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Continuation from a flat to a round Earth model in the coplanar orbit transfer problem

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SUMMARY

In this article, we focus on the problem of minimization of the fuel consumption for the coplanar orbit transfer problem. This problem is usually solved numerically by a shooting method, based on the application of the Pontryagin Maximum Principle; however, the shooting method is known to be hard to initialize, and the convergence is difficult to obtain because of discontinuities of the optimal control. Several methods are known in order to overcome that problem; however, in this article, we introduce a new approach based on the following preliminary remark. When considering a 2D flat Earth model with constant gravity, the optimal control problem of passing from an initial configuration to some final configuration by minimizing the fuel consumption can be very efficiently solved, and the solution leads to a very efficient algorithm. Based on that, we propose a continuous deformation from this flat Earth model to a modified flat Earth model that is diffeomorphic to the usual round Earth model. The resulting numerical continuation process thus provides a new way to solve the problem of minimization of the fuel consumption for the coplanar orbit transfer problem. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The fuel efficient orbit transfer of a satellite is a widely studied problem (see [1, 2]). We can distinguish mainly between two formulations of this problem. The first one considers that the vehicle produces instantaneous change of velocity and is referred to as the impulse orbit transfer (see [3–5]). The second formulation takes into account the fact that all engines have a limited thrust, and that the vehicle's dynamics has to be continuous in the position and velocity coordinates (see [6]). In this continuous approach, we also separate the high-thrust and low-thrust transfer, depending on the magnitude of the available acceleration.

In this paper, we focus on the high-thrust orbit transfer that we furthermore restrict to be coplanar. This problem can be naturally written as an optimal control problem. There exist various numerical methods to solve such a problem, and we usually separate them in two classes: direct and indirect methods. Direct methods (e.g. surveyed in [7]) consist in discretizing the optimal control problem in order to rewrite it as a parametric optimization problem. Then, a nonlinear large-scale optimization solver is applied. The advantage of this approach is that, it is straightforward and is usually quite robust. The main drawback is that, because of the discretization step, those methods are

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computationally demanding and that they are not very accurate in general when compared with the indirect approach (see [7]). Indirect methods are based on the Pontryagin Maximum Principle (PMP, see [8]) that is a set of necessary conditions for a candidate trajectory and control strategy to be optimal. The idea is to use those necessary conditions to reduce the search of a solution to the search of the zero of the so-called shooting function (indirect methods are also called shooting methods in this context). The advantage is that shooting methods are very fast when they converge and that they produce high accuracy solutions. Their main drawback is that they typically use a Newton-like algorithm to look for the zero of the shooting function and thus, they may be hard to successfully initialize. We can also mention mixed methods that use a discretization of the PMP necessary conditions and then apply a large-scale optimization solver (see e.g. [9]).

Because of its fast convergence and high accuracy, we will turn to a shooting method to solve the coplanar orbit transfer problem with minimal fuel consumption. There already exist some methods to cope with the initialization drawback of this method. In [10], the authors use the impulse transfer solution to provide a good initial guess to the shooting algorithm. This method is based on the fact that limited thrust orbit transfer try to mimic impulse transfer, as outlined in [4, 11]. However, this approach is only valid for nearly circular initial and final orbits. In [12], a multiple shooting method parameterized by the number of thrust arcs is used to solve an Earth–Mars transfer, and the solving of an orbit transfer with n thrust arcs is based on the solution of the transfer with $n-1$ thrust arcs; however, no specific method to initialize this iterative process is discussed. In [13, 14], differential or simplicial continuation methods linking the minimization of the L^2-norm of the control to the minimization of the fuel consumption is used to solve the low-thrust orbit transfer around the Earth. However, this approach is not adapted for high-thrust transfer. In [15], simplified formulas are established by interpolating many numerical experiments, which permit to initialize successfully the shooting method for the minimal time orbit transfer problem, in a certain range of values and for nearly circular initial and final orbits. Based on that initial guess and on averaging techniques, the authors of [16] implement in the software T3D continuation and smoothing processes in order to solve minimal time or minimal fuel consumption orbit transfer problems.

In this article, we propose a novel way to initialize a shooting method for high-thrust coplanar orbit transfer with fixed final time. It is based on a continuation method starting with the solving of a simplified transfer on a flat Earth model and then continuously adding curvature to end up with the model we want to solve. Note that we restrict ourselves to fixed final time problems because it has already been numerically shown that the continuous transfer with maximization of the final mass does not have any solution (see [11, 12]).

This paper is organized as follows. First, we state the optimal control problem we want to solve along with the necessary conditions given by the PMP. Then, we introduce the simplified flat Earth model and modify it so as to introduce curvature and make it diffeomorphic to the round Earth model. The next section presents the continuation procedure and explains how to pass from the simplified model to the targeted optimal control problem. A refined analysis is then carried out to provide a robust and efficient algorithm to solve the simplified flat Earth model, which consists in simplifying and specializing the application of the shooting method, because of the particular structure of the problem. Finally, we give a numerical example in which we solve an orbit transfer from an unstable (on a collision course) Sun Synchronous Orbit to a nearly circular final orbit. Because our approach involves diffeomorphic changes of coordinates, we explain in the Appendix the impact of a change of coordinates onto the set of adjoint vectors of the PMP.

1.1. The round Earth model and the optimal control problem

The model that we use for the coplanar orbit transfer problem is the following. Assume that the Earth is spherical with center O and consider an inertial geocentric frame $(O, \vec{i}, \vec{j}, \vec{k})$. Because we consider the coplanar orbit transfer problem, we assume that the whole trajectory lies in the plane $O + \mathbb{R}\vec{i} + \mathbb{R}\vec{j}$. The satellite is modeled as a mass point $M(t)$, with $\overrightarrow{OM}(t) = r(t)\vec{e}_r$, where $(\vec{e}_r, \vec{e}_\varphi)$ denotes the usual Frénet frame defined by

$$\vec{e}_r = \sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}, \quad \vec{e}_\varphi = \cos\varphi\,\vec{i} - \sin\varphi\,\vec{j}.$$

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It is subject to the central gravitational field $g(r) = \frac{\mu}{r^2}$, where μ is the Earth gravitational parameter and to the thrust $\overrightarrow{T}(t) \in \mathbb{R}^2$. The mass of the satellite is denoted by $m(t)$. The vehicle follows the two-dimensional controlled Kepler equation

$$\frac{d^2\overrightarrow{OM}}{dt}(t) = -g(r(t))\vec{e}_r + \frac{\overrightarrow{T}(t)}{m(t)}, \quad \dot{m}(t) = -\beta\|\overrightarrow{T}(t)\|, \tag{1}$$

where $\beta > 0$ is the inverse of the thruster exhaust velocity. Moreover, the control $\overrightarrow{T}(\cdot)$ must satisfy the constraint

$$\|\overrightarrow{T}(\cdot)\| \leqslant T_{\max}, \tag{2}$$

where $T_{\max}$ is the maximal allowed thrust, and $\| \ \|$ denotes the usual Euclidean norm. Note that we do not consider any constraint on the direction of the thrust. However, such a constraint can be verified a posteriori, and the numerical results show that the thrust direction mainly lies in two narrow cones (one per thrust arc). This can lead to indications on how to design the vehicle so as to place the thrusters efficiently.

Instead of Cartesian coordinates, we next use polar coordinates whose definition is recalled. Recall that $r(t) = \|\overrightarrow{OM}(t)\|$, and set

$$v(t) = \left\|\frac{d\overrightarrow{OM}}{dt}(t)\right\| = \sqrt{\dot{r}(t)^2 + r(t)^2\dot{\varphi}(t)^2}.$$

Define the flight path angle $\gamma(t)$ by

$$\frac{d\overrightarrow{OM}}{dt}(t) = v(t)(\sin\gamma(t)\,\vec{e}_r + \cos\gamma(t)\,\vec{e}_\varphi).$$

Define the coordinates $q = (r, \varphi, v, \gamma, m)$, with (r, φ) the polar coordinates of the satellite, v its speed and γ the slope of the velocity vector. Then, the control system (1) is written in cylindrical coordinates as

$$\begin{aligned}
\dot{r}(t) &= v(t)\sin\gamma(t)\\
\dot{\varphi}(t) &= \frac{v(t)}{r(t)}\cos\gamma(t)\\
\dot{v}(t) &= -g(r(t))\sin\gamma(t) + \frac{T_{\max}}{m(t)}u_1(t)\\
\dot{\gamma}(t) &= \left(\frac{v(t)}{r(t)} - \frac{g(r(t))}{v(t)}\right)\cos\gamma(t) + \frac{T_{\max}}{m(t)v(t)}u_2(t)\\
\dot{m}(t) &= -\beta T_{\max}\|u(t)\|
\end{aligned} \tag{3}$$

where the normalized control $u(t) = (u_1(t), u_2(t))$ satisfies $T(t) = u(t)T_{\max}$ and the constraint

$$\|u(t)\| = \sqrt{u_1(t)^2 + u_2(t)^2} \leqslant 1, \tag{4}$$

for almost every t. The optimal control problem under consideration then consists in steering the control system (3) from an initial configuration

$$r(0) = r_0,\ \varphi(0) = \varphi_0,\ v(0) = v_0,\ \gamma(0) = \gamma_0,\ m(0) = m_0, \tag{5}$$

to some final configuration that is either of the form

$$r(t_f) = r_f,\ v(t_f) = v_f,\ \gamma(t_f) = \gamma_f, \tag{6}$$

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or of the form

$$\xi_{K_f} = \frac{v(t_f)^2}{2} - \frac{\mu}{r(t_f)} - K_f = 0,$$
$$\xi_{e_f} = \sin^2\gamma + \left(1 - \frac{r(t_f)v(t_f)^2}{\mu}\right)^2 \cos^2\gamma - e_f^2 = 0. \tag{7}$$

The conditions (6) mean that the satellite has to enter a specified orbit at a given point of it. The conditions (7) mean that the satellite must be steered to a final elliptic orbit of energy $K_f < 0$ and eccentricity e_f, without fixing the entry point on that orbit (see [17] for the definition of K_f and e_f and their expression in Cartesian coordinates). Note that for both final conditions, the orientation of the final orbit is not prescribed ($\varphi(t_f)$ is free). The criterion to consider is the maximization of the final mass $m(t_f)$. As mentioned in [11, 12], this problem does not have a solution for free final time[‡]; and therefore, we assume the final time t_f to be fixed. In what follows, this optimal control problem is referred to as (**OCP**).

According to the PMP, every optimal trajectory $q(\cdot)$ of (**OCP**), associated with a control $u(\cdot)$ on $[0, t_f]$, is the projection of an extremal $(q(\cdot), p(\cdot), p^0, u(\cdot))$, where $p(\cdot)\colon [0, t_f] \to \mathbb{R}^5$ is an absolutely continuous mapping called *adjoint vector*, p^0 is a nonpositive real number, with $(p(\cdot), p^0) \neq (0, 0)$, and there holds

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t), p^0, u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t), p^0, u(t)), \tag{8}$$

for almost every $t \in [0, t_f]$, where the Hamiltonian is defined by

$$H(q, p, p^0, u) = p_r v \sin\gamma + p_\varphi \frac{v}{r}\cos\gamma + p_v\left(-g(r)\sin\gamma + \frac{T_{\max}}{m}u_1\right)$$
$$+ p_\gamma\left(\left(\frac{v}{r} - \frac{g(r)}{v}\right)\cos\gamma + \frac{T_{\max}}{mv}u_2\right) - p_m\beta T_{\max}\|u\|,$$

with $p = (p_r, p_\varphi, p_v, p_\gamma, p_m)$. This yields the adjoint equations

$$\dot{p}_r = p_\varphi\frac{v}{r^2}\cos\gamma - \frac{2}{r}p_v g(r)\sin\gamma + \frac{1}{r}p_\gamma\left(\frac{v}{r} - 2\frac{g(r)}{v}\right)\cos\gamma$$
$$\dot{p}_\varphi = 0$$
$$\dot{p}_v = -p_r\sin\gamma - \frac{1}{r}p_\varphi\cos\gamma - p_v\left(\frac{1}{r} + \frac{g(r)}{v^2}\right)\cos\gamma + p_\gamma\frac{T_{\max}}{mv^2}u_2$$
$$\dot{p}_\gamma = -p_r v\cos\gamma + p_\varphi\frac{v}{r}\sin\gamma + p_v g(r)\cos\gamma + p_\gamma\left(\frac{v}{r} - \frac{g(r)}{v}\right)\sin\gamma$$
$$\dot{p}_m = \frac{T_{\max}}{m^2}\left(p_v u_1 + \frac{p_\gamma}{v}u_2\right) \tag{9}$$

Moreover, the maximization condition

$$H(q(t), p(t), p^0, u(t)) = \max_{\|w\|\leqslant 1} H(q(t), p(t), p^0, w) \tag{10}$$

holds almost everywhere on $[0, t_f]$, and this quantity is constant because the dynamics are autonomous. Furthermore, one has the transversality conditions that depend on the chosen final configuration. For (6), we simply have

$$p_\varphi(t_f) = 0, \quad p_m(t_f) = -p^0. \tag{11}$$

[‡]Indeed, it has been observed in those references that it is always possible to save some fuel by allowing a larger final time and more revolutions around the Earth. In other words, denoting by $m(t_f)$, the maximal possible value of the final mass, for the problem with a fixed final time t_f, the function $t_f \mapsto m(t_f)$ is increasing.

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For (7), the conditions (11) hold as well, and additionally, the vector $(p_r(t_f), p_v(t_f), p_\gamma(t_f))$ is a linear combination of the gradients (with respect to (r, v, γ)) of the two relations (7). This can be written as

$$\partial_r \xi_{K_f}(p_\gamma \partial_v \xi_{e_f} - p_v \partial_\gamma \xi_{e_f}) + \partial_v \xi_{K_f}(p_r \partial_\gamma \xi_{e_f} - p_\gamma \partial_r \xi_{e_f}) = 0, \tag{12}$$

where the expression has to be evaluated at the final time t_f.

The extremal $(x(\cdot), p(\cdot), p^0, u(\cdot))$ is said *normal* whenever $p^0 \neq 0$, and in that case, it is usual to normalize the adjoint vector so that $p^0 = -1$; otherwise, it is said *abnormal*.

A direct application of the maximization condition (10) leads to the definition of the so-called switching function $\Phi(\cdot)$ along a given extremal by

$$\Phi(t) = \frac{1}{m(t)} \sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}} - \beta p_m(t),$$

for every $t \in [0, t_f]$. This function is such that $u(t) = (u_1(t), u_2(t)) = (0, 0)$ whenever $\Phi(t) < 0$, and

$$u_1(t) = \frac{p_v(t)}{\sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}}, \quad u_2(t) = \frac{p_\gamma(t)}{v(t)\sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}},$$

whenever $\Phi(t) > 0$. Note that these formulas are well defined because the functions $p_v(\cdot)$ and $p_\gamma(\cdot)$ do not vanish simultaneously identically on any subinterval [§]. Note that the extremal control cannot be determined from the maximization condition in the case the switching function $\Phi(\cdot)$ vanishes on a subinterval of $[0, t_f]$. The nonoccurrence of this singular case can be checked from the numerical simulations; however, note that the controllability aspects of the orbit transfer problem have been studied in [2, 13, 18, 19], and it has been proved in these references that the singular case cannot occur in our problem.

Note that, if $p^0 = 0$, then, using the facts that $\dot{p}_m = \frac{T_{\max}}{m^2}\sqrt{p_v^2 + \frac{p_\gamma^2}{v^2}}$ and that $p_m(t_f) = 0$, it follows that $p_m(\cdot) \leqslant 0$ and thus $\Phi(\cdot) > 0$ on $[0, t_f]$, which means that there is no ballistic arc along the flight (actually such an extremal coincides with a minimal-time extremal). In practice, almost all initial and final configurations considered impose that the optimal trajectory should involve (at least) one ballistic arc, and therefore, the case $p^0 = 0$ does not occur (this can be checked further in the numerical simulations). Hence, from now on, we assume that $p^0 = -1$.

Based on these necessary conditions, recall that the (single) shooting method consists in finding a zero of the shooting function S defined as follows. Given $t_f > 0$ and $p_0 \in \mathbb{R}^5$, denote by $(q(t, p_0), p(t, p_0))$ the extremal solution of (8), starting from the initial condition $q(0)$ given by (5) and from $p(0) = p_0$. Then, the shooting function is defined by

$$S(t_f, p_0) = \begin{pmatrix} r(t_f, p_0) - r_f \\ v(t_f, p_0) - v_f \\ \gamma(t_f, p_0) - \gamma_f \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix} \text{ or } \begin{pmatrix} \xi_{K_f}(p_0) \\ \xi_{e_f}(p_0) \\ \text{Eq. (12)} \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix}, \tag{13}$$

depending on the chosen final conditions. The (single) shooting method thus consists of combining any numerical method for integrating a differential equation with a Newton-like method in order to determine a zero of the shooting function S.

As mentioned formerly, it is difficult to obtain convergence of this method, because of a difficulty of initialization and to the discontinuities of the control. However, we observe that, when assuming

[§]Indeed otherwise, it would follow from (9) combined with (11) that $p_r(\cdot)$ and $p_\varphi(\cdot)$ vanish identically as well on the same subinterval. Then, by Cauchy uniqueness, it would follow that $p_r(\cdot)$, $p_\varphi(\cdot)$, $p_v(\cdot)$, and $p_\gamma(\cdot)$ are identically equal to $\mathbf{0}$ on $[0, t_f]$, and that $p_m(\cdot)$ is constant, equal to $-p^0$. Then, necessarily, there must hold $p^0 \neq \mathbf{0}$, and we can take $p^0 = -1$. Therefore, the Hamiltonian reduces to $H = -\beta T_{\max}\|u\|$ along such an extremal, and the maximization condition implies that $u = 0$ on $[0, t_f]$. This raises a contradiction.

that the Earth is flat and the gravity is constant, the corresponding optimal control problem can be easily solved, in a very explicit way. We next introduce this very simplified model and explain our idea of passing continuously to the round Earth model.

1.2. Simplified flat Earth model

The motion of a vehicle in a flat Earth model with constant gravity is governed by the control system

$$\begin{aligned}
\dot{x}(t) &= v_x(t) \\
\dot{h}(t) &= v_h(t) \\
\dot{v}_x(t) &= \frac{T_{\max}}{m(t)} u_x(t) \\
\dot{v}_h(t) &= \frac{T_{\max}}{m(t)} u_h(t) - g_0 \\
\dot{m}(t) &= -\beta T_{\max} \sqrt{u_x(t)^2 + u_h(t)^2}
\end{aligned} \tag{14}$$

where $x(t)$ denotes the downrange (or in-track), $h(t)$ is the altitude, $v_x(t)$ is the horizontal component of the speed, $v_h(t)$ is the vertical component of the speed, and the control $(u_x(\cdot), u_h(\cdot))$ satisfies the constraint

$$u_x(\cdot)^2 + u_h(\cdot)^2 \leqslant 1. \tag{15}$$

The constant g_0 stands for the gravity $g_0 = \frac{\mu}{r_T^2}$ at zero altitude, with r_T the Earth radius. We denote by $\mathbf{(OCP)_{flat}}$ the optimal control problem of maximizing the final mass $m(t_f)$ for the control system (14), with the initial and final conditions

$$x(0) = x_0,\ h(0) = h_0,\ v_x(0) = v_{x0},\ v_h(0) = v_{h0},\ m(0) = m_0, \tag{16}$$

$$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_f) = 0. \tag{17}$$

If we had to make a connection to the round Earth model, these final conditions would correspond to (6) (and not (7)). Furthermore, contrarily to the round Earth model, here, it is not needed to assume a fixed final time t_f. Therefore, in $\mathbf{(OCP)_{flat}}$, the final time t_f is free.

It happens that $\mathbf{(OCP)_{flat}}$ can be explicitly and nearly analytically solved by applying the PMP. This is the object of Section 3 further, and this resolution leads to a very efficient algorithm based on a shooting method whose initialization is obvious. Based on that observation, it is tempting to try to use this efficient resolution in order to guess a good initialization for the shooting method applied to **(OCP)**. To this aim, the idea is to use a continuation process by introducing parameters such that, when one makes these parameters evolve continuously, one passes from the flat Earth model to the initial round Earth model. Because the coordinates of the flat Earth model are Cartesian, and the coordinates of the round Earth model are polar, this will of course require, at the end of the process, a change of coordinates.

Before going into more details, we can make one preliminary remark. In the continuation process, the gravity constant g_0 must be obviously deformed in order to end up with the gravity model $g(r)$. However, there is a serious difference between the flat Earth model (with constant or variable gravity) and the round Earth model; indeed, in the round Earth model, periodic trajectories with no thrust ($u = 0$) do exist, namely Keplerian orbits, whereas in the flat Earth model, there do not exist any 'horizontal trajectories' (that is, trajectories with a zero control having a constant altitude h), because of the presence of the gravity term. This obvious but important remark leads to the idea of deforming the flat Earth model by introducing some new terms into the dynamics, so that there may exist such horizontal trajectories with null thrust (zero control). Moreover, we would like this modified model to be equivalent, up to some change of coordinates, to the round Earth model.

This modified flat Earth model is derived in the next subsection, by defining a change of coordinates that is flattening circular orbits into horizontal trajectories, and then computing the control system from this change of coordinates.

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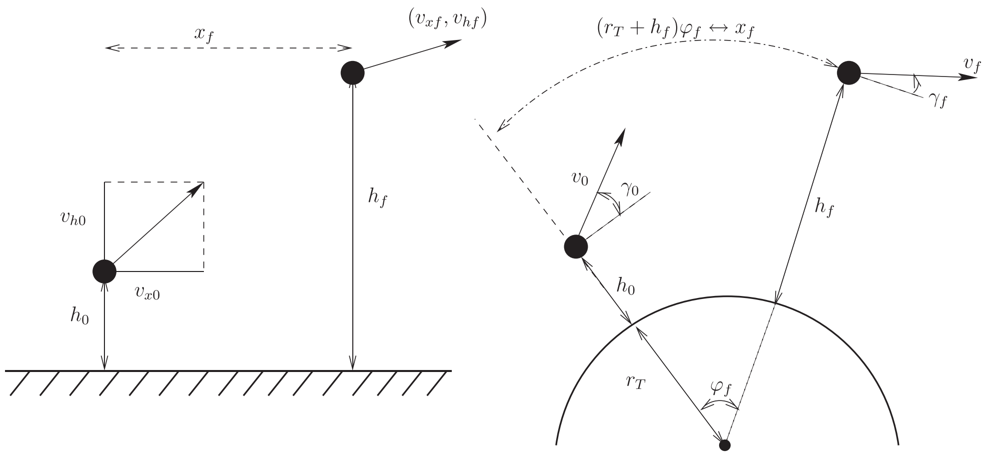

Figure 1. Correspondence between flat and round Earth coordinates.

1.3. Modified flat Earth model

The change of coordinates described in this section is illustrated in Figure 1. Starting from the polar coordinates (r, φ, v, γ) of the round Earth model, let us define some Cartesian coordinates (x, h, v_x, v_h). The idea of mapping circular orbits to horizontal trajectories leads to define $x = r\varphi$. Then, $x(t)$ represents the curvilinear abscissa of the point $M(t)$. The altitude is logically defined by $h = r - r_T$, where r_T denotes the radius of the Earth. Using the geometric definition of the angle γ, finally, one is led to define $v_x = v\cos\gamma$ and $v_y = v\sin\gamma$. Summing up, we consider the change of coordinates

$$\begin{cases} x = r\varphi \\ h = r - r_T \\ v_x = v\cos\gamma \\ v_h = v\sin\gamma \end{cases} \Longleftrightarrow \begin{cases} r = r_T + h \\ \varphi = \dfrac{x}{r_T + h} \\ v = \sqrt{v_x^2 + v_h^2} \\ \gamma = \arctan\dfrac{v_h}{v_x} \end{cases} \tag{18}$$

and denote by F the corresponding diffeomorphism, such that $F(x, h, v_x, v_h) = (r, \varphi, v, \gamma)$. For the control, the transformation from cylindrical to Cartesian coordinates is

$$\begin{pmatrix} u_x \\ u_h \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{19}$$

Applying this change of coordinates to the control system (3) now leads, after easy computations, to the control system

$$\left\{\begin{aligned} \dot r(t) &= v(t)\sin\gamma(t) \\ \dot\varphi(t) &= \frac{v(t)}{r(t)}\cos\gamma(t) \\ \dot v(t) &= -g(r(t))\sin\gamma(t) + \frac{T_{\max}}{m(t)}u_1(t) \\ \dot\gamma(t) &= \left(\frac{v(t)}{r(t)} - \frac{g(r(t))}{v(t)}\right)\cos\gamma(t) \\ &\quad + \frac{T_{\max}}{m(t)v(t)}u_2(t) \\ \dot m(t) &= -\beta T_{\max}\|u(t)\| \end{aligned}\right. \Longleftrightarrow \left\{\begin{aligned} \dot x(t) &= v_x(t) + v_h(t)\frac{x(t)}{r_T + h(t)} \\ \dot h(t) &= v_h(t) \\ \dot v_x(t) &= \frac{T_{\max}}{m(t)}u_x(t) - \frac{v_x(t)v_h(t)}{r_T + h(t)} \\ \dot v_h(t) &= \frac{T_{\max}}{m(t)}u_h(t) - g(r_T + h(t)) \\ &\quad + \frac{v_x(t)^2}{r_T + h(t)} \\ \dot m(t) &= -\beta T_{\max}\|u(t)\| \end{aligned}\right. \tag{20}$$

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This modified formulation (20) in Cartesian coordinates is equivalent (up to change of coordinates) to the initial formulation (3) in cylindrical coordinates. Therefore, it still represents the true round Earth transfer problem. In particular, it admits the (null thrust) Keplerian orbits. Comparing this modified formulation with the simplified flat Earth formulation (14) (with constant gravity), we have two differences: the first one is of course the gravity term, which is constant in the simplified model (14); the second difference is the presence of new terms in the dynamics of x, v_x, and v_h in the right-hand side of (20). These new terms can be seen as corrective terms in the flat Earth model, which make possible in particular the existence of horizontal trajectories with no thrust.

2. THE CONTINUATION PROCEDURE

To pass from the simplified flat Earth model (14) to the modified flat Earth model (20), we introduce two parameters. One of them permits to pass continuously from the constant gravity term to the variable gravity term, and the other introduces continuously the corrective terms. In brief, we consider the family of control systems

$$\begin{aligned}
\dot{x}(t) &= v_x(t) + \lambda_2 v_h(t)\frac{x(t)}{r_T + h(t)}\\
\dot{h}(t) &= v_h(t)\\
\dot{v}_x(t) &= \frac{T_{\max}}{m(t)}u_x(t) - \lambda_2\frac{v_x(t)v_h(t)}{r_T + h(t)}\\
\dot{v}_h(t) &= \frac{T_{\max}}{m(t)}u_h(t) - \frac{\mu}{(r_T + \lambda_1 h(t))^2} + \lambda_2\frac{v_x(t)^2}{r_T + h(t)}\\
\dot{m}(t) &= -\beta T_{\max}\sqrt{u_x(t)^2 + u_h(t)^2}
\end{aligned} \tag{21}$$

parameterized by the parameters λ_1 and λ_2, themselves varying between 0 and 1. For $\lambda_1 = \lambda_2 = 0$, one recovers the simplified flat Earth model (14) with constant gravity, and for $\lambda_1 = \lambda_2 = 1$, one recovers the modified flat Earth model (20), which is diffeomorphic to the initial round Earth model (3).

Now, for all $(\lambda_1, \lambda_2) \in [0, 1]^2$, denote by $\mathbf{(OCP)}_{\lambda_1,\lambda_2}$ the optimal control problem of steering the system (21) from (16) to (17) and maximizing the final mass $m(t_f)$. In what follows, we will explain how to implement a continuation procedure to pass from $\mathbf{(OCP)}_{0,0}$ to $\mathbf{(OCP)}_{1,1}$. In this procedure, we decide to make a first continuation on the parameter λ_1, keeping $\lambda_2 = 0$, passing from $\lambda_1 = 0$ (flat Earth model with constant gravity) to $\lambda_1 = 1$ (flat Earth model with variable gravity), and then a second continuation, keeping $\lambda_1 = 1$, passing from $\lambda_2 = 0$ to $\lambda_2 = 1$ (modified flat Earth model, equivalent to the initial round Earth model). Along the first continuation, the optimal control problems under consideration are with a free final time. However, because the problem $\mathbf{(OCP)}_{1,1}$ does not have any optimal solution for free final time (as already mentioned), we decide to fix the final time for the optimal control problems in consideration along the second continuation. The value chosen for t_f is the (free) final time obtained for $\mathbf{(OCP)}_{1,0}$ at the end of the first continuation. Note that this is not restrictive because numerical simulations show that the shooting method is relatively robust with respect to changes on the fixed t_f. The continuation procedure is drawn on Figure 2.

As before, the application of the PMP to $\mathbf{(OCP)}_{\lambda_1,\lambda_2}$ leads to a shooting problem as follows. For every optimal trajectory $X(\cdot) = (x(\cdot), h(\cdot), v_x(\cdot), v_h(\cdot), m(\cdot))$ of $\mathbf{(OCP)}_{\lambda_1,\lambda_2}$, associated with a control $U(\cdot) = (u_x(\cdot), u_h(\cdot))$ on $[0, t_f]$, there exists an absolutely continuous mapping $P(\cdot) = (p_x(\cdot), p_h(\cdot), p_{v_x}(\cdot), p_{v_h}(\cdot), p_m(\cdot)) : [0, t_f] \to \mathbb{R}^5$ and $p^0 \leqslant 0$, satisfying $(p(\cdot), p^0) \neq (0, 0)$, such that

$$\begin{aligned}
\dot{X}(t) &= \frac{\partial H_{\lambda_1,\lambda_2}}{\partial P}(X(t), P(t), p^0, U(t)),\\
\dot{P}(t) &= -\frac{\partial H_{\lambda_1,\lambda_2}}{\partial X}(X(t), P(t), p^0, U(t)),
\end{aligned} \tag{22}$$

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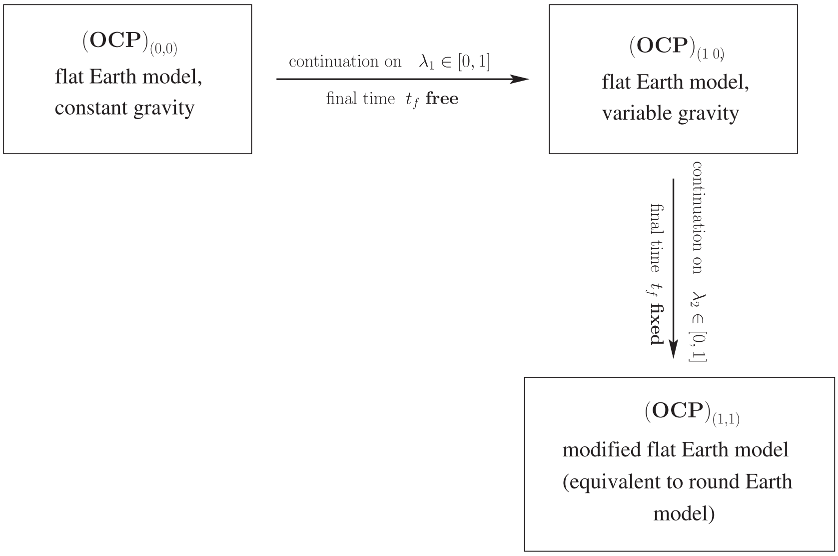

Figure 2. Continuation procedure.

for almost every $t \in [0, t_f]$, where the Hamiltonian is defined by

$$
\begin{aligned}
H_{\lambda_1,\lambda_2}(X, P, p^0, U) = {} & p_x \left(v_x + \lambda_2 v_h \frac{x}{r_T + h} \right) + p_h v_h + p_{v_x} \left(\frac{T_{\max}}{m} u_x - \lambda_2 \frac{v_x v_h}{r_T + h} \right) \\
& + p_{v_h} \left(\frac{T_{\max}}{m} u_h - \frac{\mu}{(r_T + \lambda_1 h)^2} + \lambda_2 \frac{v_x^2}{r_T + h} \right) - p_m \beta T_{\max} \|u\|.
\end{aligned}
$$

Component-wise, the adjoint equations are

$$
\begin{aligned}
\dot{p}_x &= -p_x \frac{\lambda_2 v_h}{r_T + h} \\
\dot{p}_h &= p_x \frac{\lambda_2 x v_h}{(r_T + h)^2} - p_{v_x} \frac{\lambda_2 v_x v_h}{(r_T + h)^2} - p_{v_h} \left(\frac{2\mu\lambda_1}{(r_T + \lambda_1 h)^3} - \frac{\lambda_2 v_x^2}{(r_T + h)^2} \right) \\
\dot{p}_{v_x} &= -p_x + p_{v_x} \frac{\lambda_2 v_h}{r_T + h} - 2 p_{v_h} \frac{\lambda_2 v_x}{r_T + h} \\
\dot{p}_{v_h} &= -p_x \frac{\lambda_2 x}{r_T + h} - p_h + p_{v_x} \frac{\lambda_2 v_x}{r_T + h} \\
\dot{p}_m &= \frac{T_{\max}}{m^2} (p_{v_x} u_x + p_{v_h} u_h)
\end{aligned}
\tag{23}
$$

Moreover, the maximization condition (W is a free variable)

$$
H_{\lambda_1,\lambda_2}(X(t), P(t), p^0, U(t)) = \max_{\|W\| \leqslant 1} H(X(t), P(t), p^0, W) \tag{24}
$$

holds almost everywhere on $[0, t_f]$, and this quantity is moreover equal to 0 for $(\lambda_1, \lambda_2) \in [0, 1] \times \{0\}$ (that is, along the first continuation) because the final time t_f is free, and the dynamics are autonomous. Furthermore, one has the transversality conditions

$$
p_x(t_f) = 0, \quad p_m(t_f) = -p^0. \tag{25}
$$

Defining the switching function $\Phi(\cdot)_{\lambda_1,\lambda_2}$ by

$$
\Phi_{\lambda_1,\lambda_2}(t) = \frac{1}{m(t)} \sqrt{p_{v_x}(t)^2 + p_{v_h}(t)^2} - \beta p_m(t),
$$

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for every $t \in [0, t_f]$, one has $U(t) = (u_x(t), u_h(t)) = (0, 0)$ whenever $\Phi(t) < 0$, and

$$u_x(t) = \frac{p_{v_x}(t)}{\sqrt{p_{v_x}(t)^2 + p_{v_h}(t)^2}}, \quad u_h(t) = \frac{p_{v_h}(t)}{\sqrt{p_{v_x}(t)^2 + p_{v_h}(t)^2}},$$

whenever $\Phi(t) > 0$. Note that these formulas are well defined because the functions $p_{v_x}(\cdot)$ and $p_{v_h}(\cdot)$ do not vanish simultaneously identically on any subinterval [¶].

Assuming as before $p^0 = -1$ (see comments made for the round Earth model), the continuation method (see [20, 21]) consists of solving a series of shooting problems for sequences of parameters λ_1 and λ_2, with the starting point $\lambda_1 = \lambda_2 = 0$. At each step, the previous solution is used as an initial guess for the shooting problem. For every couple (λ_1, λ_2), the shooting function S_{λ_1,λ_2} is defined as follows. Given $t_f > 0$ and $P_0 \in \mathbb{R}^5$, denote by $(X(t, P_0), P(t, P_0))$, the extremal solution of (22), starting from the initial condition $X(0)$ given by (5) up to the change of coordinates (18), and from $P(0) = P_0$. Then, the shooting function is defined by

$$S_{\lambda_1,\lambda_2}(P_0, t_f) = \begin{pmatrix} h(t_f, P_0) - h_f \\ v_h(t_f, P_0) \\ v_x(t_f, P_0) - v_{xf} \\ p_x(t_f, P_0) \\ p_m(t_f, P_0) - 1 \\ H(t_f) \text{ if } (\lambda_1, \lambda_2) \in [0,1] \times \{0\} \end{pmatrix}. \tag{26}$$

Note that in the case where t_f is fixed, the shooting function has only five components. The first part of the continuation procedure consists of solving iteratively by a Newton-like method the equation $S_{\lambda_1,0}(P_0, t_f) = 0$ for a sequence of parameters λ_1 starting from 0 and ending at 1, and the second part of that procedure consists of solving $S_{1,\lambda_2}(P_0, t_f) = 0$ for a sequence of parameters λ_2 starting from 0 and ending at 1 (see Figure 2). Note that it is possible to consider other paths of parameters (λ_1, λ_2) in the square $[0,1]^2$. Our choice here is first to introduce the variable gravity and then the correcting terms. Note, however, that using the intermediate problem $(\mathbf{OCP})_{1,0}$ in our continuation enables us to find a more reasonable final time t_f than if we had directly used as a fixed final time the optimal final time obtained from $(\mathbf{OCP})_{0,0}$. Furthermore, the numerical simulations further will show that the continuation from $(\mathbf{OCP})_{0,0}$ to $(\mathbf{OCP})_{1,0}$ is very fast when compared with the second continuation from $(\mathbf{OCP})_{1,0}$ to $(\mathbf{OCP})_{1,1}$. Hence, considering one more direct continuation path from $(\mathbf{OCP})_{0,0}$ to $(\mathbf{OCP})_{1,1}$ would not yield a significant gain in terms of execution time but might put the success of the continuation at risk.

The starting point $(0, 0)$ of the continuation process corresponds to the simplified flat Earth model with constant gravity, and hence, we initialize the procedure with the solution of that simplified model, as detailed in Section 3.

This algorithmic procedure provides a way of solving (**OCP**) without any a priori knowledge on the optimal solution. The price to pay is that, instead of solving only one optimal control problem, one has to solve a series of $(\mathbf{OCP})_{\lambda_1,\lambda_2}$. However, the whole procedure is time-efficient because the shooting method relies on a Newton-like method. Our procedure provides a way for bypassing the difficulty because of the initialization of the shooting method when applied directly to (**OCP**). Numerical simulations are given in Section 4.

One item remains, however, to be explained in our procedure. Indeed, the continuation process above leads, provided it has converged, to the solution of $(\mathbf{OCP})_{1,1}$, which corresponds to the control system (20). As explained in Section 1.3, this control system is equivalent to the initial control system (3) via the change of coordinates (18) and (19). Hence, we must explain how the change of

[¶]Indeed otherwise, it would follow from (23) combined with (25) that $p_x(\cdot)$ and $p_h(\cdot)$ vanish identically as well on the same subinterval. Then, by Cauchy uniqueness, it would follow that $p_x(\cdot)$, $p_h(\cdot)$, $p_{v_x}(\cdot)$ and $p_{v_h}(\cdot)$ are identically equal to 0 on $[0, t_f]$, and that $p_m(\cdot)$ is constant, equal to $-p^0$. Then, necessarily, there must hold $p^0 \neq 0$, and we can take $p^0 = -1$. Therefore, the Hamiltonian reduces to $H = -\beta T_{\max}\|u\|$ along such an extremal, and the maximization condition implies that $u = 0$ on $[0, t_f]$. This raises a contradiction.

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coordinates must act onto the adjoint vector, so as to recover the adjoint vector in the initial coordinates $(p_r, p_\varphi, p_v, p_\gamma, p_m)$. Recalling that F denotes the diffeomorphism defined by (18), we claim that (^{t} denotes the transpose operation)

$$ {}^t(p_r, p_\varphi, p_v, p_\gamma) = {}^t dF(x, h, v_x, v_h)^{-1} \cdot {}^t(p_x, p_h, p_{v_x}, p_{v_h}), \tag{27} $$

that is, one passes from the adjoint vector in Cartesian coordinates to cylindrical coordinates by applying the transpose of the inverse of the differential of F. After easy computations, this yields

$$ \begin{aligned} p_r &= \frac{x}{r_T + h} p_x + p_h \\ p_\varphi &= (r_T + h) p_x \\ p_v &= \cos\gamma \; p_{v_x} + \sin\gamma \; p_{v_h} \\ p_\gamma &= v(-\sin\gamma \; p_{v_x} + \cos\gamma \; p_{v_h}). \end{aligned} \tag{28} $$

This claim follows from a general fact recalled separately, in the Appendix for the sake of clarity. Note that p_m remains unchanged.

Remark 1
Here, we have considered $(\mathbf{OCP})_{\lambda_1,\lambda_2}$ with final conditions (17) that correspond to (6); that is, they correspond to the case of injecting the vehicle on a precise point of a given orbit. To handle the case of final conditions (7) (that is, a final orbit of given energy and given eccentricity), we propose to implement an additional continuation process, consisting in passing from the transversality conditions (25) to the transversality conditions (12) expressed in this Cartesian reference frame. In that case, we propose to define the shooting function of the continuation method as a convex combination of both transversality conditions.

3. ANALYSIS OF THE OPTIMAL CONTROL PROBLEM WITH THE SIMPLIFIED FLAT EARTH MODEL

For this simplified optimal control problem, we consider similar terminal conditions as (5) and (6) except that we impose γ_f to be zero and we express them in Cartesian coordinates. The terminal conditions are given by

$$ \begin{gathered} x(0) = x_0,\; h(0) = h_0,\; v_x(0) = v_{x0},\; v_h(0) = v_{h0},\; m(0) = m_0 \\ x(t_f) \text{ free},\; h(t_f) = h_f,\; v_x(t_f) = v_{xf},\; v_h(t_f) = 0,\; m(t_f) \text{ free}, \end{gathered} \tag{29} $$

Contrarily to the initial (**OCP**), we leave t_f free because the simplified problem does not allow orbits and we thus have a solution for free final time.

We assume moreover that

$$ h_f > h_0 + \frac{v_{h0}^2}{2g_0}. \tag{30} $$

Note that (30) discards the noninteresting case of a zero fuel consumption trajectory. It also discards the case of a vehicle starting with an initial velocity so large that it needs to decelerate in order to reach the final configuration. This assumption is satisfied in practice.

3.1. Application of the Pontryagin Maximum Principle

Denoting the adjoint variables by $p = (p_x, p_h, p_{v_x}, p_{v_h}, p_m)$, the Hamiltonian is

$$ H = p_x v_x + p_h v_h + p_{v_x} \frac{T_{\max}}{m} u_x + p_{v_h} \left(\frac{T_{\max}}{m} u_h - g_0 \right) - \beta T_{\max} p_m \sqrt{u_x^2 + u_h^2}, $$

and hence, the adjoint equations are

$$ \dot p_x = 0,\; \dot p_h = 0,\; \dot p_{v_x} = -p_x,\; \dot p_{v_h} = -p_h,\; \dot p_m = \frac{T_{\max}}{m^2}(p_{v_x} u_x + p_{v_h} u_h). $$

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According to the boundary conditions (29), the transversality conditions yield $p_x(t_f) = 0$ and $p_m(t_f) = -p^0$ (with $p^0 \leqslant 0$). It follows that $p_x(\cdot)$ is identically equal to 0, that $p_h(\cdot)$ is constant (denoted p_h in what follows), that $p_{v_x}(\cdot)$ is constant (denoted p_{v_x} in what follows), and that $p_{v_h}(t) = -p_h t + p_{v_h}(0)$. Because the final time t_f is free and the system is autonomous, we infer that $H = 0$ along any extremal.

Setting $\Phi(t) = \frac{1}{m(t)}\sqrt{p_{v_x}^2 + p_{v_h}(t)^2} - \beta p_m(t)$ for every $t \in [0, t_f]$, we infer from the maximization condition of the Hamiltonian that $u(t) = (u_x(t), u_h(t)) = (0, 0)$ whenever $\Phi(t) < 0$, and

$$u_x(t) = \frac{p_{v_x}}{\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}}, \quad u_h(t) = \frac{p_{v_h}(t)}{\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}},$$

whenever $\Phi(t) > 0$.

3.2. Analysis of extremal equations

First of all, notice that $\dot{p}_m(t) = \frac{\|u(t)\| T_{\max}}{m(t)^2}\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}$, hence $p_m(\cdot)$ is nondecreasing. More precisely, $p_m(\cdot)$ is increasing whenever $\Phi(\cdot) > 0$ and constant whenever $\Phi(\cdot) < 0$.

Lemma 1
The function $t \mapsto \sqrt{p_{v_x}^2 + p_{v_h}(t)^2}$ does not vanish identically on any subinterval of $[0, t_f]$.

Proof
The argument goes by contradiction. If $p_{v_x} = 0$ and $p_{v_h}(\cdot) = 0$ on a subinterval I, then differentiating with respect to t yields $p_h = 0$, and then by Cauchy uniqueness $p_{v_h}(\cdot) = 0$ on $[0, t_f]$. Hence, $p_m(\cdot)$ is constant, equal to $-p^0$. Besides, because $H = 0$, we infer that $p^0\|u(\cdot)\| = 0$ on $[0, t_f]$. From (30), the thrust $\|u(\cdot)\|$ cannot be identically equal to 0 on $[0, t_f]$, and hence $p^0 = 0$. We have proved that $(p(\cdot), p^0) = (0, 0)$, which is a contradiction with the PMP. □

This lemma implies in particular that the formulas for the extremal controls are well defined. Moreover, it follows from easy computations that the function $t \mapsto \Phi(t)$ is almost everywhere two times differentiable, and (recall that $p_x = 0$)

$$\dot{\Phi}(t) = \frac{-p_h p_{v_h}(t)}{m(t)\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}}, \tag{31}$$

$$\ddot{\Phi}(t) = \frac{\beta\|u(t)\|}{m(t)}\dot{\Phi}(t) - \frac{m(t)}{\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}}\dot{\Phi}(t)^2 + \frac{p_h^2}{m(t)\sqrt{p_{v_x}^2 + p_{v_h}(t)^2}}. \tag{32}$$

Lemma 2
The function $t \mapsto \Phi(t)$ is constant if and only if $p_h = 0$.

Proof
If $p_h = 0$, then it follows from (31) that $\dot{\Phi} = 0$. Conversely, if $\dot{\Phi} = 0$, then $p_h p_{v_h} = 0$ and differentiating with respect to time yields $p_h = 0$. □

Lemma 3
The function $t \mapsto \Phi(t)$ does not vanish identically on any subinterval of $[0, t_f]$.

Proof
The argument goes by contradiction. If $\Phi(\cdot) = 0$ on a subinterval I, then, from Lemma 2, $p_h = 0$, and then using the adjoint equations, $p_{v_h}(\cdot)$ is constant. Moreover, there holds $0 = H = -p_{v_h} g_0$,

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and hence, $p_{v_h}(\cdot) = 0$ on $[0, t_f]$. In particular, this yields $u_h(\cdot) = 0$ on $[0, t_f]$, and hence $\dot{v}_h = -g_0$ and $v_h(t) = v_{h0} - g_0 t$. From (29), $v_h(t_f) = 0$, hence, $t_f = \frac{v_{h0}}{g_0}$. Besides, integrating $\dot{x}_2 = v_h$, one gets $h(t) = h_0 + v_{h0} t - \frac{g_0}{2} t^2$, and thus $h(t_f) = h_0 + \frac{v_{h0}^2}{2g_0}$. From (29), one has $h(t_f) = h_f$, and we get a contradiction with (30). □

This lemma shows that the singular case where the extremal controls cannot be inferred directly from the maximization condition does not occur.

Lemma 4
If $p_h = 0$, then the thrust $\|u(\cdot)\| T_{\max}$ is constant on $[0, t_f]$, equal to $T_{\max}$. In other words, in that case, the thrust is always maximal and there is no switching.

Proof
If $p_h = 0$, then, from Lemma 2, $\Phi(\cdot)$ is constant, and from Lemma 3, this constant Φ cannot be equal to 0. If $\Phi < 0$, then $u = 0$ on $[0, t_f]$, which is not possible because the thrust cannot be identically zero (this would contradict (30)). Hence, $\Phi > 0$, and therefore, $\|u(\cdot)\| = 1$ on $[0, t_f]$. □

Lemma 5
If $p_h \neq 0$, then

- either $\Phi(\cdot)$ is increasing on $[0, t_f]$,
- or $\Phi(\cdot)$ is decreasing on $[0, t_f]$,
- or $\Phi(\cdot)$ has a unique minimum on $[0, t_f]$, is decreasing before that minimum and then increasing.

Proof
If $p_h \neq 0$, then from Lemma 2, $\Phi(\cdot)$ is not constant, hence, $\dot{\Phi}(\cdot)$ is not identically equal to 0. If $\dot{\Phi}(\cdot)$ does not vanish on $[0, t_f]$, then $\Phi(\cdot)$ is strictly monotone, and this yields the two first cases of the result. If $\dot{\Phi}(\cdot)$ vanishes at some point t_1 of $[0, t_f]$, then, using (32), for every $t_1 \in [0, t_f]$ such that $\dot{\Phi}(t_1) = 0$, there must hold $\ddot{\Phi}(t_1) > 0$ (because $p_h \neq 0$), and therefore, this point is a local minimum. This reasoning shows that every extremum of $\Phi(\cdot)$ is a local minimum. It follows that the function $\dot{\Phi}(\cdot)$ cannot vanish more than one time; otherwise, there would exist another local minimum, and hence, there should then exist a local maximum between those two minima; but this is a contradiction because every extremum of $\Phi(\cdot)$ is a local minimum. Therefore, the third point of the lemma follows. □

Lemmas 4 and 5 imply that the thrust $\|u(\cdot)\| T_{\max}$ of the optimal trajectory is either constant, equal to $T_{\max}$, or has exactly one switching (and in that case, passing either from 0 to $T_{\max}$, or from $T_{\max}$ to 0), or has exactly two switchings and passes from $T_{\max}$ to 0 and then from 0 to $T_{\max}$. Actually, we next prove that the latter possibility cannot occur and finally derive the following result.

We first state and prove the following lemma, useful for the proof of Theorem 1.

Lemma 6
If the modulus of the control $\|u(\cdot)\|$ has at least one switching on $[0, t_f]$, then $p^0 \neq 0$.

Proof of Lemma 6
The argument goes by contradiction. If $p^0 = 0$, then $p_m(t_f) = 0$. Because $p_m(\cdot)$ is nondecreasing, it follows that $p_m(t) \leqslant 0$ for every $t \in [0, t_f]$. From Lemma 1, $\sqrt{p_{v_x}^2 + p_{v_h}(\cdot)^2}$ does not vanish identically on any subinterval; hence, $\Phi(\cdot) = \frac{1}{m(\cdot)} \sqrt{p_{v_x}^2 + p_{v_h}(\cdot)^2} - \beta p_m(\cdot) > 0$ on $[0, t_f]$, and therefore, $\|u(\cdot)\| = 1$ on $[0, t_f]$. This contradicts the assumption of having one switching. □

We are now in a position to prove the theorem.

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Theorem 1

The optimal trajectory of $(\mathbf{OCP})_{\mathbf{flat}}$ is a succession of at most two arcs with a control modulus $\|u(\cdot)\|$ being either equal to 1 or 0. More precisely, the modulus $\|u(\cdot)\|T_{\max}$ of the thrust is

- either constant on $[0, t_f]$ and equal to $T_{\max}$,
- or of the type $T_{\max} - 0$,
- or of the type $0 - T_{\max}$.

Proof of Theorem 1

To prove the theorem, one has to prove that the strategy $T_{\max} - 0 - T_{\max}$ for the modulus of the thrust cannot occur. The argument goes by contradiction. Assume that the modulus of the thrust $\|u(\cdot)\|T_{\max}$ is of this type and denote by $t_1 < t_2$ the two switching times.

Let us first prove that the minimum of $\Phi(\cdot)$ is reached at $\bar{t} = \frac{p_{v_h}(0)}{p_h}$. Using (31), if $\dot{\Phi}(\bar{t}) = 0$, then there must hold $p_h p_{v_h}(\bar{t}) = 0$. Because $\Phi(\cdot)$ is not constant, one has $p_h \neq 0$; hence, $p_{v_h}(\bar{t}) = 0$. Integrating the differential equation satisfied by $p_{v_h}(\cdot)$, one gets $\bar{t} = \frac{p_{v_h}(0)}{p_h}$. By definition, this minimum is reached within the interval $(0, t_f)$. In particular, we deduce that

$$0 < t_1 < \bar{t} = \frac{p_{v_h}(0)}{p_h} < t_2 < t_f. \tag{33}$$

On $[t_1, t_2]$, one has $\|u(t)\| = 0$p; hence, in particular, $m(\cdot)$ and $p_m(\cdot)$ are constant on this interval, and thus $m(t_1) = m(t_2)$ and $p_m(t_1) = p_m(t_2)$. Because the function $t \mapsto \Phi(t) = \frac{\sqrt{p_{v_x}^2 + (p_{v_h}(0) - p_h t)^2}}{m(t)} - \beta p_m(t)$ vanishes by definition at t_1 and t_2, it follows that

$$\sqrt{p_{v_x}^2 + (p_{v_h}(0) - p_h t_1)^2} = \sqrt{p_{v_x}^2 + (p_{v_h}(0) - p_h t_2)^2},$$

and hence, $|p_{v_h}(0) - p_h t_1| = |p_{v_h}(0) - p_h t_2|$. Since $t_1 \neq t_2$, we infer that $t_2 = 2\frac{p_{v_h}(0)}{p_h} - t_1$.

Note that the latter equality means that the graph of $\Phi(\cdot)$ on the interval $[t_1, t_2]$ is symmetric with respect to the point $\bar{t} = \frac{p_{v_h}(0)}{p_h}$ where the minimum is reached.

Using the fact that $H = 0$ along an extremal and that $p_x = 0$, one gets

$$p_h v_h(t) + \|u(t)\|T_{\max}\Phi(t) - g_0 p_{v_h}(t) = 0, \tag{34}$$

for every $t \in [0, t_f]$. In particular, at $t = t_f$, one gets $g_0 p_{v_h}(t_f) = T_{\max}\Phi(t_f) > 0$, which implies $p_{v_h}(t_f) > 0$. Because $p_{v_h}(\cdot)$ is affine and vanishes at $\bar{t}$, we get that $p_h < 0$, $p_{v_h}(\cdot) \leqslant 0$ on $[0, \bar{t}]$ and $\geqslant 0$ on $[\bar{t}, t_f]$.

Now, note that $u_h(\cdot)$ has the same sign as $p_{v_h}(\cdot)$ during the thrust arcs; hence, it is negative on $[0, t_1]$ and positive on $[t_2, t_f]$. In particular, there holds $\dot{v}_h(t) \leqslant -g_0$, for every $t \in [0, t_2]$. Note also that, because $v_h(\bar{t}) = 0$, it follows that $v_{h0} > 0$. Set $\tilde{t} = \min(\frac{v_{h0}}{g_0}, t_f) \in [\bar{t}, t_f]$. Then,

$$h(\tilde{t}) \leqslant h_0 + \int_0^{\tilde{t}} (v_{h0} - g_0 t)\, dt + \max\left(0, \int_0^{\tilde{t} - v_{h0}/g_0} (v_{h0} - g_0 t)\, dt\right) \leqslant h_0 + \frac{v_{h0}^2}{2g_0}.$$

Furthermore, $u_h(\cdot)$ is positive and increasing on $[t_2, t_f]$, and $v_{hf} = 0$, hence, $v_h(\cdot)$ is nonpositive on $[\bar{t}, t_f]$. It then follows that $h(t) \geqslant h_f$, for every $t \in [\bar{t}, t_f]$. In particular, we have $h(\tilde{t}) \geqslant h_f$. This leads to $h_0 + \frac{v_{h0}^2}{2g_0} \geqslant h_f$ and raises a contradiction with Assumption (30). □

Remark 2

The conclusions of Theorem 1 might seem counterintuitive because we can think that a strategy $T_{\max} - 0 - T_{\max}$ should be a better choice (at least in view of the results for the round Earth model). However, we deal here with a flat Earth model with constant gravity; and thus, no gravitational or centripetal help is obtained by introducing a ballistic arc.

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Because the strategy where the thrust is maximal all along the flight is also a minimum time strategy and is not cost efficient, we next focus on the strategy $T_{\max} - 0$ (with one switching). Note, however, that the former strategy can be viewed as a particular case of the latter one. The study of the strategy $0 - T_{\max}$ is similar.

3.3. *Refined analysis of the strategy $T_{\max} - 0$ and algorithmic procedure*

Assume that we are in the case where the thrust has one switching, denoted t_1, with $0 < t_1 < t_f$ and is of the form $T_{\max} - 0$.

Lemma 7
There holds $t_f = \frac{p_{v_h}(0)}{p_h}$ and $p_h v_h(t_1) + p_h g_0 t_1 = g_0 p_{v_h}(0)$. Moreover, $p_h > 0$, $\text{sign}(p_{v_x}) = \text{sign}(v_{xf} - v_{x0})$ and $p_{v_h}(0) > 0$.

Proof
First of all, note that the identity (34) still holds in that case.

On $[t_1, t_f]$, one has $\|u(\cdot)\| = 0$; and hence, from (34), $p_h v_h(\cdot) = g_0 p_{v_h}(\cdot)$. Taking $t = t_f$ yields $p_{v_h}(t_f) = 0$ because $v_h(t_f) = 0$ from the boundary conditions (29). Because $p_{v_h}(t) = -p_h t + p_{v_h}(0)$, it follows that $t_f = \frac{p_{v_h}(0)}{p_h}$ (note that $p_h \neq 0$ from Lemma 2). Moreover, for every $t \in [t_1, t_f]$, one has $\dot{v}_h(t) = -g_0$; hence, $v_h(t) = v_h(t_1) - g_0(t - t_1)$. Because $p_{v_h}(t) = -p_h t + p_{v_h}(0)$, we infer that $p_h v_h(t_1) + p_h g_0 t_1 = g_0 p_{v_h}(0)$.

Because $t_f = \frac{p_{v_h}(0)}{p_h}$, necessarily p_h and $p_{v_h}(0)$ have the same sign. Let us prove by contradiction that, actually, $p_{v_h}(0) > 0$. If $p_{v_h}(0) < 0$ then, because the function $p_{v_h}(\cdot)$ is affine and because $p_{v_h}(t_f) = 0$, there should hold $p_{v_h}(t) \leqslant 0$ for every $t \in [0, t_f]$. Hence, $u_h(t) < 0$ and $\dot{v}_h(t) \leqslant -g_0$ on $[0, t_f]$. Integrating, we would obtain $h(t) \leqslant h_0 + v_{h0} t - \frac{g_0}{2} t^2 \leqslant h_0 + \frac{v_{h0}^2}{2g_0}$. At the final time t_f, this would contradict (30). We thus conclude that $p_h > 0$ and $p_{v_h}(0) > 0$.

For every $t \in [0, t_1[$, one has $\dot{v}_x(t) = \frac{T_{\max}}{m(t)} u_x(t)$ with $u_x(t)$ having the same sign than p_{v_x}, and for every $t \in]t_1, t_f]$, one has $\|u(\cdot)\| = 0$; and hence, $v_x(\cdot)$ remains constant. Then, we directly have $\text{sign}(v_{xf} - v_{x0}) = \text{sign}(u_x(\cdot)) = \text{sign}(p_{v_x})$. □

Lemma 7 enables to significantly simplify the application of the single shooting method to that case. We next explain the construction of this simplified algorithmic procedure. Usually, when applying the single shooting method, we have five unknowns, namely

- the initial adjoint vector $(p_h, p_{v_x}, p_{v_h}(0), p_m(0), p^0)$, defined up to a multiplicative scalar (note that $p^0 \neq 0$ from Lemma 6). This definition up to a multiplicative scalar is usually used to set $p^0 = -1$, which leaves only four components of the initial adjoint vector as unknowns. This particular normalization will however not be used here, see further.
- the final time t_f,

and five equations

$$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_f) = 0,\ p_m(t_f) = -p^0,\ H(t_f) = 0.$$

Recall that the adjoint vector (completed with p^0) is defined up to a multiplicative scalar, and instead of choosing the usual normalization $p^0 = -1$, because there holds $p_h > 0$, we rather choose to normalize the adjoint vector so that $p_h = 1$. Because the variable p^0 is only used here to tune the equation $p_m(t_f) = -p^0$, we can therefore forget about the variable p^0 and the equation $p_m(t_f) = -p^0$. This is a first simplification.

Now there remain four unknowns, $(p_{v_x}, p_{v_h}(0), p_m(0))$, and t_f, and four equations

$$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_f) = 0,\ H(t_f) = 0.$$

Note that the knowledge of the value of $p_m(0)$ permits to determine the switching function $\Phi(\cdot)$ and hence, the switching time t_1. It is therefore possible to replace the unknown $p_m(0)$ with the new

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Data: Terminal conditions $(h_0, v_{x0}, v_{h0}, h_f, v_{xf})$ and gravity constant g_0.
Result: shooting function unknowns $(t_1, p_{vx}, p_{vh}(0))$ for the simplified flat Earth problem
Initialization : choose some values $p_{v_x} > 0$ and $p_{v_h}(0) > 0$.
Integrate numerically $(h(t), v_x(t), v_h(t))$ from $t = 0$ to t_1 satisfying $v_h(t_1) + g_0 t_1 = g_0 p_{v_h}(0)$
Set $t_f = p_{v_h}(0)$.
On $[t_1, t_f]$, compute explicitly $v_x(t) = v_x(t_1),\ h(t) = h(t_1) + v_h(t_1)(t - t_1) - \frac{g_0}{2}(t - t_1)^2$.
Solve the system $h(t_f) = h_f,\ v_x(t_f) = v_{xf}$, with a Newton-like method.

Algorithm 1: *Algorithmic procedure for solving the simplified flat Earth problem.*

unknown t_1. Hence, from now on we have the four unknowns $(p_{v_x}, p_{v_h}(0), t_1, t_f)$, and the four previous equations.

Taking into account the fact that $p_h = 1$, one can see from the previous computations and from Lemma 7 that the system of equations

$$v_h(t_f) = 0,\ H(t_f) = 0,$$

is equivalent to the system of equations

$$t_f = p_{v_h}(0),\ v_h(t_1) + g_0 t_1 = g_0 p_{v_h}(0).$$

The final time t_f being then directly determined by the value of $p_{v_h}(0)$, we can reduce the problem to three unknowns $(p_{v_x}, p_{v_h}(0), t_1)$ and three equations

$$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_1) + g_0 t_1 = g_0 p_{v_h}(0).$$

We finally end up with the following simplified algorithmic procedure, described in Algorithm 1.

Note that it is possible to compute explicit expressions of $h(t)$ and of $v_x(t)$ on the whole interval $[0, t_f]$; however, it happens that, from the numerical point of view, this does not save time, and the procedure described earlier is more efficient.

The aforementioned algorithm is very easy to carry out and happens to be very efficient. The convergence is obtained instantaneously in term of execution time for almost every random choice of initialized values of $p_{v_x} > 0$ and $p_{v_h}(0) > 0$ (on a standard desktop machine, in MATLAB). Here, reaching the convergence means finding a zero of the shooting function with an accuracy of at least 10^{-8} (and 10^{-12} for the accuracy of the integration of the state and costate dynamic).

This simple code is used as a first step in our continuation procedure described in Section 2.

The algorithm provides a solution in terms of $(t_1, t_f, p_{v_x}, p_{v_h}(0))$. Then, the unknowns $(t_f, \bar{p}_x(0), \bar{p}_h(0), \bar{p}_{v_x}(0), \bar{p}_{v_h}(0), \bar{p}_m(0))$ of the shooting function (26) associated with this simplified model are computed by

$$\begin{aligned}
&\bar{p}_x(0) = 0,\ \bar{p}_h(0) = \frac{m_0 - \beta T_{\max} t_1}{\beta \sqrt{p_{v_x}^2(0) + (p_{v_h}(0) - t_1)^2}},\\
&\bar{p}_{v_h}(0) = p_{v_h}(0)\bar{p}_h(0),\ \bar{p}_{v_x}(0) = p_{v_x}(0)\bar{p}_{v_h}(0),\\
&\bar{p}_m(0) = \frac{1}{\beta T_{\max}}\left(\bar{p}_h(0) v_{h0} + \frac{T_{\max}}{m_0}\sqrt{\bar{p}_{v_x}^2(0) + \bar{p}_{v_h}^2(0)} - g_0 \bar{p}_{v_h}(0)\right).
\end{aligned} \tag{35}$$

4. NUMERICAL SIMULATIONS

4.1. Continuation procedure

In this section, we provide numerical simulations of the algorithmic procedure described in Section 2, which consists of solving, in a continuation process, a sequence of shooting problems

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initialized with the simple algorithm introduced in Section 3.3. Because the latter code converges without the need for a carefully selected initialization, we thus get a way of solving (**OCP**) without any a priori knowledge on the optimal solution.

Consider (**OCP**) with the initial conditions

$$\varphi_0 = 0,\ r_0 = 200 + r_T \text{ km},\ v_0 = 5.5 \text{ km/s},\ \gamma_0 = 2 \text{ deg},\ m_0 = 40\,000 \text{ kg},$$

(which correspond to a Sun Synchronous Orbit) and the final conditions

$$r_f = 800 + r_T \text{ km},\ v_f = 7.5 \text{ km/s},\ \gamma_f = 0 \text{ deg},$$

and problem parameters

$$T_{\max} = 180 \text{ kN},\ Isp = 450 \text{ s}.$$

To express this terminal configurations in Cartesian coordinates, one only needs to apply the change of coordinates (18). Note that, in the round Earth model, this corresponds to injecting the space engine on a precise point of a nearly circular final orbit ($v_f \approx \sqrt{\mu/r_f}$, $\gamma_f = 0$). Once this problem will be solved, we may also consider as a final condition the previous orbit, without fixing a precise point of the orbit, by passing the transversality conditions (12) by continuation (see Remark 1).

Using the code developed in Section 3.3 and the transformation (35), we directly get the zero of the shooting function associated to $(\mathbf{OCP})_{0,0}$

$$(t_{f,(0,0)}, p_{0,0}(0)) \approx (1433\ s, 0, 0.755, 72.688, 1082.328, -0.137),$$

with the final mass $m_{0,0}(t_f) \approx 1676$ kg.

This solution is used as the starting point to the continuation from $(\mathbf{OCP})_{0,0}$ to $(\mathbf{OCP})_{1,0}$, the problem for the flat Earth model with variable gravity. This leads to the following zero of the shooting function

$$(t_{f,(1,0)}, p_{1,0}(0)) \approx (1483\ s, 0, 3.851, 69.818, 2198.465, -0.236),$$

associated with the final mass $m_{1,0}(t_f) \approx 1505$ kg.

We should note that $m_{1,0}(t_f) < m_{0,0}(t_f)$ seems counterintuitive because the variable gravity is always lower than g_0. However, the gravity does not only tend to decelerate the vehicle; it also helps to flatten the trajectory in order to reach $v_h(t_f) = 0$.

At this step, we switch from $(\mathbf{OCP})_{1,0}$ with free final time t_f to $(\mathbf{OCP})_{1,0}$ with a fixed final time. As mentioned in Section 2, this simply means that the shooting function has one less unknown and thus one less relation to satisfy at the end point of the extremal flow. This final time is $t_f \approx 1483$ s. Note that the solution of $(\mathbf{OCP})_{1,0}$ with free final time is the same as the solution with fixed time t_f, there is just one less unknown. The solution $p_{1,0}(0)$ is then used to initialize the continuation from $(\mathbf{OCP})_{1,0}$ to $(\mathbf{OCP})_{1,1}$. This leads to the solution

$$p_{1,1}(0) \approx (0, 12.219, 6824.539, 5230.033, 0.310),$$

associated with the final mass $m_{1,1}(t_f) \approx 18\,922$ kg.

First, we notice that the final mass of $(\mathbf{OCP})_{1,1}$ is far better than the ones of $(\mathbf{OCP})_{0,0}$ and $(\mathbf{OCP})_{1,0}$. This could be expected because in $(\mathbf{OCP})_{1,1}$, the vehicle can use the centripetal forces that allow it to park on an intermediary orbit in between two thrust arcs.

Figure 3 shows the zero path of the shooting function from $(\mathbf{OCP})_{1,0}$ to $(\mathbf{OCP})_{1,1}$.

We can see that this zero path does not look very smooth around several values of λ_2, namely, for $\lambda_2 \approx 0.01$, $\lambda_2 \approx 0.8$, and $\lambda_2 \approx 0.82$. Actually, focusing on the zero path around these values of λ_2 by enforcing the continuation to increase λ_2 with very small steps, we observe numerically that the zero path is continuous but is not C^1 (that is, it is not continuously differentiable) at those specific values of λ_2. This phenomenon is due to the occurrence of a new switching time (that is, a zero of the switching function) along the continuation process. Indeed, when the final time coincides with a switching time, the shooting function is still continuous but is not C^1 (see [14] for more details). To be more precise, here, if $0 \leqslant \lambda_2 \lesssim 0.01$ (by $\lesssim$ we mean that $\lambda_2 \leqslant c$ with $c \approx 0.01$) then the modulus

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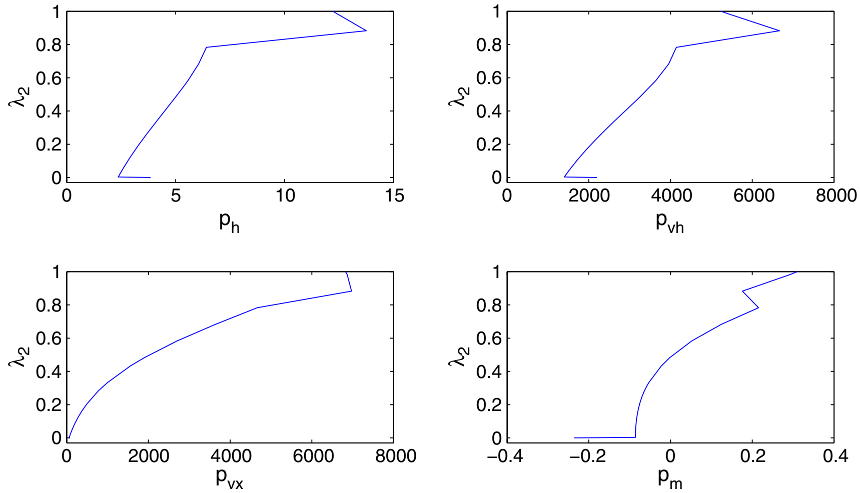

Figure 3. Evolution of the shooting function unknowns $(p_h, p_{v_x}, p_{v_h}, p_m)$ (abscissa) with respect to homotopic parameter λ_2 (ordinate).

of the thrust $\|u(\cdot)\|T_{\max}$ is of the kind $T_{\max}-0$. For $\lambda_2 \approx 0.01$, the final time coincides with a switching time after which we observe the appearance of a new thrust arc; indeed, if $0.01 \lesssim \lambda_2 \lesssim 0.8$, then the modulus of the thrust $\|u(\cdot)\|T_{\max}$ is of the kind $T_{\max} - 0 - T_{\max}$. If $0.8 \lesssim \lambda_2 \lesssim 0.82$, then the latter thrust arc disappears and the strategy is of the kind $T_{\max} - 0$, and if $0.82 \lesssim \lambda_2 \leqslant 1$, then the strategy is again of the kind $T_{\max} - 0 - T_{\max}$ (as it could be expected for $\lambda_2 = 1$).

Figure 4 compares the trajectory and control strategy of $(\mathbf{OCP})_{1,0}$ and $(\mathbf{OCP})_{1,1}$. We observe that the solution of $(\mathbf{OCP})_{1,0}$ is clearly not acceptable because its altitude becomes negative. However, $(\mathbf{OCP})_{1,0}$ is only a fictive problem and there is no need to only accept collision-free trajectory until we solve $(\mathbf{OCP})_{1,1}$. The main difference between the two control strategies is that $(\mathbf{OCP})_{1,0}$ (and $(\mathbf{OCP})_{0,0}$) only has one thrust arc, whereas $(\mathbf{OCP})_{1,1}$ has two. Furthermore, the fact that the fuel consumption is directly proportional to the thrust duration explains that $m_{1,1}(t_f)$ could be expected to be better than $m_{1,0}(t_f)$.

As mentioned before, we decided to fix the final time t_f of $(\mathbf{OCP})_{1,1}$ to the free final time obtained while solving $(\mathbf{OCP})_{1,0}$. Notice that it is possible to solve $(\mathbf{OCP})_{1,1}$ with another value of

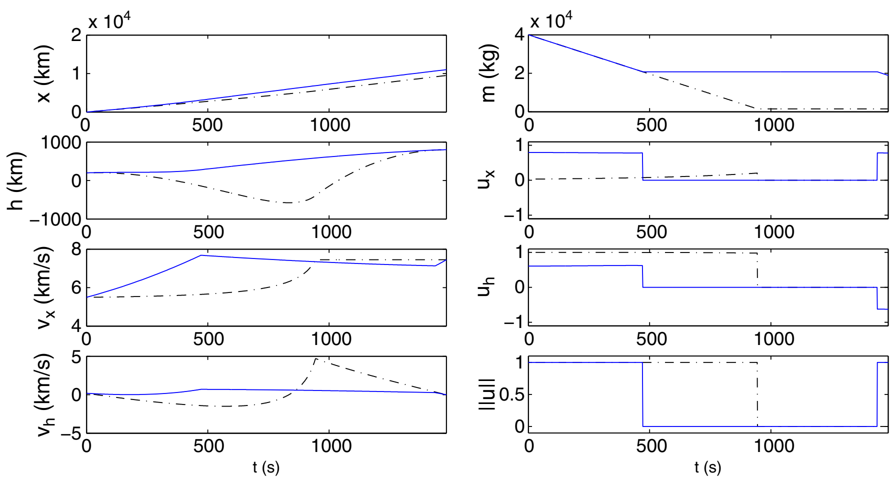

Figure 4. Trajectory and control strategy of $(\mathbf{OCP})_{1,0}$ (dashed) and $(\mathbf{OCP})_{1,1}$ (plain).

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fixed final time by using a linear continuation on t_f. For example, this continuation permits find a solution for $t_f = 2000$ s with a corresponding final mass $m_f \approx 20\,050$ kg.

Also, because our final orbit is not strictly circular, it can be interesting to consider final conditions (7) instead of (6). As mentioned in Remark 1, this can be achieved using an additional continuation on the final conditions and transversality conditions.

The whole procedure is time efficient because it only takes approximately 3 s on a standard desktop computer, without any code optimization. The execution time is roughly decomposed as follows:

- Instantaneous for the solving of $(\mathbf{OCP})_{0,0}$.
- 0.5 s for the first continuation from $(\mathbf{OCP})_{0,0}$ to $(\mathbf{OCP})_{1,0}$.
- 2.5 s for the second continuation from $(\mathbf{OCP})_{1,0}$ to $(\mathbf{OCP})_{1,1}$.
- 0.3 s for the possible additional continuation on the transversality conditions (see Remark 1).

The accuracy on the solution is 10^{-12}. Accuracy and execution time are very good because our method is based on the shooting method, which consists in particular of a Newton-like method.

4.2. Comparison with a direct method

In this section, we compare our approach with a direct method consisting of solving $(\mathbf{OCP})_{1,1}$ using a full discretization of the state and control and to a rewriting of the dynamic as nonlinear constraints of the resulting NonLinear optimization Problem (NLP) (as mentioned in the introduction, we refer to [7] for details on direct methods). We choose a discretization leading to

$$(NLP)\begin{cases} \min\limits_{z\in\mathcal{D}} \; -X_5^N, \\ X^{i+1} = X^i + \displaystyle\int_{t_i}^{t_{i+1}} \dot{X}(t, X^i, U^i, U^{i+1})dt, \; i = 0, \cdots, N-1 \\ X^0 = (x_0, h_0, v_0, \gamma_0, m_0) \in \mathbb{R}^5, \\ X_{2,3,4}^N = (h_f, v_f, \gamma_f) \in \mathbb{R}^3, \\ z = (X^0, \cdots, X^N, U^0, \cdots, U^N), \\ \mathcal{D} = \mathbb{R}^{5\times(N+1)} \times \mathcal{U}^{N+1}, \end{cases} \tag{36}$$

where $t_i = \frac{i t_f}{N}$, $i = 0, \ldots, N$ define the uniform time discretization. The optimization parameters X_i, $i = 0, \cdots, N$ are the values of the state at each t_i and $U^i \in \mathcal{U}$, $i = 0, \cdots, N$ are the values of the control at each t_i. The relation between X^{i+1} and (X^i, U^i, U^{i+1}) represents the dynamic and the integral is approximated thanks to a numerical integration scheme (for example Euler or fourth-order Runge–Kutta). Note that with this rewriting, we can take the control to be piecewise constant or piecewise linear.

To make the comparison with our continuation method, we rewrite the dynamics with a Euler or Heun integration scheme, and we set the final time to the same value as the one found with $(\mathbf{OCP})_{1,0}$. We use the modeling language *AMPL* (see [22]) combined with the optimization routine IPOPT, (COIN-OR) (see [23]) to solve the (NLP). To initialize the method, we choose to propagate a control strategy with two thrust arcs with durations and directions that are roughly the same as the one found with our method (a random initialization would not work). Starting with a coarse uniform time discretization of 100 points with the Euler integration scheme, and using the solution to initialize a time discretization of 1000 points with Heun integration scheme, we find a solution that is close (up to the accuracies of both methods) to the one we found with our approach. The execution time of this direct method is of 5 s for $N = 100$ and of 165 s for $N = 1000$. We recall that the execution time for our method on this example was 3 s. It is important to note that even with a time discretization of 1000 points, the accuracy of the solution (of the integration) of the direct approach is of the order of 10^{-6} (10^{-2} for $N = 100$ and Euler scheme), whereas the accuracy of the shooting method we used was of the order of 10^{-12} (thanks to the high-order integration method). Of course, with a different integration scheme, say a fourth-order Runge–Kutta, the accuracy of the direct method would be better but at the cost of a larger computational effort. However, even with a higher order integration scheme, the accuracy of the solution is limited by the way the control is discretized. As expected, the direct approach is computationally far more demanding.

4.3. Comparison with other initialization methods

An interesting comparison would be with the method presented in [10]. This method consists in using an approximate solution of the impulse transfer in order to explicitly compute estimates of the adjoint vector needed to perform a single shooting. Those estimates are possible when considering orbit transfer with nearly circular initial and final orbits. When dealing with this kind of Hohmann transfer, it is then preferable to use this method because the estimates are computed analytically and are enough to ensure the convergence of the shooting method. Because the shooting method converges nearly instantaneously, it seems unlikely for another method to perform better. And indeed, our approach cannot compete with [10] for Hohmann like orbit transfer. However, our approach is not restricted to nearly circular orbit transfer and is then a complement to the one of [10].

Another method, which propose initialization scheme for similar kind of problems is discussed in [13]. In this paper, the orbital transfer problem is first solved for the minimization of the square of the $L_2 - norm$ of the control. Then, a continuation is performed to link this criterion to the minimization of the $L_1 - norm$ of the control. This last criterion is equivalent to the maximization of the final mass. However, the method is restricted to low-thrust orbit transfers only, whereas our method is designed for high-thrust orbit transfers.

4.4. Restriction to high-thrust orbit transfer

Our method was designed for high-thrust orbit transfers, that is for orbit transfers with acceleration of the same order of magnitude as the Earth's gravity. It is doubtful that it can be extended to low-thrust cases. Indeed, the first step of the method, the resolution of the simplified flat Earth problem, will not converge for low thrust.

5. CONCLUSION

We have given an algorithmic procedure to solve the problem of minimization of the fuel consumption for the coplanar orbit transfer problem by a shooting method approach, without any a priori knowledge on the optimal solution (and thus on the way to initialize the shooting method). Our method relies on the preliminary remark that when studying the same problem within a simplified flat Earth model with constant gravity, the optimal control problem can be explicitly solved, and the solution leads to a very efficient algorithm that does not need any careful initial guess. Based on that remark, we have proposed a continuous deformation of this simplified model to the initial model (up to some change of coordinates), introducing continuously corrective terms into the flat Earth model. From the algorithmic point of view, the procedure then consists of solving a series of shooting problems, starting from the simplified flat Earth model which is easy to initialize and ending up with the sought solution. The whole procedure is time-efficient and provides a way for bypassing the difficulty due to the initialization of the shooting method when it is applied directly to the initial problem.

Many questions remain open and from this point of view, our work should be considered as preliminary. A first question is to investigate whether this procedure is systematically efficient, for any possible coplanar orbit transfer. Up to now, we did not make any exhaustive tests; however, it is very probable that one may encounter some difficulties, as in any continuation process, because of the intricate topology of the space of possible continuation paths, this space being not always arc-wise connected. Indeed, the flat Earth model only has one thrust arc, whereas the round Earth model has two or more. Another question is to extend our study to the three-dimensional case, the final objective for an enterprise as *Astrium Space Transportation* being to have available a reliable and efficient tool to realize any possible orbit transfer without having to spend much time on the initialization of the algorithm.

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APPENDIX: ACTION OF A CHANGE OF COORDINATES ONTO THE ADJOINT VECTOR

To understand how a change of coordinates acts onto the adjoint vector, it is useful to come back to the geometric meaning of the PMP, recalling its intrinsic character.

Let M (resp. N) be a smooth manifold of dimension n (resp. m). Consider on M the control system $\dot{x}(t) = f(x(t), u(t))$, where $f : M \times N \longrightarrow TM$ is smooth, TM is the usual tangent bundle, and the controls are bounded measurable functions taking their values in a subset U of N. Let M_0 and M_1 be two subsets of M. Consider the optimal control problem of determining a trajectory $x(\cdot)$ solution of the control system, associated with a control $u(\cdot)$ on $[0, t_f]$, so that $x(0) \in M_0$, $x(t_f) \in M_1$, and minimizing a cost function $C(t_f, u) = \int_0^{t_f} f^0(x(t), u(t))dt$,, where $f^0 : M \times N \longrightarrow \mathbb{R}$ is smooth, and the final time t_f may be fixed or not. According to the PMP, if $x(\cdot)$ is optimal, then there exists $p^0 \leqslant 0$ and an absolutely continuous mapping $p(\cdot)$ on $[0, t_f]$ (adjoint vector) satisfying $(p(\cdot), p^0) \neq (0, 0)$ and $p(t) \in T^*_{x(t)}M$, such that

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), p^0, u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), p^0, u(t)),$$

almost everywhere on $[0, t_f]$, where $H(x, p, p^0, u) = \langle p, f(x, u)\rangle + p^0 f^0(x, u)$ is the Hamiltonian, and $H(x(t), p(t), p^0, u(t)) = M(x(t), p(t), p^0)$, almost everywhere on $[0, t_f]$, where $M(x(t), p(t), p^0) = \max_{v \in U} H(x(t), p(t), p^0, v)$. If the final time t_f is not fixed, there holds moreover $M(x(t), p(t), p^0) = 0$, for every $t \in [0, t_f]$. If M_0 and M_1 (or just one of both) are regular submanifolds of M, then the adjoint vector can be chosen so that $p(0) \perp T_{x(0)}M_0$ and $p(t_f) \perp T_{x(t_f)}M_1$ (transversality conditions).

Settled in such a way on a manifold, we recall that the PMP is *intrinsic*; that is, its statement does not depend on the specific choice of coordinates. This intrinsic version has been proved, for example, in [24].

Let now M_1 (resp. N_1) be another smooth manifold of dimension n (resp. m), and let $\phi : M \to M_1$ (resp. $\psi : N \to N_1$) be a diffeomorphism. Then, it is well-known in differential geometry that the differential $d\phi$ maps diffeomorphically the tangent bundle TM into TM_1, and that the transpose (also called adjoint) of its inverse ${}^t d\Phi^{-1}$ maps diffeomorphically the cotangent bundle T^*M into T^*M_1. From this remark and from the intrinsic character of the PMP, we derive the following claim.

Let $x_1(t) = \phi(x(t))$ and $u_1(t) = \psi(u(t))$. The trajectory $x_1(\cdot)$, associated to the control $u_1(\cdot)$, is the solution of the control system $\dot{x}_1(t) = f_1(x_1(t), u_1(t)) = d\phi(x(t)).f(\phi^{-1}(x_1(t), \psi^{-1}(u_1(t))$ and corresponds to $x(\cdot)$ via the change of coordinates ϕ on the state and ψ on the control. Then, the adjoint vector $p_1(\cdot)$ associated with the trajectory $x(\cdot)$ is given by

$$p_1(\cdot) = {}^t d\phi(x(\cdot))^{-1} p(\cdot). \tag{37}$$

The formula (37) may of course be proved directly, without any geometric insight, by using Cauchy uniqueness arguments.

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