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SMOOTH CONTROL OF NANOWIRES BY MEANS OF A MAGNETIC FIELD

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Abstract. We address the problem of control of the magnetic moment in a ferromagnetic nanowire by means of a magnetic field. Based on theoretical results for the 1D Landau-Lifschitz equation, we show a robust controllability result.

1. **Model and control result.** The magnetic moment u of a ferromagnetic material is usually modeled as a unitary vector field, solution of the Landau-Lifschitz equation

$$\frac{\partial u}{\partial t} = -u \wedge H_e - u \wedge (u \wedge H_e), \tag{1}$$

where the effective field H_e is given by $H_e = \Delta u + h_d(u) + H_a$. The demagnetizing field $h_d(u)$ is solution of the magnetostatic equations

$$\text{div } B = \text{div } (H + u) = 0 \text{ and curl } H = 0,$$

where B is the magnetic induction. The applied field is denoted by H_a (see [3, 12, 17, 22] for more details on the modelization). Existence results have been established for the Landau-Lifschitz equation in [4, 5, 13, 21], numerical aspects have been investigated in [11, 15, 16], and asymptotic properties have been proved in [1, 6, 10, 18, 20]; control issues were addressed in [9].

Here we restrict ourselves to a one dimensional model, i.e., we consider a ferromagnetic nanowire, submitted to an external magnetic field applied along the axis of the wire and which is our control. The model is then written as (see [20])

$$\frac{\partial u}{\partial t} = -u \wedge h_\delta(u) - u \wedge (u \wedge h_\delta(u)), \tag{2}$$

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where $h_\delta(u) = \frac{\partial^2 u}{\partial x^2} - u_2 e_2 - u_3 e_3 + \delta e_1$. Here, (e_1, e_2, e_3) is the canonical basis of $I\!R^3$ and the nanowire is the real axis $I\!Re_1$. The magnetic field is written $\delta(t)e_1$, where the function $\delta(\cdot)$ is our control. Setting $h(u) = u_{xx} - u_2 e_2 - u_3 e_3$, this yields

$$u_t = -u \wedge h(u) - u \wedge (u \wedge h(u)) - \delta(u \wedge e_1 + u \wedge (u \wedge e_1)). \tag{3}$$

When $\delta \equiv 0$, stationary solutions do exist, of the form

$$M_0(x) = \begin{pmatrix} \operatorname{th} x \\ 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix} \tag{4}$$

and are called Bloch walls. Their stability properties were studied in [7].

When $\delta(\cdot) \equiv \delta$ is constant, the solution writes

$$u^\delta(t, x) = R_{\delta t} M_0(x + \delta t), \tag{5}$$

where

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

is the rotation of angle θ around the axis $I\!Re_1$. It corresponds to a rotation plus translation of the above wall along the nanowire.

Notice the invariance of (3) through translations $x \mapsto x - \sigma$ and rotations R_θ around the axis e_1. This generates a three-parameters family of particular solutions defined by

$$u^{\delta,\theta,\sigma}(t, x) = M_\Lambda u^\delta(t, x) = R_{\delta t+\theta} M_0(x + \delta t - \sigma) \tag{6}$$

called *travelling wall profiles.*

Controlling these walls (position plus speed) might be relevant for coding and transporting some information. This is our aim here to derive a controllability result, with an eye on possible applications such as rapid recording. In [9], control properties were proven with piecewise constant controls. However, practical applications require the control to be smooth. Recall that the control here is an external magnetic field applied along the nanowire. The main result of [9] strongly uses the fact that the control is a piecewise constant function and our aim is here to extend this result to the case of smooth controls, hence closer to practical issues.

Theorem 1.1. *There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for all $\delta_1, \delta_2 \in I\!R$ satisfying $|\delta_i| \le \delta_0$, $i = 1, 2$, for all $\sigma_1, \sigma_2 \in I\!R$, for every $\varepsilon \in (0, \varepsilon_0)$, there exist $T > 0$ and a control function $\delta(\cdot) \in C^\infty(I\!R, I\!R)$ such that, for every solution u of (3) associated with the control $\delta(\cdot)$ and satisfying*

$$\exists \theta_1 \in I\!R \mid \|u(0,\cdot) - u^{\delta_1,\theta_1,\sigma_1}(0,\cdot)\|_{H^2} \le \varepsilon, \tag{7}$$

there exists a real number θ_2 such that

$$\|u(T,\cdot) - u^{\delta_2,\theta_2,\sigma_2}(T,\cdot)\|_{H^2} \le \varepsilon. \tag{8}$$

Moreover, there exists real numbers θ_2' and σ_2', with $|\theta_2' - \theta_2| + |\sigma_2' - \sigma_2| \le \varepsilon$, such that

$$\|u(t,\cdot) - u^{\delta_2,\theta_2',\sigma_2'}(t,\cdot)\|_{H^2} \underset{t\to+\infty}{\longrightarrow} 0. \tag{9}$$

In the proof of the main result, we shall choose control laws $\delta(\cdot)$ so that

$$\delta(t) = \begin{cases} \delta_1 & \text{if } t \le 0, \\ \delta_2 + \dfrac{\sigma_1 - \sigma_2}{t} & \text{if } t \ge T, \end{cases} \tag{10}$$

where $T>0$ is large, $\delta_{|[0,T]}$ is a smooth function such that $t\dot\delta$ remains small, and the function δ is smooth overall $I\!R$.

Notice that this control shares robustness properties in H^2 norm. The time T is required to be large enough. It follows from this result that the family of travelling wall profiles (6) is approximately controllable in H^2 norm, locally in δ and globally in σ, in time sufficiently large.

2. **Proof of Theorem 1.1.**

Similarly as in [7, 8, 9], it is relevant to first reexpress the Landau-Lifschitz equation in adapted coordinates.

2.1. **Preliminaries.**

The following formulas, easy to establish, will be useful next:

- $\dfrac{d}{d\theta}R_\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{pmatrix} = R_{\theta+\frac{\pi}{2}} - e_1e_1^T = R_{\frac{\pi}{2}}R_\theta - e_1e_1^T$;
- $v\wedge e_1 = -R_{\frac{\pi}{2}}v + v_1e_1$;
- $R_\theta u \wedge R_\theta v = R_\theta(u\wedge v)$;
- $a\wedge(b\wedge c) = b(a.c) - c(a.b)$;
- $R_\theta(I\!Re_1) = I\!Re_1$.

It is clear from Equation (2) that the solution u has a constant norm. Up to normalizing, assume this norm is equal to 1. Set $v(t,x) = R_{-\delta(t)t}(u(t,x-\delta(t)t))$; then, v has a constant norm too, equal to 1. Using the above formulas, computations lead to

$$v_t = -v\wedge h(v) - v\wedge(v\wedge h(v)) - \delta(v_x + v_1v - e_1) - t\dot\delta(v_x - v_3e_2 + v_2e_3), \quad (11)$$

where we recall that $h(v) = v_{xx} - v_2e_2 - v_3e_3$. Define

$$M_1(x) = \begin{pmatrix} \frac{1}{\operatorname{ch} x} \\ 0 \\ -\operatorname{th} x \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

In the frame $(M_0(x), M_1(x), M_2)$, the solution $v : I\!R^+ \times I\!R \longrightarrow S^2 \subset I\!R^3$ writes in the form

$$v(t,x) = \sqrt{1 - r_1(t,x)^2 - r_2(t,x)^2}\,M_0(x) + r_1(t,x)M_1(x) + r_2(t,x)M_2.$$

Note that:

- $M_0'(x) = \frac{1}{\operatorname{ch} x}M_1(x)$, $M_1'(x) = -\frac{1}{\operatorname{ch} x}M_0$, $M_0''(x) = -\frac{\operatorname{sh} x}{\operatorname{ch}^2 x}M_1(x) - \frac{1}{\operatorname{ch}^2 x}M_0$;
- $e_1 = \operatorname{th} x\ M_0 + \frac{1}{\operatorname{ch} x}M_1(x)$, $e_2 = M_2$, $e_3 = \frac{1}{\operatorname{ch} x}M_0 - \operatorname{th} x\ M_1(x)$;
- $h(M_0) = -\frac{2}{\operatorname{ch}^2 x}M_0$;
- $M_0\wedge M_1 = M_2$, $M_0\wedge M_2 = -M_1$, $M_1\wedge M_2 = M_0$;

Then, easy but lengthy computations, not reported here, show that v is solution of (11) if and only if $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ satisfies

$$r_t = Ar + R(t,\delta,\dot\delta,x,r,r_x,r_{xx}), \quad (12)$$

where

$$\begin{aligned} R(t,\delta,\dot\delta,x,r,r_x,r_{xx}) = &-\delta\begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} r + G(r)r_{xx} + H_1(x,r)r_x + H_2(r)(r_x,r_x) \\ &+ P(x,r) - \delta B(x,r) - t\dot\delta C(x,r), \end{aligned} \quad (13)$$

and

- $A = \begin{pmatrix} L & L \\ -L & L \end{pmatrix}$ with $L = \partial_{xx} + (1 - 2\mathrm{th}^2 x)\mathrm{Id}$;
- $\ell = \partial_x + \mathrm{th}\, x\, \mathrm{Id}$;
- $G(r)$ is the matrix defined by

$$G(r) = \begin{pmatrix} \dfrac{r_1 r_2}{\sqrt{1-\|r\|^2}} & \dfrac{r_2^2}{\sqrt{1-\|r\|^2}} + \sqrt{1-\|r\|^2} - 1 \\ -\dfrac{r_1^2}{\sqrt{1-\|r\|^2}} - \sqrt{1-\|r\|^2} + 1 & -\dfrac{r_1 r_2}{\sqrt{1-\|r\|^2}} \end{pmatrix};$$

- $H_1(x,r)$ is the matrix defined by

$$H_1(x,r) = \frac{2}{\sqrt{1-\|r\|^2}\,\mathrm{ch}\, x} \begin{pmatrix} r_2\sqrt{1-\|r\|^2} - r_1 r_2^2 & -r_2 + r_2 r_1^2 \\ r_2 - r_2^3 & \sqrt{1-\|r\|^2}\, r_2 + r_1 r_2^2 \end{pmatrix};$$

- $H_2(r)$ is the quadratic form on $\mathbb{R}^2$ defined by

$$H_2(r)(X,X) = \frac{(1-\|r\|^2)X^T X + (r^T X)^2}{(1-\|r\|^2)^{3/2}} \begin{pmatrix} \sqrt{1-\|r\|^2}\, r_1 + r_2 \\ \sqrt{1-\|r\|^2}\, r_2 - r_1 \end{pmatrix};$$

- $P(x,r) = \begin{pmatrix} P^1(x,r) \\ P^2(x,r) \end{pmatrix}$, with

$$\begin{aligned} P(x,r) =& 2r_2(\sqrt{1-\|r\|^2} - 1)\frac{1}{\mathrm{ch}^2 x} - 2r_1 r_2 \frac{\mathrm{sh}\, x}{\mathrm{ch}^2 x} - 2r_1\|r\|^2 \frac{1}{\mathrm{ch}^2 x} \\ &- 2r_1^2\sqrt{1-\|r\|^2}\frac{\mathrm{sh}\, x}{\mathrm{ch}^2 x} + r_1^3 + r_2(1 - \sqrt{1-\|r\|^2}) + r_1 r_2^2, \end{aligned}$$

and

$$\begin{aligned} P^2(x,r) =& -2r_1(\sqrt{1-\|r\|^2} - 1)\frac{1}{\mathrm{ch}^2 x} + 2r_1^2 \frac{\mathrm{sh}\, x}{\mathrm{ch}^2 x} - 2r_2\|r\|^2 \frac{1}{\mathrm{ch}^2 x} \\ &- 2r_1 r_2\sqrt{1-\|r\|^2}\frac{\mathrm{sh}\, x}{\mathrm{ch}^2 x} + r_2\|r\|^2, \end{aligned}$$

- $B(x,r) = (\partial_x + \mathrm{th}\, x) r + \dfrac{1}{\mathrm{ch}\, x}\begin{pmatrix} \sqrt{1-\|r\|^2} - 1 + r_1^2 \\ r_1 r_2 \end{pmatrix} + \mathrm{th}\, x\, (\sqrt{1-\|r\|^2} - 1) r$,
- $C(x,r) = \left(\partial_x + \mathrm{th}\, x \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) r + \dfrac{\sqrt{1-\|r\|^2}}{\mathrm{ch}\, x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

It is clear that there holds

$$\begin{aligned} G(r) &= O(\|r\|^2), \\ H_1(x,r) &= O(\|r\|), \\ H_2(r) &= O(\|r\|), \\ P(x,r) &= O(\|r\|^2), \\ B(x,r) &= O(\|r\| + \|r_x\|), \\ C(x,r) &= O(\|r\| + \|r_x\|), \end{aligned}$$

uniformly with respect to the variable $x \in \mathbb{R}$. Then, we infer that there exists a constant $C > 0$ such that, if $\|r\|^2_{\mathbb{R}^2} = \|r\|^2 \leq \frac{1}{2}$ and $|\delta| \leq 1$, then, for all $p, q \in \mathbb{R}^2$, for all $x, t, \varepsilon \in I\!R$,

$$\|R(t,\delta,\varepsilon,x,r,p,q)\|_{I\!R^2} \leq C\Big(|\delta|\|r\|_{I\!R^2} + |\delta|\|p\|_{I\!R^2} + t|\varepsilon| + t|\varepsilon|\|p\|_{I\!R^2} + \|r\|^2_{I\!R^2}\|q\|_{I\!R^2} + \|r\|_{I\!R^2}\|p\|_{I\!R^2} + \|r\|_{I\!R^2}\|p\|^2_{I\!R^2} + \|r\|^2_{I\!R^2}\Big). \tag{14}$$

From this a priori estimate, one might consider $R(t,\delta,\dot{\delta},x,r,r_x,r_{xx})$ as a remainder term in Equation (12). The proof uses stability properties established for the linear operator A, so as to establish. We next follow the same lines as in [9].

2.2. **Change of coordinates.** The operator L is a self-adjoint operator on $L^2(I\!R)$, of domain $H^2(I\!R)$, and $L = -\ell^*\ell$ with $\ell = \partial_x + \operatorname{th} x \operatorname{Id}$ (one has $\ell^* = -\partial_x + \operatorname{th} x \operatorname{Id}$). It follows that L is nonpositive, and that $\ker L = \ker \ell$ is the one dimensional subspace of $L^2(I\!R)$ generated by $\frac{1}{\operatorname{ch} x}$. In particular, the operator L, restricted to the subspace $E = (\ker L)^\perp$, is negative.

Remark 1. On the subspace E:

- the norms $\|(-L)^{1/2}f\|_{L^2(I\!R)}$ and $\|f\|_{H^1(I\!R)}$ are equivalent;
- the norms $\|Lf\|_{L^2(I\!R)}$ and $\|f\|_{H^2(I\!R)}$ are equivalent;
- the norms $\|(-L)^{3/2}f\|_{L^2(I\!R)}$ and $\|f\|_{H^3(I\!R)}$ are equivalent.

Writing $A = JL$, with

$$J = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

it is clear that the kernel of A is $\ker A = \ker L \times \ker L$; it is the two dimensional space of $L^2(I\!R^2)$ generated by

$$a_1(x) = \begin{pmatrix} 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix} \quad \text{and} \quad a_2(x) = \begin{pmatrix} \frac{1}{\operatorname{ch} x} \\ 0 \end{pmatrix}.$$

Moreover, combining the facts that $L_{|(\ker L)^\perp}$ is negative and that $\operatorname{Spec} J = \{1+i, 1-i\}$, it follows that the operator A, restricted to the subspace $\mathcal{E} = (\ker A)^\perp$, is negative.

In what follows, solutions r of (12) are written as the sum of an element of $\ker A$ and of an element of $\mathcal{E}$. Since Equation (11) is invariant with respect to translations in x and rotations around the axis e_1, for every $\Lambda = (\theta,\sigma) \in I\!R^2$, $M_\Lambda(x) = R_\theta M_0(x-\sigma)$ is solution of (11). Define

$$R_\Lambda(x) = \begin{pmatrix} \langle M_\Lambda(x), M_1(x)\rangle \\ \langle M_\Lambda(x), M_2\rangle \end{pmatrix},$$

the coordinates of $M_\Lambda(x)$ in the mobile frame $(M_1(x), M_2(x))$.

The mapping

$$\begin{array}{rcl} \Psi : I\!R^2 \times \mathcal{E} & \longrightarrow & H^2(I\!R) \\ (\Lambda, W) & \longmapsto & r(x) = R_\Lambda(x) + W(x) \end{array}$$

is a diffeomorphism from a neighborhood $\mathcal{U}$ of zero in $I\!R^2 \times \mathcal{E}$ into a neighborhood $\mathcal{V}$ of zero in $H^2(I\!R)$. Indeed, if $r = R_\Lambda + W$ with $W \in \mathcal{E}$, then, by definition,

$$\langle r, a_1\rangle_{L^2} = \langle R_\Lambda, a_1\rangle_{L^2} \quad \text{and} \quad \langle r, a_2\rangle_{L^2} = \langle R_\Lambda, a_2\rangle_{L^2}. \tag{15}$$

Conversely, if $\Lambda \in I\!R^2$ satisfies ((15)), then $W = r - R_\Lambda \in \mathcal{E}$. The mapping $h : I\!R^2 \longrightarrow I\!R^2$, defined by $h(\Lambda) = (\langle R_\Lambda, a_1\rangle_{L^2}, \langle R_\Lambda, a_2\rangle_{L^2})$ is smooth and satisfies $dh(0) = -2\operatorname{Id}$, thus is a local diffeomorphism at $(0,0)$. It follows easily that Ψ is a local diffeomorphism at zero.

Therefore, every solution r of (12), as long as it stays[1] in the neighborhood $\mathcal{V}$, can be written as

$$r(t,\cdot) = R_{\Lambda(t)}(\cdot) + W(t,\cdot), \tag{16}$$

where $W(t,\cdot) \in \mathcal{E}$ and $\Lambda(t) \in \mathbb{R}^2$, for every $t \geq 0$, and $(\Lambda(t), W(t,\cdot)) \in \mathcal{U}$. In these new coordinates[2], Equation (12) leads to (see [7] for the details of computations)

$$\begin{aligned} W_t(t,x) &= AW(t,x) + \mathcal{R}(t,\delta,\varepsilon,\Lambda(t),x,W(t,x),W_x(t,x),W_{xx}(t,x)), \\ \Lambda'(t) &= \mathcal{M}(\Lambda(t),W(t,\cdot),W_x(t,\cdot)), \end{aligned} \tag{17}$$

where $\mathcal{R} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \left(H^2(\mathbb{R})\right)^2 \times \left(H^1(\mathbb{R})\right)^2 \times \left(L^2(\mathbb{R})\right)^2 \longrightarrow \mathcal{E}$ and $\mathcal{M} : \mathbb{R}^2 \times \left(H^1(\mathbb{R})\right)^2 \times \left(L^2(\mathbb{R})\right)^2 \longrightarrow \mathbb{R}^2$ are nonlinear mappings, for which there exist constants $K > 0$ and $\eta > 0$ such that

$$\begin{aligned} &\|\mathcal{R}(t,\delta,\varepsilon,\Lambda,\cdot,W,W_x,W_{xx})\|_{(H^1(\mathbb{R}))^2} \\ &\qquad \leq K\left(\|\Lambda\|_{\mathbb{R}^2} + |\delta| + t|\varepsilon| + \|W\|_{(H^2(\mathbb{R}))^2}\right)\|W\|_{(H^3(\mathbb{R}))^2} + Kt|\varepsilon|, \end{aligned} \tag{18}$$

$$|\mathcal{M}(\Lambda,W,W_x)| \leq K\left(\|\Lambda\|_{\mathbb{R}^2} + \|W\|_{(H^1(\mathbb{R}))^2}\right)\|W\|_{(H^1(\mathbb{R}))^2}, \tag{19}$$

for every $W \in \mathcal{E}$, every $\delta \in \mathbb{R}$, every $t \geq 0$, and every $\Lambda \in \mathbb{R}^2$ satisfying $\|\Lambda\|_{\mathbb{R}^2} \leq \eta$. Note that, since L is selfadjoint, it follows that $AW \in \mathcal{E}$, for every $W \in \mathcal{E}$, and thus (17) makes sense.

2.3. **Asymptotic estimates.** Denoting $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$, define on $\left(H^2(\mathbb{R})\right)^2 \times \mathbb{R}^2$ the function

$$\mathcal{V}(W) = \frac{1}{2}\left\|\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} W\right\|^2_{(L^2(\mathbb{R}))^2} = \frac{1}{2}\|LW_1\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|LW_2\|^2_{L^2(\mathbb{R})}. \tag{20}$$

Remark 2. It follows from Remark 1 that, on the subspace $\mathcal{E} = (\ker A)^{\perp}$, $\sqrt{\mathcal{V}(W)}$ is a norm, which is equivalent to the norm $\|W\|^2_{(H^2(\mathbb{R}^2))}$.

Consider a solution (W,Λ) of (17), such that $W(0,\cdot) = W_0(\cdot)$ and $\Lambda(0) = \Lambda_0$. Since L is selfadjoint, one has

$$\begin{aligned} &\frac{d}{dt}\mathcal{V}(W(t,\cdot)) = \left\langle AW, \begin{pmatrix} L^2W_1 \\ L^2W_2 \end{pmatrix}\right\rangle_{(L^2(\mathbb{R}))^2} \\ &+ \left\langle \begin{pmatrix} (-L)^{1/2} & 0 \\ 0 & (-L)^{1/2} \end{pmatrix} \mathcal{R}(t,\delta,\varepsilon,\Lambda,\cdot,W,W_x,W_{xx}), \begin{pmatrix} (-L)^{3/2}W_1 \\ (-L)^{3/2}W_2 \end{pmatrix}\right\rangle_{(L^2(\mathbb{R}))^2}. \end{aligned} \tag{21}$$

Concerning the first term of the right-hand side of (21), one computes

$$\left\langle AW, \begin{pmatrix} L^2W_1 \\ L^2W_2 \end{pmatrix}\right\rangle_{(L^2(\mathbb{R}^2))^2} = -\|(-L)^{3/2}W_1\|_{(L^2(\mathbb{R}))^2} - \|(-L)^{3/2}W_2\|_{(L^2(\mathbb{R}))^2},$$

and, using Remark 1, there exists a constant $C_1 > 0$ such that

$$\left\langle AW, \begin{pmatrix} L^2W_1 \\ L^2W_2 \end{pmatrix}\right\rangle_{(L^2(\mathbb{R}))^2} \leq -C_1\|W\|^2_{(H^3(\mathbb{R}))^2}. \tag{22}$$

[1] This a priori estimate will be a consequence of the stability property derived next.

[2] This decomposition is actually quite standard and has been used e.g. in [14] to establish stability properties of static solutions of semilinear parabolic equations, and in [2, 19] to prove stability of travelling waves.

Concerning the second term of the right-hand side of (21), one deduces from the Cauchy-Schwarz inequality, from Remark 1, and from the estimate (18), that

$$\begin{aligned}
&\left|\left\langle \begin{pmatrix} (-L)^{1/2} & 0 \\ 0 & (-L)^{1/2} \end{pmatrix} \mathcal{R}(t,\delta,\varepsilon,\Lambda,\cdot,W,W_x,W_{xx}), \begin{pmatrix} (-L)^{3/2}W_1 \\ (-L)^{3/2}W_2 \end{pmatrix} \right\rangle_{(L^2(I\!R))^2}\right| \\
&\leq \ \|\mathcal{R}(t,\delta,\varepsilon,\Lambda,\cdot,W,W_x,W_{xx})\|_{(H^1(I\!R))^2}\|W\|_{(H^3(I\!R))^2} \\
&\leq \ K\left(\|\Lambda\|_{I\!R^2}+|\delta|+t|\varepsilon|+\|W\|_{(H^2(I\!R))^2}\right)\|W\|^2_{(H^3(I\!R))^2}+t|\varepsilon|\|W\|_{(H^3(I\!R))^2} \\
&\leq \ K\left(\|\Lambda\|_{I\!R^2}+|\delta|+t|\varepsilon|+\|W\|_{(H^2(I\!R))^2}+\frac{1}{2\xi^2}\right)\|W\|^2_{(H^3(I\!R))^2}+\frac{\xi^2}{2}t^2\varepsilon^2,
\end{aligned} \tag{23}$$

where, to get the last line, we used the inequality

$$t|\varepsilon|\|W\|_{(H^3(I\!R))^2} \leq \frac{\xi^2}{2}t^2\varepsilon^2+\frac{1}{2\xi^2}\|W\|^2_{(H^3(I\!R))^2};$$

here, ξ denotes some real number to be chosen later.

One infers from (21), (22) and (23) that

$$\frac{d}{dt}\mathcal{V}(W) \leq \left(-C_1+K\left(\|\Lambda\|_{I\!R^2}+|\delta|+t|\dot{\delta}|+\|W\|_{(H^2(I\!R))^2}+\frac{1}{2\xi^2}\right)\right)\|W\|^2_{(H^3(I\!R))^2} + \frac{\xi^2}{2}t^2\dot{\delta}^2.$$

Fix $\epsilon > 0$; then, under the a priori estimates

$$\|\Lambda(t)\|_{I\!R^2}+|\delta|+t|\dot{\delta}|+\|W(t,\cdot)\|_{(H^2(I\!R))^2}+\frac{1}{2\xi^2} \leq \frac{C_1}{2K}$$

and

$$\frac{\xi^2}{2}t^2\dot{\delta}^2 \leq \epsilon,$$

there holds

$$\begin{aligned}
\frac{d}{dt}\mathcal{V}(W(t,\cdot)) &\leq -\frac{C_1}{2}\|W(t,\cdot)\|^2_{(H^3(I\!R))^2}+\epsilon \\
&\leq -\frac{C_1}{2}\|W(t,\cdot)\|^2_{(H^2(I\!R))^2}+\epsilon \\
&\leq -C_2\mathcal{V}(W(t,\cdot))+\epsilon.
\end{aligned}$$

The existence of a constant $C_2 > 0$ follows from Remark 2. Therefore, choosing $\xi > 0$ large enough, there exist constants $C_3 > 0$ and $C_4 > 0$ such that, if $\|W(0,\cdot)\|_{(H^2(I\!R))^2} \leq \frac{C_1}{6K}$, if the a priori estimate

$$\max_{0\leq s\leq t}\|\Lambda(s)\|_{I\!R^2} \leq \frac{C_1}{6K} \tag{24}$$

holds, and if the control function $\delta(\cdot)$ is chosen so that

$$|\delta(t)|+t|\dot{\delta}(t)| \leq \frac{C_1}{6K} \tag{25}$$

and

$$t^2\dot{\delta}(t)^2 \leq 2\epsilon/\xi^2 \tag{26}$$

for every $t \geq 0$, then

$$\|W(s,\cdot)\|_{(H^2(I\!R))^2} \leq C_3\mathrm{e}^{-C_4 s}\|W(0,\cdot)\|_{(H^2(I\!R))^2}+C_3\epsilon, \tag{27}$$

for every $s \in [0,T]$, and moreover, one deduces from (17), (19), and (27) that, if the a priori estimate (24) holds, then

$$\begin{aligned}\|\Lambda(t)\|_{\mathbb{R}^2} &\leq \|\Lambda(0)\|_{\mathbb{R}^2} + \frac{C_1C_3}{4}\|W(0,\cdot)\|_{(H^2(\mathbb{R}))^2}\int_0^t \mathrm{e}^{-C_4 s}ds \\ &\quad + KC_3^2\|W(0,\cdot)\|^2_{(H^2(\mathbb{R}))^2}\int_0^t \mathrm{e}^{-2C_4 s}ds \\ &\leq \|\Lambda(0)\|_{\mathbb{R}^2} + \frac{C_1C_3}{4C_4}\|W(0,\cdot)\|_{(H^2(\mathbb{R}))^2} + K\frac{C_3^2}{2C_4}\|W(0,\cdot)\|^2_{(H^2(\mathbb{R}))^2}.\end{aligned} \tag{28}$$

From the above a priori estimates, we infer that, if the quantity $\|\Lambda(0)\|_{\mathbb{R}^2} + \|W(0,\cdot)\|_{(H^2(\mathbb{R}))^2}$ is small enough, and if the control function δ fits the conditions (25) and (26), then $\|\Lambda(t)\|_{\mathbb{R}^2}$ remains small, for every $t \geq 0$, and $\|W(t,\cdot)\|_{(H^2(\mathbb{R}))^2}$ is exponentially decreasing to 0.

Finally we must choose a smooth control function such that $u(t,x)$ is close to $u^{\delta_1,\theta_1,\sigma_1}(t,x)$ at initial time, and close to $u^{\delta_2,\theta_2,\sigma_2}(t,x)$ for large times. Hence, we can choose the function δ such that $\delta(t) = \delta_1$ for $t \leq 0$. Then, with the reasoning above, we enforce $v(t,x)$ to remain close to $M_0(x)$, that is, the solution $u(t,x)$ follows the profile $u^{\delta(t),\theta_1,\sigma_1}(t,x)$. At times $t \geq T$, we require $u(t,x)$ to be close to $u^{\delta_2,\theta_2,\sigma_2}(t,x)$ for some θ_2; one must have, for $t \geq T$,

$$-\sigma_1 + \delta(t)t = -\sigma_2 + \delta_2 t,$$

and hence,

$$\delta(t) = \delta_2 + \frac{\sigma_1 - \sigma_2}{t}.$$

To conclude, observe that it is possible to choose a function δ and a time $T > 0$ large enough, such that δ is smooth on $\mathbb{R}$ and satisfies the above requirements and the estimates (25) and (26).

The first part of the theorem, on the interval $[0,T]$, then follows from the above considerations.

For the second part, we use a stronger version of the estimate (27), namely,

$$\|W(s,\cdot)\|_{(H^2(\mathbb{R}))^2} \leq C_3\mathrm{e}^{-C_4 s}\|W(0,\cdot)\|_{(H^2(\mathbb{R}))^2} + \xi^2 t^2 \dot{\delta}(t)^2.$$

Since $t^2\dot{\delta}(t)^2$ is integrable, it follows from the above estimate, and from (17) and (19), that $\|\Lambda'(t)\|_{\mathbb{R}^2}$ is integrable on $[0,+\infty)$. Hence, $\Lambda(t)$ has a limit in $\mathbb{R}^2$, denoted $\Lambda_\infty = (\theta_\infty, \sigma_\infty)$, as t tends to $+\infty$. The theorem follows with $\theta_2' = \theta_2 + \theta_\infty$ and $\sigma_2' = \sigma_2 + \sigma_\infty$.

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