Without loss of generality, we suppose that $n \geq n_0$. Therefore, from Lemma 5, each eigenvalue $\lambda_k(\tau)$, with $|k| > n$, is algebraically simple, and satisfies $\mathrm{Re}(\lambda_k(\tau)) < -1$.

Remark 12. Note that the integer n can be arbitrarily large. For example, if $f(y) = y^3$ and if $y_1'(0) \to +\infty$, then $n \to +\infty$.

Every solution $W(t,\cdot) \in D(\widetilde{A}(\tau))$ of (15) can be expanded as series in the Riesz basis $(e_j(\varepsilon t,\cdot))_{j\in\mathbb{Z}}$ of H, convergent in H,

$$W(t,\cdot) = \begin{pmatrix} w^1(t,\cdot) \\ w^2(t,\cdot) \end{pmatrix} = \sum_{j=-\infty}^{\infty} w_j(t) e_j(\varepsilon t,\cdot). \tag{36}$$

Note that, for integers k satisfying $|k| \leq n$, the eigenvalue $\lambda_k(\tau)$ may be real, and/or nonalgebraically simple. Since $W(t,x) \in \mathbb{R}^2$, one has $w_j(t) = \overline{w_{-j}(t)}$, for every $j > n$, and hence

$$W(t,\cdot) = \pi_1(\varepsilon t)W(t,\cdot) + 2\mathrm{Re}\left(\sum_{j=n+1}^{+\infty} w_j(t)e_j(\varepsilon t,\cdot)\right),$$

where $\pi_1(\tau)$ denotes the projection from H onto $\mathrm{Span}\{e_p(\tau,\cdot) \mid |p| \leq n\}$, defined by

$$\pi_1(\tau)h = \sum_{j=-n}^{n} \langle f_j(\tau,\cdot), h\rangle e_j(\tau,\cdot),$$

for every $h \in H$. By construction, it is quite clear that $\widetilde{A}(\tau)$ and $\pi_1(\tau)$ commute. In what follows, $\mathrm{Im}\,\pi_1(\tau)$ is identified to $\mathbb{R}^{2n+1}$, and we denote by $\widetilde{A}_1(\tau)$ the $(2n+1) \times (2n+1)$ matrix (with real coefficients) representing the restriction of $\widetilde{A}(\tau)$ on $\mathrm{Im}\ \pi_1(\tau)$.

Lemma 6. *The mapping*

$$\begin{aligned} [0,1] &\to L(H,H) \\ \tau &\mapsto \pi_1(\tau) \end{aligned}$$

is of class C^1, and one has

$$\pi_1'(\tau)h = -\sum_{|j|>n} \langle f_{j_\tau}(\tau,\cdot), h\rangle_H e_j(\tau,\cdot) - \sum_{|j|>n} \langle f_j(\tau,\cdot), h\rangle_H e_{j_\tau}(\tau,\cdot), \tag{37}$$

for every $h \in H$.

Proof. For every $h \in H$, and every $\tau \in [0,1]$, one has, using Lemma 5,

$$\pi_1(\tau)h = h - \sum_{|j|>n} \langle f_j(\tau,\cdot), h\rangle_H e_j(\tau,\cdot).$$

For $|j| > n \geq n_0$, the eigenfunctions $e_j(\tau,\cdot)$ and $f_j(\tau,\cdot)$ are C^1 functions of τ. Using the estimates (25) of Lemma 4, the sum

$$-\sum_{|j|>n} \langle f_{j_\tau}(\tau,\cdot), h\rangle_H e_j(\tau,\cdot) - \sum_{|j|>n} \langle f_j(\tau,\cdot), h\rangle_H e_{j_\tau}(\tau,\cdot)$$

converges normally, and the conclusion follows. □

In the sequel, we are going to move, by means of an appropriate *feedback control*, the $2n+1$ eigenvalues $\lambda_0(\tau), \ldots, \lambda_n(\tau)$, whose real part may be nonnegative, without moving the others, so that all eigenvalues then have a negative real part. This pole-shifting process is the first part of the stabilization procedure (see, for instance, [13, 16] for details on this standard theory).

Set $W_1(t) = \pi_1(\varepsilon t)W(t,\cdot)$. Then, from (15),

$$W_1'(t) = \widetilde{A}_1(\varepsilon t)W_1(t) + v(t)a_1(\varepsilon t) + v''(t)b_1(\varepsilon t) + r_1(\varepsilon, t), \tag{38}$$

where

$$\begin{aligned} a_1(\varepsilon t) = \pi_1(\tau)a(\varepsilon t,\cdot), \quad b_1(\varepsilon t) = \pi_1(\varepsilon t)b(\cdot), \\ r_1(\varepsilon, t) = \pi_1(\varepsilon t)R(\varepsilon, t,\cdot) + \varepsilon\pi'(\varepsilon t)W(t,\cdot). \end{aligned} \tag{39}$$

Lemma 7. *There exists a constant C_3 such that, if $|v(t)| + \|w(t,\cdot)\|_{L^\infty(0,L)} \leq 1$, then*

$$\|r_1(\varepsilon, t)\|_H \leq C_3(\varepsilon^2 + v(t)^2 + \|W(t,\cdot)\|_H^2), \tag{40}$$

for every $t \in [0, 1/\varepsilon]$.

Proof. The estimate follows from Lemma 6, from the definition (16) of $R(\varepsilon, t,\cdot)$, and from the estimate (9). □

The system (38) is a differential system in $\mathbb{R}^{2n+1}$ controlled by v, v', v''. Set

$$\beta(t) := v'(t), \quad \gamma(t) := v''(t), \tag{41}$$

and consider now $v(t)$ and $\beta(t)$ as state coordinates, and $\gamma(t)$ as a control. Notice that $v(t)$, $\beta(t)$ and $\gamma(t)$ are *real numbers*. Then, the former finite dimensional system may be rewritten as

$$\begin{cases} v'(t) = \beta(t), \\ \beta'(t) = \gamma(t), \\ W_1'(t) = \widetilde{A}_1(\varepsilon t)W_1(t) + a_1(\varepsilon t)v(t) + b_1(\varepsilon t)\gamma(t) + r_1(\varepsilon, t). \end{cases} \tag{42}$$

Introducing the matrix notations

$$X_1(t) = \begin{pmatrix} v(t) \\ \beta(t) \\ W_1(t) \end{pmatrix}, \quad A_1(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ a_1(\tau) & 0 & \widetilde{A}_1(\tau) \end{pmatrix},$$

$$B_1(\tau) = \begin{pmatrix} 0 \\ 1 \\ b_1(\tau) \end{pmatrix}, \quad R_1(\varepsilon, t) = \begin{pmatrix} 0 \\ 0 \\ r_1(\varepsilon, t) \end{pmatrix},$$

we obtain

$$X_1'(t) = A_1(\varepsilon t)X_1(t) + B_1(\varepsilon t)\gamma(t) + R_1(\varepsilon, t). \tag{43}$$

Lemma 8. *For each* $\tau \in [0,1]$, *the pair* $(A_1(\tau), B_1(\tau))$ *satisfies the Kalman condition*, *i.e.*

$$\det(B_1(\tau), A_1(\tau)B_1(\tau), \ldots, A_1(\tau)^{2n+2}B_1(\tau)) \neq 0. \tag{44}$$

Proof. Let $\tau \in [0,1]$ be fixed. Consider the infinite dimensional linear control system

$$\begin{cases} v'(t) = \beta(t), \\ \beta'(t) = \gamma(t), \\ w_t(t,x) = \widetilde{A}(\tau)w(t,x) + v(t)a(\tau,x) + \gamma(t)b(x), \end{cases} \tag{45}$$

where the state is $(v(t), \beta(t), w(t,\cdot)) \in \mathbb{R} \times \mathbb{R} \times \mathrm{H}$, and the control is $\gamma(t) \in \mathbb{R}$. It is clear from Sec. 2.2 that this control system is equivalent to the boundary control system

$$\begin{cases} z_{tt}(t,x) = z_{xx}(t,x) + f'(\bar{y}(\tau,x))z(t,x), \\ z(t,0) = 0, \quad z_x(t,L) = -\alpha z_t(t,L) + v(t), \\ v'(t) = \beta(t), \quad \beta'(t) = \gamma(t), \end{cases} \tag{46}$$

which is a classical linear wave equation. Let $T > 2L$. It is well known that the linear control system (46), and hence the linear control system (45), is exactly controllable in time T (see [2]), namely, for all (v_0, β_0, w_0) and (v_1, β_1, w_1) in $\mathbb{R} \times \mathbb{R} \times \mathrm{H}$, there exists a control $\gamma \in L^2(0,T)$ such that the solution (v, β, w) of (45) associated to this control, starting from $(v(0), \beta(0), w(0,\cdot)) = (v_0, \beta_0, w_0)$, satisfies $(v(T), \beta(T), w(T,\cdot)) = (v_1, \beta_1, w_1)$. This implies in particular that the finite dimension linear control system

$$\dot{X}_1(t) = A_1(\tau)X_1(t) + B_1(\tau)\gamma(t)$$

is controllable in time T. Hence, the Kalman condition (44) holds. $\square$

It is well known that, for an autonomous finite dimensional linear control system, the Kalman condition, equivalent to the controllability of the system, implies the stabilizability of the system. This is not longer true for nonautonomous linear systems; however, this holds provided that the system is *slowly time varying*, whence the importance of the parameter ε. In these conditions, Lemma 8 implies the following result (see [13, Chap. 9.6]).

Corollary 2. *There exists a C^1 mapping $\tau \mapsto K_1(\tau)$ on $[0,1]$, where $K_1(\tau)$ is a $1 \times (2n+1)$ matrix with real coefficients, such that the matrix $A_1(\tau) + B_1(\tau)K_1(\tau)$ admits -1 as an eigenvalue of order $2n+3$, for every $\tau \in [0,1]$.*

Moreover, there exists a C^1 mapping $\tau \mapsto P(\tau)$ on $[0,1]$, where $P(\tau)$ is a $(2n+3) \times (2n+3)$ symmetric positive definite real matrix, such that the identity

$$P(\tau)(A_1(\tau) + B_1(\tau)K_1(\tau)) + {}^t(A_1(\tau) + B_1(\tau)K_1(\tau))P(\tau) = -I \tag{47}$$

holds, for every $\tau \in [0,1]$.

The gain matrix $K_1(\tau)$ permits to construct on $[0, 1/\varepsilon]$ the feedback control function

$$\gamma(t) = K_1(\varepsilon t)X_1(t), \tag{48}$$

that stabilizes the finite dimensional control system (43). We next prove that this feedback actually stabilizes the whole infinite dimensional system (15), provided $\varepsilon > 0$ is small enough.

2.5. *Construction of a Lyapunov functional*

Let us first write the differential equation satisfied by each (complex) coordinate $w_j(t) = \langle f_j(\varepsilon t, \cdot), W(t, \cdot)\rangle_H$, with $j > n$. There holds

$$w_j'(t) = \langle f_j(\varepsilon t, \cdot), W_t(t, \cdot)\rangle_H + \varepsilon \langle f_{j_\tau}(\varepsilon t, \cdot), W(t, \cdot)\rangle_H. \tag{49}$$

Since

$$\widetilde{A}(\varepsilon t)W(t, \cdot) = \pi_1(\varepsilon t)W(t, \cdot) + \sum_{|j|>n} \lambda_j(\varepsilon t)w_j(t)e_j(\varepsilon t, \cdot),$$

we get, using (15) and (16),

$$\langle f_j(\varepsilon t, \cdot), W_t(t, \cdot)\rangle_H = \lambda_j(\varepsilon t)w_j(t) + a_j(\varepsilon t)v(t) + b_j(\varepsilon t)v''(t) + \langle f_j(\varepsilon t, \cdot), R(\varepsilon, t, \cdot)\rangle_H,$$

where

$$\begin{aligned} a_j(\varepsilon t) &:= \langle f_j(\varepsilon t, \cdot), a(\varepsilon t, \cdot)\rangle_H = \int_0^L \overline{f_j^2(\varepsilon t, x)} \left(\frac{x(x-L)}{L} f'(\hat{y}(\varepsilon t, x)) + \frac{2}{L} \right) dx, \\ b_j(\varepsilon t) &:= \langle f_j(\varepsilon t, \cdot), b(\cdot)\rangle_H = -\int_0^L \overline{f_j^2(\varepsilon t, x)} \frac{x(x-L)}{L}\, dx. \end{aligned} \tag{50}$$

Equation (49) thus yields, for every $j > n$,

$$w_j'(t) = \lambda_j(\varepsilon t)w_j(t) + a_j(\varepsilon t)v(t) + b_j(\varepsilon t)v''(t) + r_j(\varepsilon, t), \tag{51}$$

where

$$r_j(\varepsilon, t) := \langle f_j(\varepsilon t, \cdot), R(\varepsilon, t, \cdot)\rangle_H + \varepsilon\langle f_{j_\tau}(\varepsilon t, \cdot), W(t, \cdot)\rangle_H. \tag{52}$$

The aim is now to construct a control Lyapunov functional in order to stabilize system (15), using the feedback control (48). For every $t \in [0, 1/\varepsilon]$, all $v, \beta \in \mathbb{R}$, and every $W(\cdot) = \binom{w^1(\cdot)}{w^2(\cdot)} \in H$, we set

$$E(t, v, \beta, W(\cdot)) := {}^t\overline{X_1(t)}P(\varepsilon t)X_1(t), \tag{53}$$

where $X_1(t)$ denotes the matrix vector in $\mathbb{C}^{2n+3}$

$$X_1(t) := \begin{pmatrix} v \\ \beta \\ \pi_1(\varepsilon t)W(\cdot) \end{pmatrix}.$$

For every $j \in \mathbb{Z}$, set

$$w_j(t) := \langle f_j(\varepsilon t, \cdot), W(\cdot)\rangle_H,$$

and define

$$N(t, W(\cdot)) := \frac{1}{2}\sum_{|j|>n} |w_j(t)|^2, \tag{54}$$

where $|\cdot|$ denotes the complex modulus. Finally, introduce

$$V(t, v, \beta, W(\cdot)) := cE(t, v, \beta, W(\cdot)) + N(t, W(\cdot)), \tag{55}$$

where c is a positive real number to be fixed later.

The rest of the section is devoted to prove that V is a Lyapunov functional for the system (15), with the feedback control (48).

In what follows, we will repeatedly use the equivalence of norms in finite dimension. The following notation will thus happen to be useful.

Notation. Let Λ be a set and $\Delta = \{(\varepsilon, t) \mid 0 < \varepsilon \leq 1,\ 0 \leq t \leq 1/\varepsilon\}$. Let F_1, F_2 and F_3 be real functions defined on $\Delta \times \Lambda$, and let $\theta \in [0, +\infty]$. The notation $F_1 \lesssim F_2$ on $F_3 \leq \theta$ means that $F_2 \geq 0$ and that there exists a positive constant C such that

$$\forall\, (\varepsilon, t) \in \Delta, \quad \forall\, \lambda \in \Lambda, \quad (F_3(\varepsilon, t, \lambda) \leq \theta) \Rightarrow (|F_1(\varepsilon, t, \lambda)| \leq CF_2(\varepsilon, t, \lambda)).$$

We say that $F_1 \sim F_2$ if both $F_1 \lesssim F_2$ and $F_2 \lesssim F_1$ hold on $F_3 \leq \theta$.

For the sake of simplicity, when the set Λ is clear from the context it will not be given explicitly.

Let $\|\cdot\|_2$ denote the Hermitian norm in $\mathbb{C}^{2n+3}$. Since $P(\tau)$ is real symmetric positive definite, we can write (with $\Lambda = \mathbb{C}^{2n+3}$)

$$E(t, v, \beta, W(\cdot)) = {}^t\overline{X_1(t)}P(\varepsilon t)X_1(t) \sim \|X_1(t)\|_2^2.$$

Since $W(\cdot) = \sum_{j\in\mathbb{Z}} w_j(t)e_j(\varepsilon t, \cdot)$, by definition of a Riesz basis (see (17)), and using the uniform property (20), we have

$$\begin{aligned} V(t, v, \beta, W(\cdot)) &\sim v^2 + \beta^2 + \sum_{j\in\mathbb{Z}} |w_j(t)|^2 \\ &\sim v^2 + \beta^2 + \|W(t,\cdot)\|_H^2 \\ &\sim v^2 + \beta^2 + \|w_x^1(\cdot)\|_{L^2(0,L)}^2 + \|w^2(\cdot)\|_{L^2(0,L)}^2. \end{aligned} \tag{56}$$

Remark 13. The meaning of V is the following. Except the first eigenmodes, the term N is equivalent to the classical energy of the wave equation, as explained in the introduction. As was shown previously, there exists a finite number of unstable modes. The term E is used to stabilize this unstable finite dimensional part of the system, and appears as a term of correction.

Let now $(v(t), \beta(t), W(t,\cdot))$ denote a solution of (15), in which we choose the control $\gamma(t)$ in the feedback form (48). Then,

$$W_t(t,\cdot) = \widetilde{A}(\varepsilon t)W(t,\cdot) + a(\varepsilon t,\cdot)v(t) + b(\cdot)K_1(\varepsilon t)X_1(t) + R(\varepsilon, t, \cdot). \tag{57}$$

Set

$$\begin{aligned} E_1(t) &:= E(t, v(t), \beta(t), W(t,\cdot)), \\ N_1(t) &:= N(t, w(t,\cdot)), \\ V_1(t) &:= V(t, W(t,\cdot)) = cE_1(t) + N_1(t). \end{aligned}$$

Let us compute $V_1'(t)$ and state a differential inequality satisfied by V_1. First of all, from (43) and (47), we get

$$\begin{aligned} E_1'(t) &= {}^t\overline{X_1'(t)}P(\varepsilon t)X_1(t) + {}^t\overline{X_1(t)}P(\varepsilon t)X_1'(t) + \varepsilon {}^t\overline{X_1(t)}P'(\varepsilon t)X_1(t) \\ &= -\|X_1(t)\|_2^2 + {}^t\overline{R_1(\varepsilon,t)}P(\varepsilon t)X_1(t) \\ &\quad + {}^t\overline{X_1(t)}P(\varepsilon t)R_1(\varepsilon,t) + \varepsilon {}^t\overline{X_1(t)}P'(\varepsilon t)X_1(t). \end{aligned} \tag{58}$$

Using the *a priori* estimate (40), we infer that, if

$$|v(t)| + \|w^1(t,\cdot)\|_{L^\infty(0,L)} \leq 1, \tag{59}$$

then

$$\|R_1(\varepsilon,t)\|_2 \lesssim \varepsilon^2 + v(t)^2 + N_1(t).$$

Hence, if (59) holds, then

$$\begin{aligned} |{}^t\overline{R_1(\varepsilon,t)}P(\varepsilon t)X_1(t) + {}^t\overline{X_1(t)}P(\varepsilon t)R_1(\varepsilon,t)| &\lesssim \|X_1(t)\|_2(\varepsilon^2 + v(t)^2 + N_1(t)) \\ &\lesssim \sqrt{E_1(t)}(\varepsilon^2 + E_1(t) + N_1(t)). \end{aligned}$$

On the other part, from Corollary 2, the mapping $\tau \mapsto P'(\tau)$ is bounded on $[0,1]$, hence

$$|\varepsilon {}^t\overline{X_1(t)}P'(\varepsilon t)X_1(t)| \lesssim \varepsilon\|X_1(t)\|_2^2 \lesssim \varepsilon E_1(t) \lesssim \varepsilon^2 + E_1(t)^2.$$

Therefore, using (58), there exists $\delta_1 > 0$ such that, if (59) holds, then

$$E_1'(t) + \delta_1 E_1(t) \lesssim \varepsilon^2 + E_1(t)^2 + \sqrt{E_1(t)}\left(\varepsilon^2 + E_1(t) + N_1(t)\right). \tag{60}$$

Hence, there exists $\rho_1 > 0$ such that, for every $\varepsilon \in (0,1]$, and for every $t \in [0, 1/\varepsilon]$ so that $E_1(t) + N_1(t) \leq \rho_1$,

$$E_1'(t) + \frac{\delta_1}{2} E_1(t) \lesssim \varepsilon^2 + N_1(t)^2. \tag{61}$$

Let us now handle $N_1(t)$. From (51), we have

$$\begin{aligned} N_1'(t) &= \mathrm{Re} \sum_{|j|>n} \overline{w_j(t)} w_j'(t) \\ &= \sum_{|j|>n} \mathrm{Re}(\lambda_j(\varepsilon t))|w_j(t)|^2 + \mathrm{Re} \sum_{|j|>n} \overline{w_j(t)}(a_j(\varepsilon t)v(t) \\ &\quad + b_j(\varepsilon t) K_1(\varepsilon t) X_1(\varepsilon t) + r_j(\varepsilon, t)). \end{aligned} \tag{62}$$

Clearly,

$$\begin{aligned} &\left| \sum_{|j|>n} \overline{w_j(t)}(a_j(\varepsilon t)v(t) + b_j(\varepsilon t) K_1(\varepsilon t) X_1(\varepsilon t)) \right| \\ &\quad \lesssim \sqrt{N_1(t)}(|v(t)| \|a(\varepsilon t, \cdot)\|_H + \|X_1(t)\|_2 \|b(\cdot)\|_H) \lesssim \sqrt{N_1(t)}\sqrt{E_1(t)}. \end{aligned} \tag{63}$$

The term $\sum \overline{w_j} r_j$ is more difficult to handle. First, from (52), we have

$$\begin{aligned} \sum_{|j|>n} \overline{w_j(t)} r_j(\varepsilon, t) = &\sum_{|j|>n} \overline{w_j(t)} \langle f_j(\varepsilon t, \cdot), R(\varepsilon, t, \cdot)\rangle_H \\ &+ \varepsilon \sum_{|j|>n} \overline{w_j(t)} \langle f_{j_\tau}(\varepsilon t, \cdot), W(t, \cdot)\rangle_H. \end{aligned} \tag{64}$$

Since $(f_j(\varepsilon t, \cdot))_{j \in \mathbb{Z}}$ is a Riesz basis of H, the first term is easily estimated by

$$\left| \sum_{|j|>n} \overline{w_j(t)} \langle f_j(\varepsilon t, \cdot), R(\varepsilon, t, \cdot)\rangle_H \right| \lesssim \sqrt{N_1(t)} \|R(\varepsilon, t, \cdot)\|_H,$$

and using the *a priori* estimate (9), and (16), we infer that

$$\left| \sum_{|j|>n} \overline{w_j(t)} \langle f_j(\varepsilon t, \cdot), R(\varepsilon, t, \cdot)\rangle_H \right| \lesssim \sqrt{N_1(t)}(\varepsilon^2 + v(t)^2 + \|W(t, \cdot)\|_H^2), \tag{65}$$

provided $|v(t)| + \|w^1(t, \cdot)\|_{L^\infty(0,L)} \leq 1$. Concerning the second term, we get from Lemma 4 the estimate

$$\left| \sum_{|j|>n} \overline{w_j(t)} \langle f_{j_\tau}(\varepsilon t, \cdot), W(t, \cdot)\rangle_H \right| \lesssim \sqrt{N_1(t)}\, \|W(t, \cdot)\|_H. \tag{66}$$

It follows from (64)–(66), that

$$\left| \sum_{|j|>n} \overline{w_j(t)} r_j(\varepsilon, t) \right| \lesssim \sqrt{N_1(t)}(\varepsilon^2 + v(t)^2 + \|W(t,\cdot)\|_H^2) + \varepsilon \sqrt{N_1(t)}\, \|W(t,\cdot)\|_H. \tag{67}$$

From (62), (63) and (67), we get, if (59) holds,

$$\begin{aligned} N_1'(t) - \sum_{|j|>n} \mathrm{Re}(\lambda_j(\varepsilon t))|w_j(t)|^2 &\lesssim \sqrt{N_1(t)}\sqrt{E_1(t)} + \varepsilon\sqrt{N_1(t)}\, \|W(t,\cdot)\|_H \\ &\quad + \sqrt{N_1(t)}(\varepsilon^2 + v(t)^2 + \|W(t,\cdot)\|_H^2). \end{aligned} \tag{68}$$

Using the estimates

$$\begin{gathered} v(t)^2 + \|W(t,\cdot)\|_H^2 \lesssim E_1(t) + N_1(t), \\ W(t,\cdot)\|_H \lesssim \sqrt{E_1(t)} + \sqrt{N_1(t)}, \end{gathered}$$

and the estimate (35) on the eigenvalues, namely, $\mathrm{Re}(\lambda_j(\varepsilon t)) \leq -1$ for $|j| > n$, we get from (68),

$$\begin{aligned} N_1'(t) + N_1(t) &\lesssim \sqrt{E_1(t)}\sqrt{N_1(t)} + \sqrt{N_1(t)}(\varepsilon^2 + E_1(t) + N_1(t)) \\ &\quad + \varepsilon\sqrt{N_1(t)}\big(\sqrt{E_1(t)} + \sqrt{N_1(t)}\big). \end{aligned} \tag{69}$$

Note that, for every $\theta \in (0, +\infty)$,

$$\begin{gathered} \sqrt{E_1(t)}\sqrt{N_1(t)} \leq \frac{\theta}{2} N_1(t) + \frac{1}{2\theta} E_1(t), \\ \varepsilon^2 \sqrt{N_1(t)} \leq \frac{\theta}{2} N_1(t) + \frac{1}{2\theta} \varepsilon^4, \\ \sqrt{N_1(t)} E_1(t) \leq \frac{\theta}{2} N_1(t) + \frac{1}{2\theta} E_1(t)^2. \end{gathered}$$

Hence, taking $\theta > 0$ small enough, using (69), we can assert the existence of positive real numbers $\varepsilon_0 > 0$ and $\rho_2 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and for every $t \in [0, 1/\varepsilon]$ so that $E_1(t) + N_1(t) \leq \rho_2$,

$$N_1'(t) + \frac{1}{2} N_1(t) \lesssim E_1(t) + \varepsilon^2. \tag{70}$$

Using (61), and setting $\rho = \min(\rho_1, \rho_2)$, there exists $\sigma_1 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and for every $t \in [0, 1/\varepsilon]$ so that $E_1(t) + N_1(t) \leq \rho$, there holds, for every $c > 0$,

$$cE_1'(t) + N_1'(t) + \frac{\delta_1 c}{2} E_1(t) + \frac{1}{2} N_1(t) \leq \sigma_1((1+c)\varepsilon^2 + E_1(t) + cN_1(t)^2).$$

Define the constant c by

$$c := \frac{2\sigma_1}{\delta_1}.$$

Then, the function $V_1(t) = cE_1(t) + N_1(t)$ satisfies the following estimate: there exists $\rho' > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, and for every $t \in [0, 1/\varepsilon]$ so that $V_1(t) \leq \rho'$, there holds

$$V_1'(t) \leq \sigma_1(1+c)\varepsilon^2.$$

Since $v(0) = 0$ and $\beta(0) = 0$, one has $V_1(0) \lesssim \varepsilon^2$ (see (12)), and thus there exist $\varepsilon_1 > 0$ and $\sigma_2 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_1]$, and for every $t \in [0, 1/\varepsilon]$,

$$V_1(t) \leq \sigma_2 \varepsilon.$$

In particular (see (4) and (6)),

$$\|y(1/\varepsilon, \cdot) - y_1(\cdot)\|_{H^1(0,L)} + \|y_t(1/\varepsilon, \cdot)\|_{L^2(0,L)} \leq \gamma\varepsilon,$$

where $\gamma > 0$ is a real number not depending on $\varepsilon \in (0, \varepsilon_1]$. This ends the proof of Theorem 1.

2.6. *Proof of Corollary 1*

The proof consists in solving a local exact controllability result. From the previous section, $y(1/\varepsilon, \cdot)$ belongs to an arbitrarily small neighborhood of $y_1(\cdot)$ in H^1-topology if ε is small enough, and our aim is now to construct a trajectory $q(t,x)$ solution of the control system steering $y(1/\varepsilon, \cdot)$ to $y_1(\cdot)$ in some time $T > 0$ (for instance $T = 1$), i.e.

$$\begin{cases} q_{tt} = q_{xx} + f(q), \\ q(t,0) = 0, \quad q_x(t,L) = u(t), \\ q(0,x) = y(1/\varepsilon, x), \quad q(T,x) = y_1(x). \end{cases}$$

Existence of such a solution q is given by [28]. Actually in [28] the function f is assumed to be globally Lipschitzian, but the local result we need here readily follows from the proofs and the estimates contained in this paper.

Indeed, let $T > 0$ and let $\tilde{f}$ be a globally Lipschitzian mapping such that

$$\tilde{f}(s) = f(s), \quad \forall\, s \in [-\|y_1\|_{L^\infty} - 1, \|y_1\|_{L^\infty} + 1]. \tag{71}$$

From the proof of [28], we get the existence of $\mu > 0$ such that there exists $z \in Y_T$ satisfying

$$\begin{cases} z_t = z_{xx} + \tilde{f}(z + y_1) - \tilde{f}(y_1), \\ z(t,0) = 0, \\ z(0,x) = y(1/\varepsilon, x) - y_1(x), \quad z(T,x) = 0, \end{cases}$$

and the estimate

$$\|z\|_{Y_T} \leq \mu \, \|y(1/\varepsilon, \cdot) - y_1(\cdot)\|_{H^1(0,L)}\,, \tag{72}$$

which leads, with $q = z + \tilde{y}_1$, to

$$\begin{cases} q_t = q_{xx} + \tilde{f}(q), \\ q(t,0) = 0, \\ q(0,x) = y(1/\varepsilon, x), \quad q(T,x) = y_1(x), \end{cases}$$

and

$$\|q - \tilde{y}_1\|_{Y_T} \leq \mu \|y(1/\varepsilon, \cdot) - y_1(\cdot)\|_{H^1(0,L)}, \tag{73}$$

where $\tilde{y}_1(t,x) := y_1(x)$. From (72) and (73), we get

$$\|q - \tilde{y}_1\|_{L^\infty((0,T)\times(0,L))} \leq 1 \tag{74}$$

for $\|y(1/\varepsilon, \cdot) - y_1(\cdot)\|_{H^1(0,L)}$ small enough. From (71) and (74), we infer that $\tilde{f}(q) = f(q)$, which ends the proof.

3. Numerical Simulations

Numerical simulations are led, using Matlab, with the function $f(y) = y^3$, that is, we deal with the boundary control system

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} + y^3, \\ y(t,0) = 0, \quad y_x(t,L) = u(t), \\ y(0,\cdot) = y_0(\cdot), \quad y_t(0,\cdot) = y_1(\cdot), \end{cases} \tag{75}$$

Fix $L = 1$. The set $\mathcal{S}$ of steady-states consists of all solutions $y(\cdot)$ of class C^2 on $[0,1]$ such that

$$y''(x) + y(x)^3 = 0, \quad y(0) = 0. \tag{76}$$

Since f is odd, this set is connected (see Remark 4). For the numerical simulations, we choose two steady-states y_0 and y_1, namely, $y_0 = 0$, and y_1 denotes the solution of (76) vanishing at 0, 1/2 and 1, and having no other zero on $[0,1]$ (see Fig. 2). Notice that all solutions of (76) can be explicitly computed using elliptic functions.

For every $\tau \in [0,1]$, define the function $\bar{y}(\tau,.)$ on $[0,1]$ as the solution of (76) such that

$$\frac{\partial \bar{y}}{\partial x}(\tau, 0) = \tau y_1'(0),$$

and set $\bar{u}(\tau) = \bar{y}(\tau, 1)$. The one-parameter family of linear operators (13) we have to deal with writes

$$\widetilde{A}(\tau) = \begin{pmatrix} 0 & 1 \\ \triangle + 3\bar{y}(\tau,\cdot)^2 \mathrm{Id} & 0 \end{pmatrix},$$

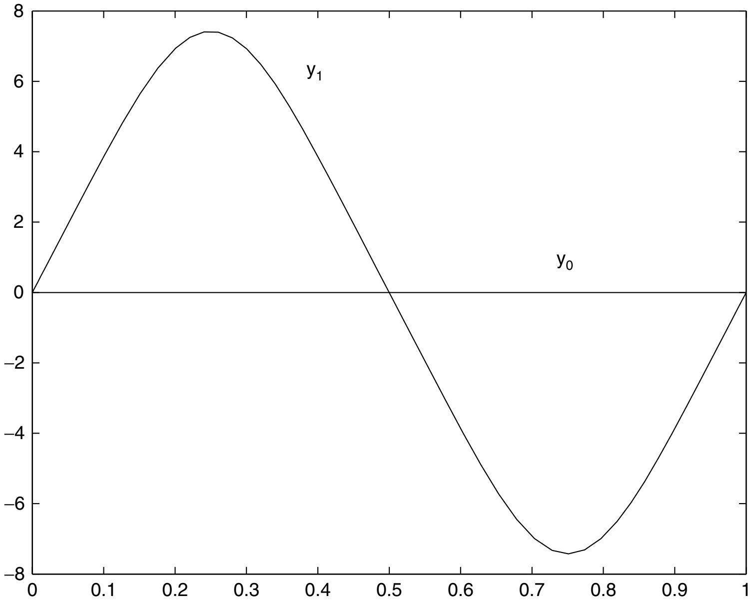

Fig. 2. Definition of the steady-states y_0 and y_1.

on the domain $D(\widetilde{A}(\tau))$ given by (14). For $\tau = 0$, there holds

$$\widetilde{A}(0) = \begin{pmatrix} 0 & 1 \\ \triangle & 0 \end{pmatrix},$$

and the eigenvalues and eigenvectors of $\widetilde{A}(0)$ are

$$\lambda_k(0) = \frac{1}{2L}\ln\frac{\alpha-1}{\alpha+1} + i\frac{k\pi}{L},$$

$$e_k(0,x) = \frac{1}{A_k}(\text{sh } \lambda_k(0)x, \lambda_k(0)\text{sh } \lambda_k(0)x),$$

where

$$A_k = \frac{1}{2L\sqrt{-\text{Re}(\lambda_k(0))}}\sqrt{(\mathrm{e}^{-2\text{Re}(\lambda_k(0))L} - \mathrm{e}^{2\text{Re}(\lambda_k(0))L})(k^2\pi^2 + (\text{Re}(\lambda_k(0)))^2L^2)}.$$

The dual Riesz basis $(f_k(0,\cdot))_{k\in\mathbb{Z}}$ is given by

$$f_k(0,x) = \frac{A_k}{B_k}(\overline{e_k^1(0,x)}, -\overline{e_k^2(0,x)}),$$

where

$$B_k = 2\sqrt{-\text{Re}(\lambda_k(0))}\frac{(\text{Re}(\lambda_k(0))L - ik\pi)^2}{\sqrt{(k^2\pi^2 + (\text{Re}(\lambda_k(0)))^2L^2)(\mathrm{e}^{-2\text{Re}(\lambda_k(0))L} - \mathrm{e}^{2\text{Re}(\lambda_k(0))L})}}.$$

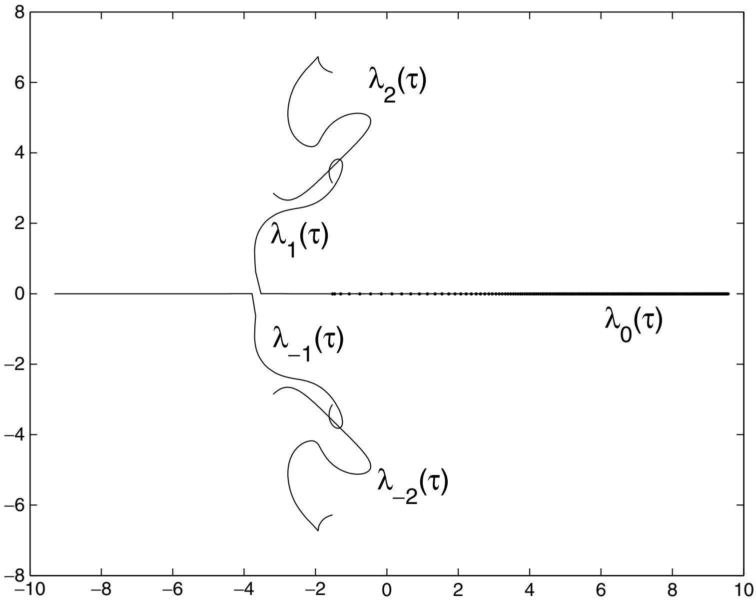

Fig. 3. First eigenvalues.

Then, solving by continuation as $\tau \in [0,1]$ boundary value problems, we compute numerically, using a standard finite difference code implemented in Matlab, or a simple shooting method, the first eigenvalues $\lambda_k(\tau)$.

Figure 3 represents the eigenvalues $\lambda_{-2}(\tau)$, $\lambda_{-1}(\tau)$, $\lambda_0(\tau)$, $\lambda_1(\tau)$, $\lambda_2(\tau)$, for $\tau \in [0,1]$. Numerically, we choose $L = 1$ and $\alpha = 1.1$. Then, the eigenvalue $\lambda_0(\tau)$ is real, passing from about -1.52 when $\tau = 0$, to about 9.66 when $\tau = 1$. The eigenvalues $\lambda_1(\tau)$ and $\lambda_{-1}(\tau)$ are complex and conjugate, up to about $\tau_0 = 0.31$. For $\tau = \tau_0$, the eigenvalue $\lambda_1(\tau_0)$ is double, and the corresponding eigenspace is of dimension one. For $\tau > \tau_0$, both eigenvalues are real, $\lambda_{-1}(\tau)$ is negative, whereas $\lambda_1(\tau)$ becomes positive. Finally, if $|k| \geq 2$, the eigenvalue $\lambda_k(\tau)$ is algebraically simple, complex, and of negative real part.

Hence, in this particular case, only the modes corresponding to $\lambda_0(\tau)$ and $\lambda_1(\tau)$ may become unstable.

From the algorithmic point of view, in order to avoid technical difficulties related to the computation of a Jordan normal form for the matrix $A_1(\tau)$ of the finite dimensional system (43), we compute numerically, by continuation, and using a simple shooting method, a basis of the three-dimensional real vector space

$$\ker(A_1(\tau) - \lambda_1(\tau)I)(A_1(\tau) - \lambda_0(\tau)I)(A_1(\tau) - \lambda_{-1}(\tau)I),$$

and a dual Riesz basis. Then, we implement a standard pole shifting procedure on this finite dimensional system (see, for instance, [13]).

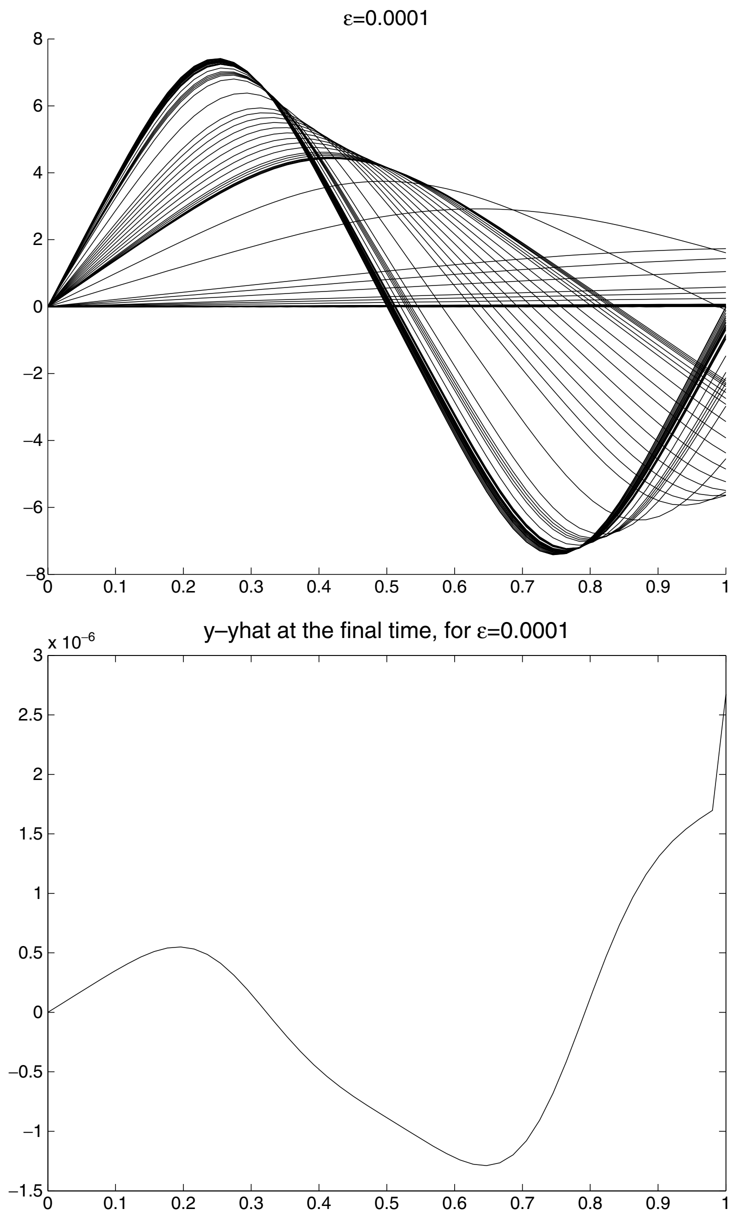

Fig. 4. Numerical simulations for $y(t, \cdot)$, where $t \in [0, 1/\varepsilon]$.

Results are drawn on Fig. 4, for $\varepsilon = 0.0001$. On the top figure is drawn $y(t, \cdot)$, for $t \in [0, 1/\varepsilon]$; on the bottom figure is represented $y(1/\varepsilon, \cdot) - \hat{y}(1/\varepsilon, \cdot)$. Notice that, if ε is not small enough, then the solution blows up, as expected (for example, $\varepsilon = 0.001$).

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