

CONTROL OF TRAVELLING WALLS IN A FERROMAGNETIC NANOWIRE

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ABSTRACT. We investigate the problem of controlling the magnetic moment in a ferromagnetic nanowire submitted to an external magnetic field in the direction of the nanowire. The system is modeled with the one dimensional Landau-Lifschitz equation. In the absence of control, there exist particular solutions, which happen to be relevant for practical issues, called travelling walls. In this paper, we prove that it is possible to move from a given travelling wall profile to any other one, by acting on the external magnetic field. The control laws are simple and explicit, and the resulting trajectories are shown to be stable.

1. Introduction and main result. The most common model used to describe the behavior of ferromagnetic materials, called micromagnetism, was introduced by W.-F. Brown in the 60's (see [3]). It is based on a thermodynamic approach, and the main idea is that equilibrium states of the magnetization minimize a given energy functional, consisting of several components. The main components, which permit an accurate description of the behavior of ferromagnetic materials, are the magnetostatic one (electromagnetism), the exchange one (spin-spin interactions), the anisotropy one (crystal shape influence) and the Zeeman one (external influences). This point of view permits to recover the standard dynamical approach of ferromagnetic phenomena, based on the so-called *Landau-Lifschitz equation*, which was introduced in the 30's in [16]. This equation contains a hamiltonian term stemming from the Larmor microscopic spin precession equation, and a purely dissipative term, perpendicular to the precession component and related to the Euler equation of the static energy functional.

More precisely, ferromagnetic materials are characterized by a spontaneous magnetization described by the magnetic moment u which is a unitary vector field linking the magnetic induction B with the magnetic field H by the relation $B = H + u$.

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The magnetic moment u is solution of the Landau-Lifschitz equation

$$\frac{\partial u}{\partial t} = -u \wedge H_e - u \wedge (u \wedge H_e), \quad (1)$$

where the effective field is given by $H_e = \Delta u + h_d(u) + H_a$, and the demagnetizing field $h_d(u)$ is solution of the magnetostatic equations

$$\operatorname{div} B = \operatorname{div} (H + u) = 0 \text{ and } \operatorname{curl} H = 0,$$

where H_a is an applied magnetic field. More details on the ferromagnetism model are provided in [3, 11, 16, 21]. Existence results have been established for the Landau-Lifschitz equation in [4, 5, 12, 20], numerical aspects have been investigated in [10, 14, 15], and asymptotic properties have been proved in [1, 6, 9, 17, 19].

In this article, we consider an asymptotic one dimensional model of ferromagnetic nanowire submitted to an applied field along the axis of the wire. Let (e_1, e_2, e_3) denote the canonical basis of \mathbb{R}^3 . The ferromagnetic nanowire is assumed to have an infinite length, and is represented by the axis $\mathbb{R}e_1$. The demagnetizing energy is given by $h_d(u) = -u_2e_2 - u_3e_3$ where $u = (u_1, u_2, u_3)$ (see [19] where this formula has been derived using a BKW method, by considering a wire of nonzero diameter, and taking the limit when the diameter tends to zero). In addition, we assume that an external magnetic field $\delta(t)e_1$ is applied along the wire axis. The real-valued function $\delta(\cdot)$ is our control.

The Landau-Lifschitz equation writes

$$\frac{\partial u}{\partial t} = -u \wedge h_\delta(u) - u \wedge (u \wedge h_\delta(u)), \quad (2)$$

where $h_\delta(u) = \frac{\partial^2 u}{\partial x^2} - u_2e_2 - u_3e_3 + \delta e_1$.

The magnetic field $\delta(\cdot)e_1$ is generated by a coil winding up around the nanowire. Note that it would be extremely costly to generate a magnetic field in other directions, all along the nanowire. By the way, such other controls do not lead actually to relevant solutions.

In terms of control system, setting $h(u) = u_{xx} - u_2e_2 - u_3e_3$, Equation (2) yields

$$u_t = -u \wedge h(u) - u \wedge (u \wedge h(u)) - \delta(u \wedge e_1 + u \wedge (u \wedge e_1)). \quad (3)$$

For $\delta \equiv 0$, physical experiments demonstrate the existence of a particular stationary solution, splitting the nanowire into two parts. The magnetic moment is almost equal to e_1 in one of them, and to $-e_1$ in the other. This fundamental stationary solution, called a *wall*, is analytically given by

$$M_0(x) = \begin{pmatrix} \operatorname{th} x \\ 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix}. \quad (4)$$

Here, and throughout the paper, the notations ch , sh , and th , respectively stand for the hyperbolic cosine, sine, and tangent functions.

Stability properties of the solution M_0 for System (3) with $\delta \equiv 0$ have been established in [7].

When applying a constant magnetic field in the direction e_1 (i.e., with a constant control function $\delta(\cdot) \equiv \delta$), physical experiments show a translation/rotation of the above wall along the nanowire. The corresponding mathematical solution of (3), associated with the constant control δ , is

$$u^\delta(t, x) = R_{\delta t} M_0(x + \delta t), \quad (5)$$

where

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

is the rotation of angle θ around the axis $\mathbb{R}e_1$. Furthermore, Equation (3) is invariant with respect to:

- translations $x \mapsto x - \sigma$, along the nanowire;
- rotations R_θ around the axis e_1 .

Hence, denoting $\Lambda = (\theta, \sigma)$, one has a two parameter family of symmetries given by $M_\Lambda u(t, x) = R_\theta u(t, x - \sigma)$. Therefore, we have a three-parameters family of particular solutions of (3) defined by

$$u^{\delta, \theta, \sigma}(t, x) = M_\Lambda u^\delta(t, x) = R_{\delta t + \theta} M_0(x + \delta t - \sigma) \quad (6)$$

and called *travelling wall profiles*.

Theorem 1. *There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for all $\delta_1, \delta_2 \in \mathbb{R}$ satisfying $|\delta_i| \leq \delta_0$, $i = 1, 2$, for all $\sigma_1, \sigma_2 \in \mathbb{R}$, for every $\varepsilon \in (0, \varepsilon_0)$, there exist $T > 0$ and a control function $\delta(\cdot) \in L^\infty(\mathbb{R}, \mathbb{R})$ such that, for every solution u of (3) associated with the control $\delta(\cdot)$ and satisfying*

$$\exists \theta_1 \in \mathbb{R} \quad \|u(0, \cdot) - u^{\delta_1, \theta_1, \sigma_1}(0, \cdot)\|_{H^2} \leq \varepsilon, \quad (7)$$

there exists a real number θ_2 such that

$$\|u(T, \cdot) - u^{\delta_2, \theta_2, \sigma_2}(T, \cdot)\|_{H^2} \leq \varepsilon. \quad (8)$$

Moreover, there exists real numbers θ'_2 and σ'_2 , with $|\theta'_2 - \theta_2| + |\sigma'_2 - \sigma_2| \leq \varepsilon$, such that

$$\|u(t, \cdot) - u^{\delta_2, \theta'_2, \sigma'_2}(t, \cdot)\|_{H^2} \xrightarrow[t \rightarrow +\infty]{} 0. \quad (9)$$

The control law $\delta(\cdot)$ realizing the conclusion of the theorem is actually given by the piecewise constant function

$$\delta(t) = \begin{cases} \delta_2 - \frac{\sigma_2 - \sigma_1}{T} & \text{if } 0 \leq t \leq T, \\ \delta_2 & \text{if } t \geq T. \end{cases} \quad (10)$$

It combines the advantages of being very simple to implement, and of sharing robustness properties in H^2 norm, as claimed in the theorem.

The time T of the theorem is arbitrary, but must be large enough so that

$$\left| \delta_2 - \frac{\sigma_2 - \sigma_1}{T} \right| \leq \delta_0.$$

This theorem shows that the family of travelling wall profiles (6) is approximately controllable in H^2 norm, locally in δ and globally in σ , in time sufficiently large. The controllability property with respect to θ is not clear. Intuitively the system should not be controllable in θ , however this question is not very relevant from the physical point of view, since it is the position of the wall which is physically interesting. In particular, our result asserts that it is possible to pass approximately (up to the variable θ from a wall profile $u^{0, \theta, \sigma}$ to any other by means of a scalar control of the form (10). This approximate controllability result may have applications for magnetic recording. Note that, on the one part, an exact controllability result does not seem to be reachable, due to the physical properties of the system, and on

the other part, this approximate controllability property is sufficient for practical interest.

Up to now, only the one dimensional case, that is, a ferromagnetic nanowire, has been considered for control applications. What happens in the two dimensional case is an open question.

2. Proof of Theorem 1. We follow the same lines as in [7, 8], and first express the Landau-Lifschitz equation in convenient coordinates. This permits to establish stability properties, and then to derive the result.

2.1. Expression of the system in adapted coordinates. The control function (10) considered here being piecewise constant, it suffices to consider Equation (3) on each subinterval. Hence, we assume hereafter that the control function $\delta(\cdot)$ is constant, equal to δ . Let u be a solution of (3). Set $v(t, x) = R_{-\delta t}(u(t, x - \delta t))$. It is not difficult to check that

$$v_t = -v \wedge h(v) - v \wedge (v \wedge h(v)) - \delta(v_x + v_1 v - e_1). \quad (11)$$

Consider the mobile frame $(M_0(x), M_1(x), M_2)$, where $M_1(\cdot)$ and M_2 are defined by

$$M_1(x) = \begin{pmatrix} 1 \\ \text{ch } x \\ 0 \\ -\text{th } x \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

In what follows, we will prove that v is close to M_0 . This allows to decompose $v : \mathbb{R}^+ \times \mathbb{R} \rightarrow S^2 \subset \mathbb{R}^3$ in the mobile frame as

$$v(t, x) = \sqrt{1 - r_1(t, x)^2 - r_2(t, x)^2} M_0(x) + r_1(t, x) M_1(x) + r_2(t, x) M_2.$$

Easy but lengthy computations show that v is solution of (11) if and only if $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ satisfies

$$r_t = Ar + R_\delta(x, r, r_x, r_{xx}), \quad (12)$$

where

$$R_\delta(x, r, r_x, r_{xx}) = -\delta \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} r + G(r)r_{xx} + H_1(x, r)r_x + H_2(r)(r_x, r_x) + P_\delta(x, r), \quad (13)$$

and

- $A = \begin{pmatrix} L & L \\ -L & L \end{pmatrix}$ with $L = \partial_{xx} + (1 - 2\text{th}^2 x)\text{Id}$;
- $\ell = \partial_x + \text{th } x \text{Id}$;
- $G(r)$ is the matrix defined by

$$G(r) = \begin{pmatrix} \frac{r_1 r_2}{\sqrt{1 - |r|^2}} & \frac{r_2^2}{\sqrt{1 - |r|^2}} + \sqrt{1 - |r|^2} - 1 \\ -\frac{r_1^2}{\sqrt{1 - |r|^2}} - \sqrt{1 - |r|^2} + 1 & -\frac{r_1 r_2}{\sqrt{1 - |r|^2}} \end{pmatrix};$$

- $H_1(x, r)$ is the matrix defined by

$$H_1(x, r) = \frac{2}{\sqrt{1 - |r|^2} \text{ch } x} \begin{pmatrix} r_2 \sqrt{1 - |r|^2} - r_1 r_2^2 & -r_2 + r_2 r_1^2 \\ r_2 - r_2^3 & \sqrt{1 - |r|^2} r_2 + r_1 r_2^2 \end{pmatrix};$$

- $H_2(r)$ is the quadratic form on \mathbb{R}^2 defined by

$$H_2(r)(X, X) = \frac{(1 - |r|^2)X^T X + (r^T X)^2}{(1 - |r|^2)^{3/2}} \begin{pmatrix} \sqrt{1 - |r|^2}r_1 + r_2 \\ \sqrt{1 - |r|^2}r_2 - r_1 \end{pmatrix};$$

- $P_\delta(x, r) = \begin{pmatrix} P_\delta^1(x, r) \\ P_\delta^2(x, r) \end{pmatrix}$, with

$$\begin{aligned} P_\delta^1(x, r) &= 2r_2(\sqrt{1 - |r|^2} - 1)\frac{1}{\operatorname{ch}^2 x} - 2r_1 r_2 \frac{\operatorname{sh} x}{\operatorname{ch}^2 x} - 2r_1 |r|^2 \frac{1}{\operatorname{ch}^2 x} \\ &\quad - 2r_1^2 \sqrt{1 - |r|^2} \frac{\operatorname{sh} x}{\operatorname{ch}^2 x} + r_1^3 + r_2(1 - \sqrt{1 - |r|^2}) + r_1 r_2^2 \\ &\quad - \delta \left(\frac{1}{\operatorname{ch} x} (\sqrt{1 - |r|^2} - 1 + r_1^2) + (\sqrt{1 - |r|^2} - 1)r_1 \operatorname{th} x \right) \end{aligned}$$

and

$$\begin{aligned} P_\delta^2(x, r) &= -2r_1(\sqrt{1 - |r|^2} - 1)\frac{1}{\operatorname{ch}^2 x} + 2r_1^2 \frac{\operatorname{sh} x}{\operatorname{ch}^2 x} - 2r_2 |r|^2 \frac{1}{\operatorname{ch}^2 x} \\ &\quad - 2r_1 r_2 \sqrt{1 - |r|^2} \frac{\operatorname{sh} x}{\operatorname{ch}^2 x} + r_2 |r|^2 \\ &\quad - \delta \left(\frac{1}{\operatorname{ch} x} r_1 r_2 + (\sqrt{1 - |r|^2} - 1)r_2 \operatorname{th} x \right). \end{aligned}$$

It is not difficult to prove that there exists a constant $C > 0$ such that, if $\|r\|_{\mathbb{R}^2}^2 = |r|^2 \leq \frac{1}{2}$ and $|\delta| \leq 1$, then, for every $x \in \mathbb{R}$, for all $p, q \in \mathbb{R}^2$,

$$\begin{aligned} \|R_\delta(x, r, p, q)\|_{\mathbb{R}^2} &\leq C(|\delta| \|p\|_{\mathbb{R}^2} + \|r\|_{\mathbb{R}^2}^2 \|q\|_{\mathbb{R}^2} + \|r\|_{\mathbb{R}^2} \|p\|_{\mathbb{R}^2} + \|r\|_{\mathbb{R}^2} \|p\|_{\mathbb{R}^2}^2 + \|r\|_{\mathbb{R}^2}^2). \end{aligned} \quad (14)$$

This a priori estimate shows that $R_\delta(x, r, r_x, r_{xx})$ is a remainder term in Equation (12). The rest of the proof relies on a spectral analysis of the linear operator A , so as to establish stability properties for Equation (12).

First of all, notice that L is a selfadjoint operator on $L^2(\mathbb{R})$, of domain $H^2(\mathbb{R})$, and that $L = -\ell^* \ell$ with $\ell = \partial_x + \operatorname{th} x \operatorname{Id}$ (one has $\ell^* = -\partial_x + \operatorname{th} x \operatorname{Id}$). It follows that L is nonpositive, and that $\ker L = \ker \ell$ is the one dimensional subspace of $L^2(\mathbb{R})$ generated by $\frac{1}{\operatorname{ch} x}$. In particular, the operator L , restricted to the subspace $E = (\ker L)^\perp$, is negative.

Remark 1. It is obvious that, on the subspace E :

- the norms $\|(-L)^{1/2} f\|_{L^2(\mathbb{R})}$ and $\|f\|_{H^1(\mathbb{R})}$ are equivalent;
- the norms $\|L f\|_{L^2(\mathbb{R})}$ and $\|f\|_{H^2(\mathbb{R})}$ are equivalent;
- the norms $\|(-L)^{3/2} f\|_{L^2(\mathbb{R})}$ and $\|f\|_{H^3(\mathbb{R})}$ are equivalent.

Writing $A = JL$, with

$$J = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

it is clear that the kernel of A is $\ker A = \ker L \times \ker L$; it is the two dimensional space of $L^2(\mathbb{R}^2)$ generated by

$$a_1(x) = \begin{pmatrix} 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix} \quad \text{and} \quad a_2(x) = \begin{pmatrix} \frac{1}{\operatorname{ch} x} \\ 0 \end{pmatrix}.$$

Moreover, combining the facts that $L|_{(\ker L)^\perp}$ is negative and that $\text{Spec } J = \{1 + i, 1 - i\}$, it follows that the operator A , restricted to the subspace $\mathcal{E} = (\ker A)^\perp$, is negative.

These facts suggest to decompose solutions r of (12) as the sum of an element of $\ker A$ and of an element of \mathcal{E} .

To this aim, recall that, since Equation (11) is invariant with respect to translations in x and rotations around the axis e_1 , for every $\Lambda = (\theta, \sigma) \in \mathbb{R}^2$, $M_\Lambda(x) = R_\theta M_0(x - \sigma)$ is solution of (11). Define

$$R_\Lambda(x) = \begin{pmatrix} \langle M_\Lambda(x), M_1(x) \rangle \\ \langle M_\Lambda(x), M_2(x) \rangle \end{pmatrix},$$

the coordinates of $M_\Lambda(x)$ in the mobile frame $(M_1(x), M_2(x))$.

We claim that the mapping

$$\begin{aligned} \Psi : \mathbb{R}^2 \times \mathcal{E} &\longrightarrow H^2(\mathbb{R}) \\ (\Lambda, W) &\longmapsto r(x) = R_\Lambda(x) + W(x) \end{aligned}$$

is a diffeomorphism from a neighborhood \mathcal{U} of zero in $\mathbb{R}^2 \times \mathcal{E}$ into a neighborhood \mathcal{V} of zero in $H^2(\mathbb{R})$. Indeed, if $r = R_\Lambda + W$ with $W \in \mathcal{E}$, then, by definition,

$$\langle r, a_1 \rangle_{L^2} = \langle R_\Lambda, a_1 \rangle_{L^2} \quad \text{and} \quad \langle r, a_2 \rangle_{L^2} = \langle R_\Lambda, a_2 \rangle_{L^2}. \quad (15)$$

Conversely, if $\Lambda \in \mathbb{R}^2$ satisfies ((15)), then $W = r - R_\Lambda \in \mathcal{E}$. The mapping $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, defined by $h(\Lambda) = (\langle R_\Lambda, a_1 \rangle_{L^2}, \langle R_\Lambda, a_2 \rangle_{L^2})$ is smooth and satisfies $dh(0) = -2\text{Id}$, thus is a local diffeomorphism at $(0, 0)$. It follows easily that Ψ is a local diffeomorphism at zero.

Therefore, every solution r of (12), as long as it stays¹ in the neighborhood \mathcal{V} , can be written as

$$r(t, \cdot) = R_{\Lambda(t)}(\cdot) + W(t, \cdot), \quad (16)$$

where $W(t, \cdot) \in \mathcal{E}$ and $\Lambda(t) \in \mathbb{R}^2$, for every $t \geq 0$, and $(\Lambda(t), W(t, \cdot)) \in \mathcal{U}$. In these new coordinates², Equation (12) leads to (see [7] for the details of computations)

$$\begin{aligned} W_t(t, x) &= AW(t, x) + \mathcal{R}(\delta, \Lambda(t), x, W(t, x), W_x(t, x), W_{xx}(t, x)), \\ \Lambda'(t) &= \mathcal{M}(\Lambda(t), W(t, \cdot), W_x(t, \cdot)), \end{aligned} \quad (17)$$

where $\mathcal{R} : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times (H^2(\mathbb{R}))^2 \times (H^1(\mathbb{R}))^2 \times (L^2(\mathbb{R}))^2 \longrightarrow \mathcal{E}$ and $\mathcal{M} : \mathbb{R}^2 \times (H^1(\mathbb{R}))^2 \times (L^2(\mathbb{R}))^2 \longrightarrow \mathbb{R}^2$ are nonlinear mappings, for which there exist constants $K > 0$ and $\eta > 0$ such that

$$\begin{aligned} &\|\mathcal{R}(\delta, \Lambda, \cdot, W, W_x, W_{xx})\|_{(H^1(\mathbb{R}))^2} \\ &\leq K \left(\|\Lambda\|_{\mathbb{R}^2} + |\delta| + \|W\|_{(H^2(\mathbb{R}))^2} \right) \|W\|_{(H^3(\mathbb{R}))^2}, \end{aligned} \quad (18)$$

$$|\mathcal{M}(\Lambda, W, W_x)| \leq K \left(\|\Lambda\|_{\mathbb{R}^2} + \|W\|_{(H^1(\mathbb{R}))^2} \right) \|W\|_{(H^1(\mathbb{R}))^2}, \quad (19)$$

for every $W \in \mathcal{E}$, every $\delta \in \mathbb{R}$, and every $\Lambda \in \mathbb{R}^2$ satisfying $\|\Lambda\|_{\mathbb{R}^2} \leq \eta$.

Remark 2. Using the fact that L is selfadjoint, it is obvious to prove that $AW \in \mathcal{E}$, for every $W \in \mathcal{E}$; hence, (17) makes sense.

¹This a priori estimate will be a consequence of the stability property derived next.

²This decomposition is actually quite standard and has been used e.g. in [13] to establish stability properties of static solutions of semilinear parabolic equations, and in [2, 18] to prove stability of travelling waves.

2.2. Stability properties, and proof of Theorem 1. We are now in position to establish stability properties for system (17). Denoting $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$, define on $(H^2(\mathbb{R}))^2 \times \mathbb{R}^2$ the function

$$\mathcal{V}(W) = \frac{1}{2} \left\| \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} W \right\|_{(L^2(\mathbb{R}))^2}^2 = \frac{1}{2} \|LW_1\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|LW_2\|_{L^2(\mathbb{R})}^2. \quad (20)$$

Remark 3. It follows from Remark 1 that, on the subspace $\mathcal{E} = (\ker A)^\perp$, $\sqrt{\mathcal{V}(W)}$ is a norm, which is equivalent to the norm $\|W\|_{(H^2(\mathbb{R}^2))}^2$.

Consider a solution (W, Λ) of (17), such that $W(0, \cdot) = W_0(\cdot)$ and $\Lambda(0) = \Lambda_0$. Since L is selfadjoint, one has

$$\begin{aligned} & \frac{d}{dt} \mathcal{V}(W(t, \cdot)) \\ &= \left\langle AW, \begin{pmatrix} L^2 W_1 \\ L^2 W_2 \end{pmatrix} \right\rangle_{(L^2(\mathbb{R}))^2} \\ &+ \left\langle \begin{pmatrix} (-L)^{1/2} & 0 \\ 0 & (-L)^{1/2} \end{pmatrix} \mathcal{R}(\delta, \Lambda, \cdot, W, W_x, W_{xx}), \begin{pmatrix} (-L)^{3/2} W_1 \\ (-L)^{3/2} W_2 \end{pmatrix} \right\rangle_{(L^2(\mathbb{R}))^2}. \end{aligned} \quad (21)$$

Concerning the first term of the right-hand side of (21), one computes

$$\left\langle AW, \begin{pmatrix} L^2 W_1 \\ L^2 W_2 \end{pmatrix} \right\rangle_{(L^2(\mathbb{R}))^2} = -\|(-L)^{3/2} W_1\|_{(L^2(\mathbb{R}))^2} - \|(-L)^{3/2} W_2\|_{(L^2(\mathbb{R}))^2},$$

and, using Remark 1, there exists a constant $C_1 > 0$ such that

$$\left\langle AW, \begin{pmatrix} L^2 W_1 \\ L^2 W_2 \end{pmatrix} \right\rangle_{(L^2(\mathbb{R}))^2} \leq -C_1 \|W\|_{(H^3(\mathbb{R}))^2}^2. \quad (22)$$

Concerning the second term of the right-hand side of (21), one deduces from the Cauchy-Schwarz inequality, from Remark 1, and from the estimate (18), that

$$\begin{aligned} & \left| \left\langle \begin{pmatrix} (-L)^{1/2} & 0 \\ 0 & (-L)^{1/2} \end{pmatrix} \mathcal{R}(\delta, \Lambda, \cdot, W, W_x, W_{xx}), \begin{pmatrix} (-L)^{3/2} W_1 \\ (-L)^{3/2} W_2 \end{pmatrix} \right\rangle_{(L^2(\mathbb{R}))^2} \right| \\ & \leq \|\mathcal{R}(\delta, \Lambda, \cdot, W, W_x, W_{xx})\|_{(H^1(\mathbb{R}))^2} \|W\|_{(H^3(\mathbb{R}))^2} \\ & \leq K \left(\|\Lambda\|_{\mathbb{R}^2} + |\delta| + \|W\|_{(H^2(\mathbb{R}))^2} \right) \|W\|_{(H^3(\mathbb{R}))^2}^2. \end{aligned} \quad (23)$$

Hence, from (21), (22), and (23), one gets

$$\frac{d}{dt} \mathcal{V}(W) \leq \left(-C_1 + K \left(\|\Lambda\|_{\mathbb{R}^2} + |\delta| + \|W\|_{(H^2(\mathbb{R}))^2} \right) \right) \|W\|_{(H^3(\mathbb{R}))^2}^2.$$

If the a priori estimate

$$\|\Lambda(t)\|_{\mathbb{R}^2} + |\delta| + \|W(t, \cdot)\|_{(H^2(\mathbb{R}))^2} \leq \frac{C_1}{2K}$$

holds, then

$$\frac{d}{dt} \mathcal{V}(W(t, \cdot)) \leq -\frac{C_1}{2} \|W(t, \cdot)\|_{(H^3(\mathbb{R}))^2}^2 \leq -\frac{C_1}{2} \|W(t, \cdot)\|_{(H^2(\mathbb{R}))^2}^2 \leq -C_2 \mathcal{V}(W(t, \cdot))$$

(using Remark 3 for the existence of a constant $C_2 > 0$). It follows that there exist constants $C_3 > 0$ and $C_4 > 0$ such that, if $|\delta| + \|W(0, \cdot)\|_{(H^2(\mathbb{R}))^2} \leq \frac{C_1}{4K}$ are small enough, and if the a priori estimate

$$\max_{0 \leq s \leq t} \|\Lambda(s)\|_{\mathbb{R}^2} \leq \frac{C_1}{4K} \quad (24)$$

holds, then

$$\|W(s, \cdot)\|_{(H^2(\mathbb{R}))^2} \leq C_3 e^{-C_4 s} \|W(0, \cdot)\|_{(H^2(\mathbb{R}))^2}, \quad (25)$$

for every $s \in [0, T]$, and moreover, one deduces from (17), (19), and (25) that, if the a priori estimate (24) holds, then

$$\begin{aligned} \|\Lambda(t)\|_{\mathbb{R}^2} &\leq \|\Lambda(0)\|_{\mathbb{R}^2} + \frac{C_1 C_3}{4} \|W(0, \cdot)\|_{(H^2(\mathbb{R}))^2} \int_0^t e^{-C_4 s} ds \\ &\quad + K C_3^2 \|W(0, \cdot)\|_{(H^2(\mathbb{R}))^2}^2 \int_0^t e^{-2C_4 s} ds \\ &\leq \|\Lambda(0)\|_{\mathbb{R}^2} + \frac{C_1 C_3}{4C_4} \|W(0, \cdot)\|_{(H^2(\mathbb{R}))^2} + K \frac{C_3^2}{2C_4} \|W(0, \cdot)\|_{(H^2(\mathbb{R}))^2}^2. \end{aligned} \quad (26)$$

From all previous a priori estimates, we conclude that, if $|\delta| + \|\Lambda(0)\|_{\mathbb{R}^2} + \|W(0, \cdot)\|_{(H^2(\mathbb{R}))^2}$ is small enough, then $\|\Lambda(t)\|_{\mathbb{R}^2}$ remains small, for every $t \geq 0$, and $\|W(t, \cdot)\|_{(H^2(\mathbb{R}))^2}$ is exponentially decreasing to 0.

The first part of the theorem, on the interval $[0, T]$, easily follows from the above considerations. For the second part, observe that, from (17), (19), and (25), one deduces that $\|\Lambda'(t)\|_{\mathbb{R}^2}$ is integrable on $[0, +\infty)$, and hence, $\Lambda(t)$ has a limit in \mathbb{R}^2 , denoted $\Lambda_\infty = (\theta_\infty, \sigma_\infty)$, as t tends to $+\infty$. The theorem follows with $\theta'_2 = \theta_2 + \theta_\infty$ and $\sigma'_2 = \sigma_2 + \sigma_\infty$.

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