

## Mean-field sparse Jurdjevic–Quinn control

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We consider nonlinear transport equations with non-local velocity describing the time-evolution of a measure. Such equations often appear when considering the mean-field limit of finite-dimensional systems modeling collective dynamics. We address the problem of controlling these equations by means of a time-varying bounded control action localized on a time-varying control subset of small Lebesgue measure. We first define dissipativity for nonlinear transport equations in terms of Lie derivatives of a Lyapunov function depending on the measure. Then, assuming that the uncontrolled system is dissipative, we provide an explicit construction of a control law steering the system to an invariant sublevel of the Lyapunov function. The control function and the control domain are designed in terms of the Lie derivatives of the Lyapunov function. In this sense the construction can be seen as an infinite-dimensional analogue of the well-known Jurdjevic–Quinn procedure. Moreover, the control law presents sparsity properties in the

sense that the support of the control is small. Finally, we show that our result applies to a large class of kinetic equations modeling multi-agent dynamics.

*Keywords:* Multi-agent systems; crowd control; control systems; networked control; distributed parameter systems; control of partial differential equations; Lyapunov methods.

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## 1. Introduction and Main Result

### 1.1. Context

In recent years, the study of collective behaviors in group of autonomous agents has drawn a broad interest from scientific communities, e.g. in civil engineering (for evacuation problems<sup>16,24</sup>), robotics (coordination of robots<sup>9,26,28,32</sup>), computer science and sociology (social networks<sup>23</sup>), and biology (animals groups<sup>5,14,20</sup>). In particular, it is well known how simple rules of interaction between agents can promote formation of special patterns, or complex behaviors, like lines in ants formations and migrating lobsters, or V-shaped formation in migrating birds. This phenomenon is often referred to as *self-organization*.

Beside the problem of analyzing the collective behavior of a “closed” system,<sup>15</sup> it is interesting to understand how a behavior can be influenced by an external agent (e.g. a policy maker). For instance, one can try to enforce creation of patterns when they are not formed naturally, or break the formation of such patterns.<sup>11,12,19,31,25</sup>

From the mathematical point of view, the challenge in the modeling problem is that one needs to pass from a big set of simple rules for each individual to a model capable of catching the main features of the dynamics of the whole group. This is usually achieved by using a so-called mean-field approach, which permits to consider the limit of a set of ordinary differential equations (one for each agent) to a partial differential equation representing the evolution of the whole mass. Many kinetic equations obtained via mean-field limit are transport equations with non-local velocity of the form

$$\partial_t \mu + \nabla \cdot (f[\mu]\mu) = 0, \quad (1.1)$$

where  $\mu$  is a probability measure on  $\mathbb{R}^d$  representing the density of agents,  $\nabla \cdot$  is the divergence operator, and  $f[\mu]$  is a vector field depending on the measure accounting for the interactions between agents.

Here we consider the control problem associated with (1.1). Since agents are indistinguishable, controls can only be state-dependent and cannot focus on specific agents. For this reason, we model the control action by means of a vector field  $g[\mu]$ , and a *control gain*  $u(t, x)$  localized in a small *control set*  $\omega(t)$ , modeling our choice of the gain on the vector field. Notice that the control gain  $u$  and the control set  $\omega$  are both time-dependent while the vector field  $g$  is fixed, and it depends on the density  $\mu$ . The resulting control system is given by

$$\partial_t \mu + \nabla \cdot ((f[\mu] + \chi_\omega u g[\mu])\mu) = 0. \quad (1.2)$$

Here, the function  $\chi_\omega$  is the indicator function of  $\omega$ , defined almost everywhere by  $\chi_\omega(x) = 1$  if  $x \in \omega$  and  $\chi_\omega(x) = 0$  otherwise.

In the following, we assume that  $f[\mu]$  and  $g[\mu]$  are bounded Lipschitz vector fields on  $\mathbb{R}^d$  for any  $\mu$ , Lipschitz with respect to the Wasserstein distance  $W_p$ ,  $p \in [1, +\infty)$ , as a function of  $\mu$ . This ensures existence and uniqueness of the solution of the associated Cauchy problem<sup>3,29</sup> for any Lipschitz control gain  $u$ . We recall fundamental properties for equations in the form (1.1) in Sec. 2 below and we use those results to prove existence and uniqueness for the Cauchy problem associated with (1.2).

Admissible controls satisfy the following *continuous sparse space constraint*: we assume to act only on a small portion of the configuration space and with finite strength. Here, given a measurable subset  $\omega$  of  $\mathbb{R}^d$ , we denote by  $|\omega|$  its Lebesgue measure.

**Control Constraints (U)**

Fix  $c > 0$ . For each time  $t \geq 0$ , we have:

$$\text{Sparsity space constraint: } |\omega(t)| \leq c, \tag{1.3}$$

$$\text{Finite strength: } \|u(t, \cdot)\|_{L^\infty} \leq 1. \tag{1.4}$$

Control sparsity constraints have been firstly introduced in Refs. 11 and 12, for a population with a finite number of agents. The sparsity space constraint was introduced in Ref. 31. In the mean-field approach, this is the most natural sparsity constraint when one wants to use space-dependent vector fields and to act on “small sets”. A sparsity constraint limiting the fraction of the measure on which the control may act, the *sparsity population constraint*, was also considered in Ref. 31. However, we will see in Sec. 3 that the sparsity space constraint is more appropriate for the mean-field approach. Indeed, for the sparsity space constraint, one can easily deal both with measures that are absolutely continuous with respect to the Lebesgue measure, and with measures containing singular (Dirac) parts. In Theorem 1.1 we will define a control strategy that satisfies the following property: *if the initial data, at time 0, is absolutely continuous with respect to the Lebesgue measure, then it remains absolutely continuous for any positive time*. This does not prevent  $\mu(t)$  of converging to some Dirac mass as  $t \rightarrow +\infty$ , as this is the case for consensus problems in multi-agent models and crowd dynamics.

In this paper, our objective is to generalize the Jurdjevic–Quinn stabilization method<sup>27</sup> to mean-field controlled equations, under the sparsity constraint (U) described above. Following the Jurdjevic–Quinn approach, we assume the existence of a Lyapunov function  $V$  for which:

- the uncontrolled dynamics  $f[\mu]$  gives no increase of  $V$ ;

- the control  $ug[\mu]$  allows one to increase–decrease  $V$ , except for some specific configurations of the population  $\mu$  in the subset  $\mathcal{Z}$  of the set of probability measure on  $\mathbb{R}^d$  with compact support defined as the set on which the Lie derivatives of  $V$  vanish (see the precise definition in (1.13)).

We will then define a sparse control strategy, steering the population exactly to the set  $\mathcal{Z}$ , in complete analogy with the standard finite-dimensional Jurdjević–Quinn method.

### 1.2. *Mathematical setting*

We denote by  $\mathcal{P}_c(\mathbb{R}^d)$  the space of probability measures on  $\mathbb{R}^d$  with compact support and by  $\mathcal{P}_c^{ac}(\mathbb{R}^d)$  the subspace of probability measures on  $\mathbb{R}^d$  that are absolutely continuous with respect to the Lebesgue measure. Given  $R > 0$ ,  $B(0, R)$  denotes the ball in  $\mathbb{R}^d$  centered in 0 of radius  $R$ . We denote by  $\mathcal{P}_c(\overline{B(0, R)})$  the space of probability measures on  $\mathbb{R}^d$  with support in  $\overline{B(0, R)}$  and by  $\mathcal{P}_c^{ac}(\overline{B(0, R)})$  the subspace of  $\mathcal{P}_c(\overline{B(0, R)})$  of absolutely continuous measures (with respect to the Lebesgue measure).

We consider the control system (1.2) where  $f, g$  are two vector fields on  $\mathcal{P}_c(\mathbb{R}^d)$ . An admissible control is a Lipschitz function  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the constraints (U). We denote the set of admissible controls by  $\mathcal{U}$ . We assume that  $f, g : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \text{Lip}(\mathbb{R}^d, \mathbb{R}^d)$  satisfy, for some  $R > 0$ , the following regularity assumption:

(H1) there exist  $L > 0, Q > 0$  and  $p \geq 1$  such that,

$$\begin{aligned} |f[\mu](x) - f[\mu](y)| &\leq L|x - y|, & |g[\mu](x) - g[\mu](y)| &\leq L|x - y|, \\ |f[\mu](x) - f[\nu](x)| &\leq QW_p(\mu, \nu), & |g[\mu](x) - g[\nu](x)| &\leq QW_p(\mu, \nu), \end{aligned} \tag{1.5}$$

for all  $\mu, \nu \in \mathcal{P}_c(\overline{B(0, R)})$  and for all  $x, y \in \mathbb{R}^d$ .

Thanks to Assumption (H1) Eq. (1.1) generates a semigroup, we will use the notation  $e^{tf}\mu_0$  to denote the unique solution of (1.1) at time  $t$  with initial data  $\mu_0$ . Similarly we denote by  $e^{tug}\mu_0$  the unique solution of the Cauchy problem associated with

$$\partial_t \mu + \nabla \cdot (ug[\mu]\mu) = 0, \tag{1.6}$$

with initial data  $\mu_0$  and  $u \in \mathcal{U}$ . We recall existence results for these equations in Sec. 2.

Given a Lyapunov function  $V : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \mathbb{R}$ , the vector fields  $f$  and  $ug$  defined on the space  $\mathbb{R}^d$  play the role of derivatives for the function  $V[\mu]$ , in the following sense: the vector field  $f$  induces an infinitesimal change in  $V$  that can be estimated as the derivative  $\lim_{t \rightarrow 0} \frac{V[e^{tf}\mu] - V[\mu]}{t}$ . For this limit to be well-defined we need to impose differentiability of the function  $t \mapsto V[e^{tf}\mu]$  for any  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ .

**Definition 1.1.** Assume that  $V : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \mathbb{R}$  is such that  $t \mapsto V[e^{tf}\mu]$  is of class  $C^1$  for all  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ . We define the *Lie derivative* of  $V$  along the vector field  $f$  as

the limit

$$\mathcal{L}_f V[\mu] = \lim_{t \rightarrow 0} \frac{V[e^{tf}\mu] - V[\mu]}{t}. \tag{1.7}$$

Requiring the non-increase of  $V$  along the flow of  $f$  is equivalent to require dissipativity for the system.

**Definition 1.2.** We say that the system (1.2) is dissipative if there exists a Lyapunov function  $V : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \mathbb{R}$  such that  $t \mapsto V[e^{tf}\mu]$  is of class  $C^1$  and

$$\mathcal{L}_f V[\mu] \leq 0, \quad \text{for every } \mu \in \mathcal{P}_c(\mathbb{R}^d). \tag{1.8}$$

In analogy with Definition 1.1 we can define the Lie derivative of  $V$  along the vector field  $ug$  as

$$\mathcal{L}_{ug} V[\mu] = \lim_{t \rightarrow 0} \frac{V[e^{tug}\mu] - V[\mu]}{t}, \tag{1.9}$$

provided that the function  $t \mapsto V[e^{tug}\mu]$  is of class  $C^1$  for all  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$  and  $u \in \mathcal{U}$ .

**Remark 1.1.** The notion of solution of (1.6), of associated semigroup, and of Lie derivative along  $ug$  can be easily extended to piecewise constant functions  $t \mapsto u(t, \cdot) \in \mathcal{U}$ .

**Remark 1.2.** The definition of Lie derivative implies the multiplicative property

$$\mathcal{L}_{kf} V[\mu] = k\mathcal{L}_f V[\mu] \quad \text{and} \quad \mathcal{L}_{kug} V[\mu] = k\mathcal{L}_{ug} V[\mu]. \tag{1.10}$$

This continuity condition also implies additivity of Lie derivatives. Indeed, one can easily see that  $e^{t(u+u')g} = e^{tug+o(t)}e^{tu'g}$ , which in turn implies that

$$\mathcal{L}_{(u+u')g} V[\mu] = \mathcal{L}_{ug} V[\mu] + \mathcal{L}_{u'g} V[\mu].$$

While conditions (1.7)–(1.9) are equivalent to differentiability of  $V$  along  $f$  and  $ug$ , we also need a kind of differentiability for  $V$  along directions of the dynamics. By the additivity property, we can state it as follows: there exists  $K > 0$  such that, for all  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$  and  $u \in \mathcal{U}$ , we have

$$|\mathcal{L}_{ug} V[\mu]| \leq K\|u\|_{L^1(\mu)}. \tag{1.11}$$

This yields a metric for the space of controls  $u$  similar to the zero-order metric in a more general sub-Riemannian structure for metrics on the space of diffeomorphisms on a manifold<sup>4</sup> (see also Ref. 1). The main difference here is that we choose the  $L^1$ -norm weighted with respect to the measure  $\mu(t)$ , and not with respect to the Lebesgue measure.

While conditions (1.7)–(1.11) hold for a fixed  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ , we also require the continuity of the Lie derivatives of first and second order. Namely we require the following assumption

(H2) Assume that  $V : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \mathbb{R}$  is such that  $t \mapsto V[e^{t\mathbf{f}}\mu]$  and  $t \mapsto V[e^{t\mathbf{u}g}\mu]$  are of class  $C^2$  for all  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$  and  $u \in \mathcal{U}$ . For all  $\mu^i$  and  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ , with  $\mu^i \rightharpoonup \mu$  (weak convergence of measures), i.e.  $\lim_{i \rightarrow +\infty} \int \phi d\mu^i = \int \phi d\mu$  for every  $\phi \in C_c^0(\mathbb{R}^d, \mathbb{R})$ , one has

$$\begin{aligned} \lim_{i \rightarrow +\infty} \mathcal{L}_f V[\mu^i] &= \mathcal{L}_f V[\mu], & \lim_{i \rightarrow +\infty} \mathcal{L}_f^2 V[\mu^i] &= \mathcal{L}_f^2 V[\mu], \\ \lim_{i \rightarrow +\infty} \mathcal{L}_{ug} V[\mu^i] &= \mathcal{L}_{ug} V[\mu], & \lim_{i \rightarrow +\infty} \mathcal{L}_{ug}^2 V[\mu^i] &= \mathcal{L}_{ug}^2 V[\mu], \\ \lim_{i \rightarrow +\infty} \mathcal{L}_f \mathcal{L}_{ug} V[\mu^i] &= \mathcal{L}_f \mathcal{L}_{ug} V[\mu]. \end{aligned} \tag{1.12}$$

The convergences (1.12) imply, in particular, that  $\mathcal{L}_{f+ug} V[\mu]$  exists and satisfies  $\mathcal{L}_{f+ug} V[\mu] = \mathcal{L}_f V[\mu] + \mathcal{L}_{ug} V[\mu]$ , i.e. that additivity holds also for the vector field  $f + ug$ .

**Remark 1.3.** Clearly, the choice of the set of admissible controls  $\mathcal{U}$  has an impact on the set of admissible functionals  $V$  for which (1.9)–(1.12) are satisfied. We choose here the set of Lipschitz functions because existence is then ensured for (1.2) (see Refs. 3 and 29).

Note that reducing the space of admissible controls to some proper subset of  $\text{Lip}(\mathbb{R}^d, \mathbb{R})$  (such as  $C_c^\infty(\mathbb{R}^d, \mathbb{R})$ ) may enlarge the set of functionals  $V$  for which the regularity conditions (1.7)–(1.12) are satisfied. In Sec. 3, we will enforce the decrease of the functional  $V$  by a steepest descent method on the space  $\mathcal{U}$ , by (approximately) solving an optimization problem in the space of Lipschitz functions.

Following the classical Lyapunov theory for finite-dimensional systems, we need to impose some conditions ensuring compactness of trajectories. In finite dimension, this is often imposed by requiring  $V$  to be proper, i.e.  $\lim_{|x| \rightarrow +\infty} V(x) = +\infty$ , hence the fact that  $\frac{d}{dt} V(x(t)) \leq 0$  implies compactness. In the present mean-field setting, instead, such a condition cannot be imposed by a simple evaluation of the function  $V$ : in the case where  $V$  is the variance of the measure, one can have measures  $\mu$  with arbitrarily small variance and arbitrarily large support. For this reason, we impose compactness of trajectories by assuming that the dynamics of the system, i.e. the vector fields  $f$  and  $g$ , have a compact support. More precisely, we assume that  $f, g : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \text{Lip}(\mathbb{R}^d, \mathbb{R}^d)$  satisfy, for some  $R > 0$ , the following compactness assumption:

$$(H3) \text{ for all } \mu \in \overline{\mathcal{P}_c(B(0, R))} \quad f[\mu](x) = g[\mu](x) = 0 \quad \text{for all } x \notin \overline{B(0, R)}.$$

Note that this condition implies that  $\mu(t) \in \overline{\mathcal{P}_c(B(0, R))}$  for any  $t \geq 0$ .

Assumption (H3) implies that we can restrict our analysis to measures belonging to  $\overline{\mathcal{P}_c(B(0, R))}$ .

**Remark 1.4.** As a consequence of assumption (H3) the results of this paper can be stated for transport equations with non-local terms (1.2) defined on bounded

manifolds without boundary, or on bounded manifold with no-flux boundary condition.

**Remark 1.5.** The uniform Lipschitz property of  $f$  and  $g$  in (1.5) and the uniform compactness of their support in Assumption (H3) imply that there exists  $M > 0$  such that  $\|f[\mu]\|_{L^\infty} \leq M$  and  $\|g[\mu]\|_{L^\infty} \leq M$  for any  $\mu \in \mathcal{P}_c(\overline{B(0, R)})$ . These facts imply existence and uniqueness of the solution of the Cauchy problem for (1.2) (see, e.g. Refs. 3 and 29).

Summing up, we make the following assumptions on the system (1.2) and on the Lyapunov functional  $V$ :

**Assumptions (H)**

The vector fields  $f, g : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \text{Lip}(\mathbb{R}^d, \mathbb{R}^d)$  and the Lyapunov function  $V : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \mathbb{R}$  satisfy Assumptions (H) if the system is dissipative and there exists  $R > 0$  such that (H1)–(H2)–(H3) hold.

**1.3. The main result**

The main idea of our control strategy is to choose the controller to make  $V$  decrease along trajectories. We will do this choice with a steepest descent method, similarly to the finite-dimensional approach described in Refs. 11–13. The result control will steer the system to the set  $\mathcal{Z}$  on which the Lie derivatives of  $V$  vanish. More precisely we have the following definition.

**Definition 1.3.** Assume that  $V : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \mathbb{R}$  is such that  $t \mapsto V[e^{tf}\mu]$  and  $t \mapsto V[e^{tug}\mu]$  are of class  $C^1$  for all  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$  and  $u \in \mathcal{U}$ . We define

$$\mathcal{Z} = \{\mu \in \mathcal{P}_c(\mathbb{R}^d) \mid \mathcal{L}_f V[\mu] = \mathcal{L}_{ug} V[\mu] = 0 \ \forall u \in \mathcal{U}\}. \tag{1.13}$$

Since the space of admissible controls  $\chi_\omega u g[\mu]$  is infinite-dimensional, we restrict ourselves to a finite-dimensional set by imposing the following structure. Consider the class of Lipschitz mollified indicator functions  $\chi_{[a,b]}^\eta : \mathbb{R} \rightarrow \mathbb{R}$ , defined by

$$\chi_{[a,b]}^\eta(x) = \begin{cases} 1 & \text{for } x \in [a, b], \\ 0 & \text{for } x \notin [a - \eta, b + \eta], \\ \frac{x - a + \eta}{\eta} & \text{for } x \in [a - \eta, a], \\ \frac{-x + b + \eta}{\eta} & \text{for } x \in [b, b + \eta], \end{cases}$$

and then, consider the  $d$ -dimensional version of such functions. Given  $a = (a^1, \dots, a^d), b = (b^1, \dots, b^d)$  and  $x = (x^1, \dots, x^d)$  in  $\mathbb{R}^d$ , we define

$$\chi_{[a,b]}^\eta(x) = \min_{i=1, \dots, d} \chi_{[a^i, b^i]}^\eta(x^i). \tag{1.14}$$

Now, for any choice of the three parameters  $(a, b, \eta)$ , we take  $\omega = \omega(a, b, \eta)$  as the multi-interval  $[a^1 - \eta, b^1 + \eta] \times [a^2 - \eta, b^2 + \eta] \times \dots \times [a^d - \eta, b^d + \eta]$ . Then we reduce the choice of the sparse control in an infinite-dimensional space of controls to the choice of three parameters  $(a, b, \eta)$ . In what follows, we set

$$U(a, b, \eta) = \chi_{[a^1 - \eta, b^1 + \eta] \times [a^2 - \eta, b^2 + \eta] \times \dots \times [a^d - \eta, b^d + \eta]} \chi_{[a, b]}^\eta. \tag{1.15}$$

We then define the “slope function” by

$$s_t(a, b, \eta) = |\mathcal{L}_{U(a, b, \eta)g[\mu(t)]} V[\mu(t)]|,$$

which describes the instantaneous variation of  $V$  in  $\mu(t)$  as a consequence of the action of the control  $U(a, b, \eta)$ . Note that (1.11) and the fact that the function  $t \mapsto V[e^{tug}\mu]$  is of class  $C^1$  imply the continuity of the slope function with respect to its arguments  $(t, a, b, \eta)$ .

We then apply a steepest descent method by choosing the control corresponding to one of the maximizers<sup>a</sup>  $(a^*, b^*, \eta^*)$  of  $s_t$  in the space

$$\Omega_t = \{(a, b, \eta) \mid |\omega(a, b, \eta)| \leq c \text{ and } \eta \geq t^{-1}\}. \tag{1.16}$$

The condition  $|\omega(a, b, \eta)| \leq c$  in (1.16) ensures that the space constraint (1.3) is satisfied. We will see in Lemma 4.2 that the condition  $\eta \geq t^{-1}$  implies that the control function is uniformly Lipschitz for any bounded time interval  $[0, \theta]$ , thus ensuring that  $\mu(t)$  remains absolutely continuous with respect to the Lebesgue measure. At the same time, when  $t \rightarrow +\infty$ , this constraint allows to consider controls with an arbitrarily large Lipschitz constant, since  $\text{Lip}(\chi_{[a, b]}^\eta) = \frac{1}{\eta}$ . This Lipschitz constraint is somehow unavoidable if one wants to ensure regularity of the measure  $\mu(t)$  within finite time; otherwise, the steepest descent method might either generate a non-Lipschitz vector field (for which existence for (1.2) holds for small times only) or a time-varying Lipschitz vector field converging to a non-Lipschitz vector field within finite time (see an example for a problem of crowd dynamics in Sec. 3).

Choosing the control as the instantaneous maximizer of  $s_t$  may cause chattering (in time) phenomena, as it has been already noticed in finite dimension (see Ref. 13). For this reason, we regularize the control by means of an *hysteresis*: we introduce a parameter  $h \in (0, 1)$  and, given the control  $U(a^*, b^*, \eta^*)$ , maximizer of  $s_t$  at time  $t_n$ , we keep it constant over an interval  $[t_n, t_n + \delta]$  along which  $(s_t(a^*, b^*, \eta^*) \geq (1 - h)s_t(a, b, \eta))$ .

Summing up, the combination of a *steepest descent method* with an *hysteresis* provides a control making  $V$  decrease and steering the density  $\mu(t)$  to  $\mathcal{Z}$ . Our main result is the following.

**Theorem 1.1.** (Main theorem) *Let  $f, g : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \text{Lip}(\mathbb{R}^d, \mathbb{R}^d)$  and  $V : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \mathbb{R}$  satisfy Assumptions (H). Consider the controlled transport equation*

<sup>a</sup>The method used to select a maximizer plays no role in the convergence of the method. One may consider for instance the lexicographic order in  $\mathbb{R}^{2n+1}$ , and choose the smallest maximizer.

with non-local velocity

$$\partial_t \mu + \nabla \cdot ((f[\mu] + \chi_{\omega(t,y)} u(t,y)g[\mu])\mu) = 0, \quad \mu(0) = \mu_0, \tag{1.17}$$

where  $\mu_0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$  is such that  $\text{supp}(\mu_0) \subset \overline{B(0,R)}$ . Fix the hysteresis parameter  $h \in (0,1)$ . Fix the initial parameters  $n = 0$  and  $t_0 = 0$ . Define the following algorithm step:

**Step  $n$ :** At time  $t_n$ , choose one of the maximizers  $(a^*, b^*, \eta^*)$  of  $s_{t_n}(a, b, \eta)$  in the set  $\Omega_{t_n}$  defined in (1.16).

Then, we have two cases:

- If either  $s_{t_n}(a^*, b^*, \eta^*) < t_n^{-1}$  or  $\Omega_{t_n}$  is empty, then choose the zero control

$$\chi_{\omega(t)} u(t, x) \equiv 0, \tag{1.18}$$

(thus,  $\omega(t)$  need not be defined) and let the measure  $\mu(t)$ , starting at  $\mu(t_n)$ , evolve according to (1.17) over the time interval  $[t_n, t_{n+1}]$ , where  $t_{n+1}$  is the smallest time greater than  $t_n$  for which there exists  $(\bar{a}, \bar{b}, \bar{\eta}) \in \Omega'_t$  such that  $s_{t_{n+1}}(\bar{a}, \bar{b}, \bar{\eta}) \geq 2t_{n+1}^{-1}$ , where

$$\Omega'_t = \{(a, b, \eta) \in \Omega_t \mid \eta \geq 2t^{-1}\}. \tag{1.19}$$

- If  $s_{t_n}(a^*, b^*, \eta^*) \geq t_n^{-1}$ , then choose the control defined by

$$\begin{aligned} \omega(t) &= [a^{*1} - \eta^*, b^{*1} + \eta^*] \times \dots \times [a^{*d} - \eta^*, b^{*d} + \eta^*], \\ u(t, \cdot) &= -\chi_{[a^*, b^*]}^{\eta^*} \text{sign}(\mathcal{L}_{U(a,b,\eta)g[\mu(t)]} V[\mu(t)]), \end{aligned} \tag{1.20}$$

where  $U$  is given in (1.15), and let the measure  $\mu(t)$ , starting at  $\mu(t_n)$ , evolve according to (1.17) over the time interval  $[t_n, t_{n+1}]$ , where  $t_{n+1}$  is the smallest time greater than  $t_n$  satisfying at least one of the following conditions:

- either  $s_{t_{n+1}}(a^*, b^*, \eta^*) \leq \frac{t_{n+1}^{-1}}{2}$ ;
- or there exists  $(\bar{a}, \bar{b}, \bar{\eta}) \in \Omega'_{t_{n+1}}$  such that

$$s_{t_{n+1}}(a^*, b^*, \eta^*) \leq (1 - h)s_{t_{n+1}}(\bar{a}, \bar{b}, \bar{\eta}). \tag{1.21}$$

If  $t_{n+1}$  is finite, then go to Step  $(n + 1)$ .

If  $t_{n+1} = +\infty$ , then keep the control (1.18) or (1.20) over the time interval  $[t_n, +\infty)$ .

For this control strategy, the control  $\chi_{\omega} u$  satisfies the control constraint **(U)**, the unique solution  $\mu(t)$  of (1.17) is such that  $\mu(t) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$  for any  $t \in [0, +\infty)$ , and  $\mu(t)$  converges to  $\mathcal{Z} \cap \mathcal{P}_c(\overline{B(0,R)})$ , i.e.

- $\lim_{t \rightarrow +\infty} \inf_{\nu \in \mathcal{Z} \cap \mathcal{P}_c(\overline{B(0,R)})} W_p(\mu(t), \nu) = 0$ ;
- or equivalently, there exists a choice  $\nu(t) \in \mathcal{Z} \cap \mathcal{P}_c(\overline{B(0,R)})$  for each  $t \geq 0$  such that  $\mu(t) \rightharpoonup \nu(t)$ , i.e. for all  $\phi \in C_c^\infty(\mathbb{R}^d)$  it holds  $\lim_{t \rightarrow \infty} \int \phi d(\mu(t) - \nu(t)) = 0$ .

**Remark 1.6.** The three threshold time-dependent functions used in the definition of control algorithm in Theorem 1.1 satisfy  $t^{-1}/2 < t^{-1} < 2t^{-1}$ . One can easily see that they can be replaced with three positive functions satisfying  $\phi_1(t) < \phi_2(t) < \phi_3(t)$  converging to 0 as  $t \rightarrow +\infty$ . In particular, the functions can take finite values for  $t = 0$ , by maybe allowing one control to be active on the starting interval  $[0, t_1]$ .

The rest of the paper is structured as follows. In Sec. 2, we recall the main definitions and results for transport partial differential equation with non-local velocities as (1.1) and (1.2). In Sec. 3, we discuss some examples of dynamics of the form (1.2), and we explain some differences with respect to the finite-dimensional setting. Theorem 1.1 is proved in Sec. 4. In Sec. 5, we study a generalization of Theorem 1.1 to a system of the form (1.2) with several control potentials. Finally, in Sec. 6, we present an application of Theorem 1.1 to the control of kinetic multi-agent systems.

## 2. Transport Equations with Non-Local Velocities

In this section, we recall existence and uniqueness results for (1.1) and (1.2). In (1.1), the variable  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$  is a probability measure on  $\mathbb{R}^d$ . The term  $f[\mu]$  is called the *velocity field* and it is a non-local term. Since the value of a measure at a single point is not well defined, it is important to observe that  $f[\mu]$  is not a function depending on the value of  $\mu$  in a given point, as it is often the case in the setting of hyperbolic equations in which  $f[\mu](x) = f(\mu(x))$ . Instead, one has to consider  $f$  as an operator taking as input the whole measure  $\mu$  and giving as an output a global vector field  $f[\mu]$  on the whole space  $\mathbb{R}^d$ . These operators are often called “non-local”, as they consider the density not only at a given point, but in a whole neighborhood.

We first recall two useful definitions to deal with measures and solutions of (1.1), namely the Wasserstein distance and the push-forward of measures (for more details see, e.g. Ref. 34).

**Definition 2.1.** Given two probability measures  $\mu$  and  $\nu$  on  $\mathbb{R}^d$  and  $p \in [1, +\infty)$ , the  $p$ -Wasserstein distance between  $\mu$  and  $\nu$  is

$$W_p(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^{2d}} |x - y|^p d\pi(x, y) \mid \pi \in \Pi(\mu, \nu) \right\}^{1/p},$$

where  $\Pi(\mu, \nu)$  is the set of transference plans from  $\mu$  to  $\nu$ , i.e. of the probability measures  $\pi$  on  $\mathbb{R}^d \times \mathbb{R}^d$  such that  $\text{Proj}_x \# \pi = \mu$  and  $\text{Proj}_y \# \pi = \nu$  with  $\text{Proj}_x : (x, y) \mapsto x$  and  $\text{Proj}_y : (x, y) \mapsto y$ .

The topology induced by  $W_p$  on the space of probability measures  $\mathcal{P}(X)$  on a compact space  $X$  coincides with the weak-\* topology of measures (see Ref. 34). As a consequence of Assumption (H3), each trajectory  $\mu(t)$  of the controlled system (1.2) is contained in the compact space  $\mathcal{P}_c(\overline{B(0, R)})$  (compact if endowed with the Wasserstein topology). Thus, *from now on, we will state equivalently convergence*

with respect to the weak- $*$  topology of measures and with respect to the Wasserstein distance. We now define the push-forward of measures.

**Definition 2.2.** Given a Borel map  $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , the push-forward of a measure  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$  is defined by  $\gamma\#\mu(A) = \mu(\gamma^{-1}(A))$  for every measurable subset  $A$  of  $\mathbb{R}^d$ .

We now recall an existence and uniqueness result for (1.1) (see a complete proof in Ref. 31).

**Theorem 2.1.** *We assume that, for every  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ , the velocity field  $f[\mu]$  is a function of  $(t, x)$  with the regularity*

$$f[\cdot] : \mathcal{P}(\mathbb{R}^d) \rightarrow \text{Lip}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),$$

$$\mu \mapsto f[\mu],$$

satisfying the following assumptions:

- there exist functions  $L(\cdot)$  and  $M(\cdot)$  in  $L^\infty_{\text{loc}}(\mathbb{R})$  such that

$$\|f[\mu](t, x) - f[\mu](t, y)\| \leq L(t)\|x - y\|, \quad \|f[\mu](t, x)\| \leq M(t)(1 + \|x\|),$$

for every  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ , every  $t \in \mathbb{R}$  and all  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ ;

- for a given  $p \in [1, +\infty)$ , there exists a function  $K(\cdot)$  in  $L^\infty_{\text{loc}}(\mathbb{R})$  such that

$$\|f[\mu] - f[\nu]\|_{L^\infty(\mathbb{R}; C^0(\mathbb{R}^d))} \leq K(t)W_p(\mu, \nu),$$

for all  $(\mu, \nu) \in (\mathcal{P}_c(\mathbb{R}^d))^2$ .

Then, for every  $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ , the Cauchy problem

$$\partial_t \mu + \nabla \cdot (f[\mu]\mu) = 0, \quad \mu|_{t=0} = \mu_0 \tag{2.1}$$

has a unique solution  $\mu(\cdot) \in C^0(\mathbb{R}; \mathcal{P}_c(\mathbb{R}^d))$ , where  $\mathcal{P}_c(\mathbb{R}^d)$  is endowed with the weak- $*$  topology of measures. Moreover,  $t \mapsto \mu(t)$  is Lipschitz in the sense of the Wasserstein distance  $W_p$ . Moreover, if  $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ , then  $\mu(t) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$  for every  $t \in \mathbb{R}$ .

Furthermore, for every  $T > 0$ , there exists  $C_T > 0$  such that

$$W_p(\mu(t), \nu(t)) \leq e^{C_T t} W_p(\mu(0), \nu(0)), \tag{2.2}$$

for all solutions  $\mu$  and  $\nu$  of (2.1) in  $C^0([0, T]; \mathcal{P}_c(\mathbb{R}^d))$ .

Finally, the solution  $\mu$  of the Cauchy problem (2.1) can be made explicit as follows. Let  $\Phi(t)$  be the flow of diffeomorphisms of  $\mathbb{R}^d$  generated by the time-dependent vector field  $f[\mu]$ , defined as the unique solution of the Cauchy problem  $\dot{\Phi}(t) = f[\mu(t)] \circ \Phi(t)$ ,  $\Phi(0) = \text{Id}_{\mathbb{R}^d}$ , or in other words,

$$\partial_t \Phi(t, x) = f[\mu(t)](t, \Phi(t, x)), \quad \Phi(0, x) = x.$$

Then, we have

$$\mu(t) = \Phi(t)\#\mu_0,$$

that is,  $\mu(t)$  is the push-forward of  $\mu_0$  under  $\Phi(t)$ .

Theorem 2.1 can be generalized to mass-varying transport PDEs, that is, in presence of sources (see Ref. 30). We now observe that Theorem 2.1 can be applied to (1.2) as well, under Assumptions **(H)** and provided that the control  $u$  be a Lipschitz function of the space variable for all times.

**Corollary 2.1.** *Under Assumptions **(H)**, if  $u$  is a uniformly Lipschitz function of  $x$  on the time interval  $[0, \theta]$ , satisfying the constraint **(U)**, then, given any initial data  $\mu(0) = \mu_0 \in \mathcal{P}_c(\overline{B(0, R)})$ , Eq. (1.2) has a unique solution  $\mu(\cdot) \in C^0([0, \theta], \mathcal{P}_c(\overline{B(0, R)}))$ . Moreover, if  $\mu_0 \in \mathcal{P}_c^{ac}(\overline{B(0, R)})$ , then  $\mu(t) \in \mathcal{P}_c^{ac}(\overline{B(0, R)})$  for every  $t \in [0, \theta]$ . Denoting by  $\Psi$  the flow of diffeomorphisms of  $\mathbb{R}^d$  generated by the time-dependent vector field  $f[\mu] + u(t, x)g[\mu]$ , we have  $\mu(t) = \Psi(t)\#\mu_0$ .*

**Proof.** It suffices to check that Theorem 2.1 can be applied to the vector field  $f[\mu] + u(t, x)g[\mu]$ . As already stated, the existence of a uniform bound  $M$  for  $\|f + ug\|_{L^\infty}$  is a consequence of the uniform Lipschitz property and of the uniform boundedness of the support of both  $f[\mu]$  and  $g[\mu]$ , together with the bound  $\|u\|_{L^\infty} \leq 1$  imposed by **(U)** in (1.4). Similarly, we have a uniform bound on the Lipschitz constant  $\text{Lip}(f + ug)$ . Indeed, by (1.3),

$$\begin{aligned} \text{Lip}(f[\mu(t)] + ug[\mu(t)]) &\leq L + \text{Lip}_x(u)\|g[\mu(t)]\|_{L^\infty} + \|u\|_{L^\infty}\text{Lip}(g[\mu(t)]) \\ &\leq 2L + \text{Lip}_x(u)M. \end{aligned} \tag{2.3}$$

Finally, we have

$$\begin{aligned} \|f[\mu] + ug[\mu] - f[\nu] + ug[\nu]\|_{L^\infty} &\leq \|f[\mu] - f[\nu]\|_{L^\infty} + \|u\|_{L^\infty}\|g[\mu] - g[\nu]\|_{L^\infty} \\ &\leq W_p(\mu, \nu) + 1 \cdot QW_p(\mu, \nu) \\ &= 2QW_p(\mu, \nu). \end{aligned}$$

This proves the corollary. □

We end this section with an estimate of the  $L^\infty$ -norm of the solution  $\mu(t)$  to (1.1), when it is absolutely continuous with respect to the Lebesgue measure.

**Proposition 2.1.** *Let  $\mu(\cdot)$  be the unique solution of (1.1) for a given Lipschitz vector field  $f$  with  $\mu_0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ . Then*

$$\frac{d}{dt}\|\mu(t)\|_{L^\infty} \leq \|\mu(t)\|_{L^\infty}\|\nabla \cdot f\|_{L^\infty}. \tag{2.4}$$

**Proof.** The proof follows Proposition 3.1 in Ref. 22. Let  $\rho(t)$  be the density of  $\mu(t)$  with respect to the Lebesgue measure. For each  $p \in [1, +\infty)$ , by dropping the dependence with respect to time, we write

$$\frac{d}{dt} \int \rho^p dx = -p \int \rho^{p-1} \nabla \cdot (f\rho) dx = -p \int (\rho^p \nabla \cdot f + \rho^{p-1} \langle f, \nabla \rho \rangle) dx.$$

Since  $\nabla \cdot (f\rho^p) = \rho^p \nabla \cdot f + \langle f, \nabla(\rho^p) \rangle = \rho^p \nabla \cdot f + p\rho^{p-1} \langle f, \nabla \rho \rangle$ , we infer that

$$\frac{d}{dt} \int \rho^p dx = - \int (p-1)\rho^p \nabla \cdot f dx - \int \nabla \cdot (f\rho^p) dx.$$

The last term is zero as a consequence of the divergence theorem. Then  $\frac{d}{dt} \|\rho\|_{L^p}^p \leq (p-1) \|\rho\|_{L^p}^p \|\nabla \cdot f\|_{L^\infty}$ , which in turn implies (2.4) as  $p \rightarrow +\infty$ .  $\square$

### 3. Steepest Descent Under Population Constraint Induces Mass Concentration

In this section, we discuss a remarkable phenomenon for controlled equations of the form (1.2): starting from a measure  $\mu_0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ , i.e. a measure that is absolutely continuous with respect to the Lebesgue measure in  $\mathbb{R}^d$ , a time-dependent choice of the control might drive the measure outside  $\mathcal{P}_c^{ac}(\mathbb{R}^d)$  in finite time, in particular with emergence of Dirac deltas. In fact, we will show that *such a phenomenon arises when trying to minimize a Lyapunov function  $V$* , in particular when one chooses the control  $u(t)$  as the instantaneous minimizer of the Lie derivative of  $V$  as time evolves. This example also shows that some key ideas coming from control of finite-dimensional systems cannot be extended straightforwardly to infinite dimension.

In this section, we discuss the interest and the drawbacks of a control constraint different than **(U)**, namely the following:

#### Alternative Control Constraints (**U'**)

Fix  $c > 0$ . For each time  $t \geq 0$  it holds:

**Sparsity population constraint:** 
$$\int_{\omega(t)} d\mu(t) \leq c, \quad (3.1)$$

**Finite strength:** 
$$\|u(t, \cdot)\|_{L^\infty} \leq 1. \quad (3.2)$$

The population constraint represents the idea of acting on a small part of the mass itself, and not on a small part of the configuration space, as we require in the space constraint in **(U)**. Even though the sparse population constraint is interesting from the theoretical point of view, it has a surprising drawback on the modeling point of view: when a measure is extremely concentrated, the constraint **(U')** implies that the control cannot act on the whole mass anymore. This is somehow unnatural, since a crowd that is already concentrated is the best configuration to steer. On the other hand, the space constraint **(U)** permits to act on the whole measure, when it is concentrated in a set of size  $c$ , i.e. exactly when it is concentrated.

We now show that the population constraint also induces some formal mathematical problems when using a sparse Jurdjevic–Quinn approach. Consider the

following system on the real line:

$$\partial_t \mu + \partial_x (u\mu) = 0. \tag{3.3}$$

This is a particular case of (1.2) with  $f = 0$  and  $g = 1$ . We consider the initial data  $\mu_0 = \chi_{[0,1]}$ , i.e. a uniform probability density on the interval  $[0, 1]$ . We consider the Lyapunov function

$$V[\mu] = \int x^2 d\mu(x),$$

i.e. the second moment with respect to zero. We have  $\mathcal{L}_f V = 0$ , and we have  $\mathcal{L}_{ug} V[\mu] = 0$  for  $\mu = \delta_0$  only, i.e.  $\mathcal{Z} = \{\delta_0\}$ . Then, minimizing  $V$  is equivalent to steer  $\mu(t)$  to the Dirac mass  $\delta_0$ .

We now apply a rough form of the steepest descent method to the problem of minimizing  $V$ : given the initial measure  $\mu_0$ , we look for a control function  $u$  that maximizes the descent  $\mathcal{L}_{ug} V[\mu]$ , while taking into account the control constraints  $(\mathbf{U}')$ . An easy computation shows that no optimal choice for  $u$  exists. Indeed, for every  $\varepsilon > 0$ , consider the  $C^\infty$ -function

$$u_\varepsilon(x) = \begin{cases} -1 & \text{for } x \in [1 - c + \varepsilon, 1], \\ 0 & \text{for } x \in (-\infty, 1 - c] \cup [1 + \varepsilon, +\infty), \\ C^\infty\text{-spline with values in } [-1, 0] & \text{for } x \in [1 - c, 1 - c + \varepsilon] \cup [1, 1 + \varepsilon]. \end{cases}$$

Then, for a sufficiently small time  $t > 0$ , each particle  $x \in (1 - c + \varepsilon, 1]$  is displaced to  $x - t$  while each particle  $x \in [0, 1 - c]$  undergoes no displacement. The particles in the small interval  $[1 - c, 1 - c + \varepsilon]$  are displaced toward  $1 - c$ , then giving a reduction of the value of the functional  $V$ . Then, we have

$$\begin{aligned} \mathcal{L}_{u_\varepsilon g} V[\mu] &= \frac{d}{dt} \Big|_{t=0} \int_{1-c}^{1-c+\varepsilon} (x + tu(x) + o(t))^2 d\mu(x) + \frac{d}{dt} \Big|_{t=0} \int_{1-c+\varepsilon}^1 (x - t)^2 d\mu(x) \\ &\leq -2 \int_{1-c+\varepsilon}^1 x d\mu(x). \end{aligned}$$

As a consequence, by decreasing the parameter  $\varepsilon > 0$ , one can reach a larger decrease of  $V$ . Nevertheless, the limit for  $\varepsilon \rightarrow 0$  would result in the discontinuous control function  $u_0 = \chi_{[1-c,1]}$ , for which the solution to the corresponding dynamics (3.3) does not satisfy Assumptions<sup>b</sup>  $(\mathbf{H})$ .

The fact that a maximizer of the steepest descent does not exist in the space of Lipschitz functions can be overcome by fixing a value  $\varepsilon_0 > 0$  and applying the control  $u_{\varepsilon_0}$  over a small interval of time  $[0, t_0]$  with  $t_0 < \varepsilon_0$ . As a result, the component of the measure  $\mu$  with  $x \in [1 - c, 1]$  concentrates in the interval  $[1 - c, 1 - t_0]$ , while its density keeps being constantly equal to 1 for  $x \in [0, 1 - c]$ . At time  $t$ , one can

<sup>b</sup>Existence and uniqueness for the solution of (1.2) in small times, with possibly discontinuous controls, can be derived from results in Refs. 2 and 21. Nevertheless, it is shown in Ref. 21 that, under such assumptions, one can have formation of singularities such as Dirac deltas in finite time.

observe that the largest contribution to  $V$  comes anyway from the mass in the interval  $[1 - c, 1 - t_0]$ , on which the control already acted. Moreover, the mass in such interval keeps being  $c$ .

For the reasons described above, any strategy maximizing the descent of  $V$  acts on the mass in the interval  $[1 - c, 1]$  for all times. In particular, we can choose a sequence  $\varepsilon_i$  acting on the time interval  $[t_{i-1}, t_i]$ , with the condition  $\varepsilon_i < c - t_i$  for all  $t \in [0, c)$ . Applying the time-dependent control  $u(t, x) = u_{\varepsilon_i}(x)$  for  $t \in [t_{i-1}, t_i)$ , the corresponding solution  $\mu(t)$  of (3.3) has support in  $[0, 1 - t]$ . More precisely, it has the following structure: the measure keeps having density 1 in the interval  $[0, 1 - c]$ , while the rest of the mass  $c$  is contained in the interval  $[1 - c, 1 - t]$ .

Then, the solution  $\mu(t)$  converges as  $t \nearrow c$  to the singular measure

$$\mu(c) = \chi_{[0, 1-c]} + c\delta_{1-c}. \tag{3.4}$$

This is not in contradiction with the fact that any solution of (1.1) with initial data  $\mu_0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$  and Lipschitz vector field  $f$  satisfies  $\mu(t) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ . Indeed, the condition  $\varepsilon(t) < c - t$  implies  $\lim_{t \rightarrow c} \varepsilon(t) = 0$ , hence the control  $u(t, x)$  converges to a non-Lipschitz function.

Starting from the singular measure (3.4) at time  $t = c$ , one finds several problems to steer it toward the minimizer  $\delta_0$  of  $V$ . First, the main contribution to  $V$  is given by the Dirac delta  $c\delta_{1-c}$ : then, the control set  $\omega$  chosen to maximize the descent for  $V$  would certainly contain such mass. But the condition  $1 - c \in \omega$  together with  $u(1 - c) \neq 0$  would directly impose to choose  $\omega$  containing a whole neighborhood of  $1 - c$ . This would imply  $\int_{\omega} \mu(c) > c$ , hence the population constraint would be automatically violated.

This would in turn enforce us to focus our control on the absolutely continuous part, possibly leading to the formation of a new Dirac delta  $c\delta_{1-2c}$ , and so on. The final result would be a set of Dirac deltas, not concentrated at 0, on which control with population constraint cannot be applied.

Summing up, the steepest descent method with population constraint in  $(U')$  might not steer the measure to a configuration in  $\mathcal{Z}$ , but rather to a configuration in which the population constraint itself may not be satisfied.

#### 4. Proof of Theorem 1.1

For the moment, we assume that the solution  $\mu(\cdot)$  of (1.17), with the control strategy defined by Theorem 1.1, is well defined on  $[0, \theta]$  with  $\theta \in (0, +\infty]$ , and we establish some lemmas describing its evolution. Recall that  $\text{supp}(\mu_0) \subset \overline{B(0, R)}$ .

**Lemma 4.1.** *We have  $\text{supp}(\mu(t)) \subset \overline{B(0, R)}$  for every  $t \in [0, \theta]$ .*

**Proof.** Since the vector field  $f + ug$  is zero outside  $\overline{B(0, R)}$ , the corresponding flow  $\phi^t(\cdot)$  coincides with the identity in  $\mathbb{R}^d \setminus \overline{B(0, R)}$ . Since we have  $\mu(t) = \phi_{\#}^t \mu_0$  by Theorem 2.1, we get that, for any Borel set  $E$  satisfying  $E \cap \overline{B(0, R)} = \emptyset$ , we have  $\mu(t)(E) = \mu_0(\phi^{-t}(E)) = \mu_0(E) = 0$ . The lemma follows.  $\square$

Recall that  $L$  is the Lipschitz constant for  $f[\mu]$  and  $g[\mu]$  given in (1.5), and recall that  $\|f[\mu]\|_{L^\infty} \leq M$  and  $\|g[\mu]\|_{L^\infty} \leq M$  for every  $\mu \in \mathcal{P}_c^{ac}(\overline{B(0, R)})$  (as in Remark 1.5).

**Lemma 4.2.** *We have  $\|\mu(t)\|_{L^\infty} \leq e^{d\theta(2L+M\theta)}\|\mu(0)\|_{L^\infty}$ , for every  $t \in [0, \theta]$ .*

**Proof.** Since the vector field  $f + ug$  satisfies (2.3), and since  $\text{Lip}_x(u(t, \cdot)) \leq \frac{1}{\eta} \leq t \leq \theta$ , the lemma follows from Proposition 2.1.  $\square$

**Lemma 4.3.** *The function  $(t, a, b, \eta) \mapsto s_t(a, b, \eta)$  is continuous with respect to  $t$ , and uniformly Lipschitz with respect to  $(a, b, \eta)$  on  $\cup\{\Omega_t, t \in [0, \theta]\}$ .*

**Proof.** Let us first establish the Lipschitz property for  $(a, b, \eta) \in \cup\{\Omega_t, t \in [t_n, t_{n+1}] \cap [0, \theta]\}$ . Note that the condition  $|\omega(a, b, \eta)| \leq c$  implies that  $\eta \leq \frac{c}{2}$ . Besides, we have  $\eta \geq t^{-1} \geq \theta^{-1}$ . By definition of  $U(a, b, \eta)$  in (1.15), with simple geometric arguments, it is clear that in the 1D case we have

$$\begin{aligned} \|U(a, b, \eta) - U(a', b', \eta')\|_{L^1(dx)} &\leq \|U(a, b, \eta) - U(a, b, \eta')\|_{L^1(dx)} \\ &\quad + \|U(a, b, \eta') - U(a', b', \eta')\|_{L^1(dx)} \\ &\leq |\eta - \eta'| + |a - a'| + |b - b'|, \end{aligned}$$

where  $dx$  is the standard Lebesgue measure on  $\mathbb{R}$ . By applying the estimate componentwise, the same result follows in dimension  $d$ . By Lemma 4.2, there exists  $P > 0$  such that  $\|\mu(t)\|_{L^\infty} \leq M$  for every  $t \in [0, \theta]$ , hence

$$\|U(a, b, \eta) - U(a', b', \eta')\|_{L^1(\mu(t))} \leq P(|a - a'| + |b - b'| + |\eta - \eta'|),$$

and thus (1.11) implies that  $s_t(a, b, \eta)$  is Lipschitz with respect to  $(a, b, \eta)$ , with Lipschitz constant  $KP$ .

The function  $s_t(a, b, \eta)$  is continuous with respect to  $t$ , as a consequence of the continuity of  $\mathcal{L}_{ug}V$  given by (1.12).  $\square$

**Lemma 4.4.** *Assume that  $\mu(t_n) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ . If  $\Omega_{t_n}$  is nonempty then  $s_{t_n}(a, b, \eta)$  has a maximizer in  $\Omega_{t_n}$ .*

**Proof.** It suffices to observe that  $\Omega_{t_n}$  can be considered as being compact: indeed, each choice  $(a, b, \eta) \in \Omega_{t_n}$  can be replaced by an equivalent choice  $(a', b', \eta)$  with  $a, b \in B_0(R + 2c)$  since  $\|U(a, b, \eta) - U(a', b', \eta)\|_{L^1(\mu)} = 0$ , since  $\mu$  has zero mass outside of  $\overline{B(0, R)}$ . In other terms, one can restrict the choice of the parameters  $a, b$  to a compact set. Similarly, we have  $\eta \in [t_n^{-1}, \frac{c}{2}]$ . Since, by Lemma 4.3,  $s_{t_n}$  is a continuous function of its arguments, then it admits a maximizer.  $\square$

We are now in a position to prove Theorem 1.1. We split the proof into three steps:

- Step 1.** For each time  $t_n$ , the  $n$ th step of the algorithm univocally determines a control satisfying the constraint **(U)**, the corresponding solution of (1.2) and a time  $t_{n+1} > t_n$ .
- Step 2.** We have  $t_n \rightarrow +\infty$ .
- Step 3.** We have  $\lim_{t \rightarrow +\infty} \mathcal{L}_{f+ug} V[\mu(t)] = 0$ . This fact, together with the choice of maximizing controls and of the hysteresis, provides convergence to the sets in which the maximizers of  $s_t(a, b, \eta)$  give zero control. Since the constraint  $\eta \geq t^{-1}$  is negligible for  $t \rightarrow +\infty$ , the strategy provides convergence of  $\mu(t)$  to  $\mathcal{Z}$ .

**Proof of Step 1.** Let us prove that the algorithm of Theorem 1.1 univocally defines a control strategy, by induction. We have  $t_0 = 0$  and  $\mu(0) = \mu_0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ . Let us prove that, for a given time  $t_n$ , the time  $t_{n+1}$  is well defined and satisfies  $t_{n+1} > t_n$ .

We first observe that the control  $\chi_\omega u$  is a well-defined function, Lipschitz with respect to  $x$ . Setting  $\tilde{t} = \frac{1}{2} \left( \frac{c}{|B_0(1)|} \right)^{1/d}$ , we note that  $\Omega_t = \emptyset$  for every  $t \in [0, \tilde{t})$  since any function of the form  $\chi_{[a,b]}^\eta$  has a support of size larger than  $(2\eta)^d |B_0(1)|$ . For  $t \geq \tilde{t}$ , the set  $\Omega_t$  is nonempty and Lemma 4.4 yields the existence of a maximizer  $(a^*, b^*, \eta^*)$  in  $\Omega_t$ . We thus have two cases:

- If  $s_{t_n}(a^*, b^*, \eta^*) < t_n^{-1}$  or  $\Omega_t$  empty, then the control  $\chi_\omega u = 0$  is well defined and is Lipschitz.
- If  $s_{t_n}(a^*, b^*, \eta^*) \geq t_n^{-1}$ , then the control  $\chi_\omega u = U(a^*, b^*, \eta^*)$  is well defined and is Lipschitz, as a consequence of the Lipschitz property in (1.14).

Let us now prove that there exists a unique minimum  $t_{n+1}$  defined by the algorithm, and that it satisfies  $t_{n+1} > t_n$ . For  $t \in [0, \tilde{t})$ , there is nothing to prove, since  $t_1 \geq \tilde{t}$ . For  $t \geq \tilde{t}$ , we have two cases:

- If  $s_{t_n}(a^*, b^*, \eta^*) < t_n^{-1}$ , then the set

$$A = \{t \geq t_n \mid s_t(a, b, \eta) \geq t^{-1} \text{ for some } (a, b, \eta) \in \Omega'_t\}$$

is closed or empty. If it is nonempty, there exists a minimal element  $t_{n+1} \geq t_n$ . Moreover,  $t_{n+1} \neq t_n$ , since  $s_{t_n}(a, b, \eta) \leq s_{t_n}(a^*, b^*, \eta^*) < t_n^{-1}$  for all  $(a, b, \eta) \in \Omega_{t_n}$ . If  $A$  is empty then  $+\infty = t_{n+1} > t_n$ .

- If  $s_{t_n}(a^*, b^*, \eta^*) \geq t_n^{-1}$ , then, similarly to the previous case, since the function  $s_t(a^*, b^*, \eta^*)$  is continuous with respect to the time  $t$ , the set

$$A' = \left\{ t \geq t_n \mid s_t(a^*, b^*, \eta^*) \leq \frac{t_n^{-1}}{2} \right\}$$

is closed or empty, and it does not contain  $t_n$ . We now consider the set

$$B' = \left\{ t \geq t_n \text{ such that there exists } (\bar{a}, \bar{b}, \bar{\eta}) \in \Omega'_t \text{ for which } \right. \\ \left. s_t(\bar{a}, \bar{b}, \bar{\eta}) \geq (1 - h)^{-1} s_t(a^*, b^*, \eta^*) \geq (1 - h)^{-1} \frac{t^{-1}}{2} \right\}.$$

Let us prove that it is closed and that  $t_n \notin B'$ . Take a sequence  $(t^i, a^i, b^i, \eta^i)$  such that  $t^i \in B'$  is a sequence converging to some  $\hat{t}$ , and  $(a^i, b^i, \eta^i)$  satisfy  $s_{t^i}(a^i, b^i, \eta^i) \geq (1 - h)^{-1} s_{t^i}(a^*, b^*, \eta^*) \geq \frac{\tau(t^i)}{2}$ . Observing that the compact set  $\Omega'_t$  varies smoothly with respect to time, we can restrict ourselves to a sequence  $(a^i, b^i, \eta^i)$  converging to some  $(\bar{a}, \bar{b}, \bar{\eta}) \in \Omega'_{\hat{t}}$ . Then, by continuity of  $s_t$ , we have  $s_{\hat{t}}(\bar{a}, \bar{b}, \bar{\eta}) \geq (1 - h)^{-1} s_{\hat{t}}(a^*, b^*, \eta^*)$ , hence  $\hat{t} \in B'$ . Moreover,  $\hat{t} \neq t_n$ , otherwise  $(a^*, b^*, \eta^*)$  would not be a maximizer of  $s_{t_n}$ .

Since both  $A'$  and  $B'$  are closed or empty, not containing  $t_n$ , then  $A' \cup B'$  is closed or empty and does not contain  $t_n$ . If it is closed, then it admits a minimal element  $t_{n+1} > t_n$ ; if it is empty, then we have  $+\infty = t_{n+1} > t_n$ . □

**Proof of Step 2.** We now prove that the sequence  $t_n$  of times given by the algorithm converges to  $+\infty$ . Since  $t_n$  is increasing, it has a limit  $T$ . By contradiction, if  $T < +\infty$ , then  $\mu(t)$  is defined for every  $t \in [0, T]$ . Indeed, since  $\|f + ug\|_{L^\infty} \leq 2M$ , the curve  $t \mapsto \mu(t)$  is Lipschitz, and thus  $\mu(T)$  is well defined.

If we have  $s_{t_n}(a^*, b^*, \eta^*) < t_n^{-1}$  at time  $t_n$ , then at the next time  $t_{n+1}$  we must have  $s_{t_{n+1}}(\bar{a}, \bar{b}, \bar{\eta}) \geq t_{n+1}^{-1}$  for some  $(\bar{a}, \bar{b}, \bar{\eta}) \in \Omega_{t_{n+1}}$ , by definition of the algorithm itself. As a consequence, the sequence  $t_n$  converging to  $T$  contains an infinite number of times  $t_{n_i}$  such that  $s_{t_{n_i}}(a^i, b^i, \eta^i) \geq t_{n_i}^{-1}$ , where  $(a^i, b^i, \eta^i)$  is a maximizer of  $s_{t_{n_i}}$  in  $\Omega_{t_{n_i}}$ .

The sequence  $(a^i, b^i, \eta^i)$  is bounded, and its converging subsequences have their limit in  $\Omega_T$ . Indeed, we can restrict ourselves to  $(a^i, b^i) \in B_0(R + 2c)$ , and we have  $\eta^i \geq T^{-1}$  and  $\eta^i \leq \frac{c}{2}$ . Hence, taking a subsequence if necessary, we have the existence of a limit  $(\hat{a}, \hat{b}, \hat{\eta}) \in \Omega_T$ .

Observe now that, at time  $t_{n_i+1}$ , one of the two conditions leading to switching of the control holds. Since the sequence  $t_{n_i+1}$  has an infinite number of terms, at least one of the conditions holds for an infinite subsequence (that we do not relabel). We show now that this is in contradiction with the fact that  $t_n$  converges to a finite time  $T$ . We have two cases:

- If  $s_{t_{n_i}}(a^i, b^i, \eta^i) \geq t_{n_i}^{-1}$  and  $s_{t_{n_i+1}}(a^i, b^i, \eta^i) \leq \frac{t_{n_i}^{-1}}{2}$ , then, taking a subsequence converging to  $(\hat{a}, \hat{b}, \hat{\eta}) \in \Omega_T$ , we have a contradiction with the continuity of  $s_T$  in  $(\hat{a}, \hat{b}, \hat{\eta})$ . Indeed, we have

$$s_T(\hat{a}, \hat{b}, \hat{\eta}) = \lim_{i \rightarrow +\infty} s_{t_{n_i}}(a^i, b^i, \eta^i) \geq t_{n_i}^{-1} > \frac{t_{n_i}^{-1}}{2} \\ \geq \lim_{i \rightarrow +\infty} s_{t_{n_i+1}}(a^i, b^i, \eta^i) = s_T(\hat{a}, \hat{b}, \hat{\eta}).$$

- If there exists  $(\bar{a}^i, \bar{b}^i, \bar{\eta}^i) \in \Omega'_{t_{n_i+1}}$  such that

$$s_{t_{n_i+1}}(a^i, b^i, \eta^i) \leq (1 - h)s_{t_{n_i+1}}(\bar{a}^i, \bar{b}^i, \bar{\eta}^i), \tag{4.1}$$

then, for  $n_i \rightarrow +\infty$ , we have  $\Omega'_{t_{n_i+1}} \subset \Omega_{t_{n_i}}$ , since  $2t_{n_i+1}^{-1} \geq t_{n_i}^{-1}$ , as a consequence of the fact that  $\lim_{i \rightarrow +\infty} (t_{n_i+1} - t_{n_i}) = 0$ . Since  $(a^i, b^i, \eta^i)$  is a maximizer of  $s_{t_{n_i}}$  in  $\Omega_{t_{n_i+1}}$ , we have

$$s_{t_{n_i}}(\bar{a}^i, \bar{b}^i, \bar{\eta}^i) \leq s_{t_{n_i}}(a^i, b^i, \eta^i). \tag{4.2}$$

One can take a converging subsequence of  $(\bar{a}^i, \bar{b}^i, \bar{\eta}^i)$ , for the same reasons given above for the sequence  $(a^i, b^i, \eta^i)$ . Denoting by  $(\bar{a}^*, \bar{b}^*, \bar{\eta}^*)$  and  $(a^*, b^*, \eta^*)$  the two limits, and using continuity of  $s_t(a, b, \eta)$  with respect to all its arguments, we get from (4.1)–(4.2) that

$$s_T(a^*, b^*, \eta^*) \leq (1 - h)s_T(\bar{a}^*, \bar{b}^*, \bar{\eta}^*) \leq (1 - h)s_T(a^*, b^*, \eta^*),$$

which is in contradiction with  $s_T(a^*, b^*, \eta^*) \geq T^{-1} > 0$ .

Then  $t_n$  cannot converge to a finite value  $T$ . Therefore either  $t_n \rightarrow +\infty$  or there exists a  $t_n$  such that  $t_{n+1} = +\infty$ . In both cases, the control strategy is defined for every  $t \in [0, +\infty)$ . □

**Proof of Step 3.** It remains to prove that  $\mu(t)$  converges to  $\mathcal{Z}$ . This is the hardest part of the proof, in which the choice of the admissible controls in  $\Omega_t$  plays a crucial role.

Thanks to Step 2, we have, for every time,  $\mu(t) \in \mathcal{P}_c(\overline{B(0, R)})$ , that is compact with respect to the weak topology, which coincides with the topology of the Wasserstein distance. Then Assumptions **(H)** imply that  $V$  is a continuous function and thus is bounded below.

We now prove that the function  $V(t) = V[\mu(t)]$  is differentiable for almost every  $t$ , and that it satisfies  $\dot{V}(t) = \lim_{t \rightarrow 0} \frac{V[e^{t(f+ug)}\mu] - V[\mu]}{t} \leq 0$ . Differentiability on the open time interval  $(t_n, t_{n+1})$  follows from the fact that  $\dot{V}(t) = \mathcal{L}_{f+ug}V[\mu(t)]$  is continuous, as a consequence of Assumptions **(H)**. Clearly, the set of times  $t_n$  on which differentiability is not ensured is countable, hence  $V(t)$  is differentiable for almost every  $t$ .

For  $t \in (t_n, t_{n+1})$ , we have  $\dot{V} = \mathcal{L}_{f+ug}V[\mu] = \mathcal{L}_fV[\mu] + \mathcal{L}_{ug}V[\mu]$ . If at time  $t_n$  the algorithm defines the control  $\chi_\omega u \equiv 0$ , then clearly

$$\dot{V} = \mathcal{L}_fV[\mu] \leq 0, \tag{4.3}$$

for every  $t \in (t_n, t_{n+1})$ . If instead the control given by the algorithm is  $\chi_\omega u$  in (1.20), we have

$$\begin{aligned} \dot{V} &= \mathcal{L}_fV[\mu] + \mathcal{L}_{\chi_\omega ug}V[\mu] \\ &\leq -\text{sign}(\mathcal{L}_{U(a^*, b^*, \eta^*)g}[\mu(t_n)]V[\mu(t_n)])(\mathcal{L}_{U(a^*, b^*, \eta^*)g}[\mu(t)]V[\mu(t)]). \end{aligned} \tag{4.4}$$

It is clear that, at the beginning of the interval, we have

$$\begin{aligned} \lim_{t \rightarrow t_n^+} \operatorname{sign}(\mathcal{L}_{U(a^*, b^*, \eta^*)g[\mu(t_n)]}V[\mu(t_n)])(\mathcal{L}_{U(a^*, b^*, \eta^*)g[\mu(t_n)]}V[\mu(t_n)]) &\geq s_{t_n}(a^*, b^*, \eta^*) \\ &\geq \tau(t_n) > 0, \end{aligned}$$

and hence  $\dot{V}(t_n^+) < 0$ . Since  $\dot{V}(t)$  is a continuous function, we either have  $\dot{V}(t) < 0$  for every  $t \in (t_n, t_{n+1})$ , or there exists  $t \in (t_n, t_{n+1})$  such that  $\mathcal{L}_{U(a^*, b^*, \eta^*)g[\mu(t)]} \times V[\mu(t)] = 0$ . This is equivalent to state that  $s_t(a^*, b^*, \eta^*) = 0$ , which is in contradiction with  $s_t(a^*, b^*, \eta^*) > \frac{\tau(t)}{2} > 0$  for every  $t \in (t_n, t_{n+1})$ , by definition of the time  $t_{n+1}$ .

We now prove that  $\lim_{t \rightarrow \infty} \mu(t) \in \mathcal{Z}$ . Since  $\mathcal{P}_c(\overline{B(0, R)})$  is compact, all sequences have limits. Consider a sequence  $t^j \rightarrow \infty$  such that  $\lim_{j \rightarrow \infty} \mu(t^j) = \mu^*$ . We are going to prove that  $\mu^* \in \mathcal{Z}$ .

Since  $V$  is continuous, bounded below and  $\dot{V}(t) \leq 0$  for almost every  $t$ , we have  $\lim_{t \rightarrow +\infty} V(t) = V^*$  for some  $V^*$ . The existence and continuity of the second-order derivatives  $\mathcal{L}_{f+ug}\mathcal{L}_{f+ug}V[\mu]$  on the compact space  $\mathcal{P}_c(\overline{B(0, R)})$  implies the existence of a uniform bound on  $\ddot{V}$ . As a consequence, we have  $\lim_{t \rightarrow \infty} \dot{V}(t) = 0$ . Since  $\dot{V} \leq \mathcal{L}_f V \leq 0$  by either (4.3) or (4.4), this in turn implies  $\lim_{j \rightarrow \infty} \mathcal{L}_f V[\mu(t^j)] = 0$ , hence  $\mathcal{L}_f V[\mu^*] = 0$  by continuity of  $\mathcal{L}_f V$ .

We now prove that  $\mathcal{L}_{ug}V[\mu^*] = 0$  for all  $u \in \mathcal{U}$ . By contradiction, assume that there exists  $u^* \in \mathcal{U}$  such that  $|\mathcal{L}_{u^*g}V[\mu^*]| \neq 0$ . Without loss of generality, by using (1.10), we assume that  $\|u^*\|_{L^\infty} = 1$ . Similarly, by decomposing  $u^* = u^+ - u^-$  with  $u^+, u^-$  non-negative Lipschitz functions, and using additivity of the Lie derivative, we can replace  $u^*$  with either  $u^+$  or  $u^-$  and assume that it is non-negative and that  $|\mathcal{L}_{u^*g}V[\mu^*]| = C^* \neq 0$ . Finally, by observing that  $\mu^*$  has compact support, we can replace  $u^*$  with a non-negative Lipschitz function with compact support.

We now approximate  $u^*$  in  $L^\infty$  by a family of functions of the form  $\sum_{i=1}^I k^i \chi_{[a^i, b^i]}^\eta$ , where the number  $I$  of terms depends on the approximation error, but not on the (sufficiently small) parameter  $\eta$ .

For simplicity, we only give the construction in the 1D case.

Since  $u^*$  is Lipschitz with bounded support, it is Riemann integrable. In particular, by using an approximation of  $u^*$  from below, we have the following: take a grid step  $\Delta x$  and define the rectangles  $k^i \chi_{[\tilde{a}^i, \tilde{b}^i]}$  with  $\tilde{b}^i - \tilde{a}^i = \Delta x$ , for which

$$\sum_{i=1}^I k^i \chi_{[\tilde{a}^i, \tilde{b}^i]} \leq u^* \quad \text{and} \quad \left\| u^* - \sum_{i=1}^I k^i \chi_{[\tilde{a}^i, \tilde{b}^i]} \right\|_{L^\infty} \leq \varepsilon, \tag{4.5}$$

for some  $\varepsilon$ . The Riemann integrability of  $u^*$  implies that, for any  $\varepsilon > 0$  there exists  $\Delta x$  such that (4.5) is satisfied. Note that  $\|u^*\|_{L^\infty} = 1$  also implies  $k^i \leq 1$ .

We now prove that we can replace  $k^i \chi_{[\tilde{a}^i, \tilde{b}^i]}$  with their mollified version  $k^i \chi_{[a^i, b^i]}^\eta$  (see Fig. 1) for any sufficiently small  $\eta$ , while keeping (4.5) satisfied. We provide here the explicit construction. First denote with  $L'$  the Lipschitz constant of  $u^*$ . To

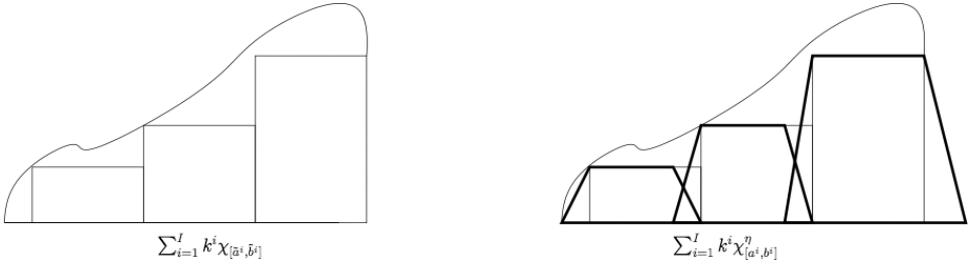


Fig. 1. Construction of the approximation  $k^i \chi_{[a^i, b^i]}^\eta$ .

replace  $k^i \chi_{[\tilde{a}^i, \tilde{b}^i]}$  with  $k^i \chi_{[a^i, b^i]}^\eta$ , we have two cases:

- If  $\tilde{b}^{i-1} < \tilde{a}^i$ , then keep both  $b^{i-1} = \tilde{b}^{i-1}$  and  $a^i = \tilde{a}^i$ , and choose  $\eta \leq \min\{\frac{\tilde{a}^i - \tilde{b}^{i-1}}{2}, \frac{k^{i-1}}{L'}, \frac{k^i}{L'}\}$ .
- If  $\tilde{b}^{i-1} = \tilde{a}^i$ , then choose  $\eta \leq \min\{\frac{k^{i-1}}{L'}, \frac{k^i}{L'}\}$ . If  $k^{i-1} > k^i$ , then define  $b^{i-1} = \tilde{b}^i$  and  $a^i = \tilde{a}^i + \eta$ . Otherwise, take  $b^{i-1} = \tilde{b}^i - \eta$  and  $a^i = \tilde{a}^i$ .

Note that the constraints imposed on  $\eta$  are higher bounds, and they are in finite number. Then,  $\eta$  can be chosen in a whole interval  $(0, \eta']$ , where  $\eta'$  depends on  $\varepsilon$  only. By construction, we have  $\sum_{i=1}^I k^i \chi_{[\tilde{a}^i, \tilde{b}^i]} \leq \sum_{i=1}^I k^i \chi_{[a^i, b^i]}^\eta \leq u^*$ , thus both conditions in (4.5) are satisfied. Note that this construction depends only on  $u^*$  and not on  $\mu^*$ . We denote this new function by  $u^\eta = \sum_{i=1}^I k^i \chi_{[a^i, b^i]}^\eta$ .

Consider now the sequence of measures  $\mu(t^j)$  converging to  $\mu^*$ . If  $t^j$  is a switching time  $t_n$  for the algorithm, replace it with a slightly larger time. Then we can assume that  $\mu(t^j)$  keeps converging to  $\mu^*$ , with no switching times. By continuity of  $\mathcal{L}_{u^*g}V[\mu]$  given by (1.12), we have  $|\mathcal{L}_{u^*g}V[\mu(t^j)]| \geq \frac{C^*}{2}$  for sufficiently large indices. Note that

$$|\mathcal{L}_{(u^* - u^\eta)g}V[\mu(t^j)]| \leq K \|u - u^\eta\|_{L^1(\mu(t^j))} \leq K \|u - u^\eta\|_{L^\infty} \leq K\varepsilon.$$

Then, choose a sufficiently small  $\varepsilon \leq \frac{C^*}{2K}$  and a corresponding  $\Delta x > 0$  such that (4.5) is satisfied. If  $\Delta x \geq \frac{\varepsilon}{2}$ , then replace it with  $\frac{\varepsilon}{2}$ : by integrability of  $u^*$ , (4.5) is still satisfied when refining the grid. Then, the previous construction shows that there exists  $\eta'$  such that  $u^\eta$  satisfies (4.5) for every  $\eta \in (0, \eta']$ . Choose then  $\eta^* \in (0, \eta']$ . For a sufficiently large  $j$ , we have  $|\mathcal{L}_{u^{\eta^*}g}V[\mu(t^j)]| \geq \frac{C^*}{2}$ .

Note now that  $u^{\eta^*}$  is a sum of  $I$  terms of the form  $k^i \chi_{[a^i, b^i]}^{\eta^*}$ . Then, for each  $j$  there exists at least one term such that

$$\left| \mathcal{L}_{\chi_{[a^i, b^i]}^{\eta^*}g}V[\mu(t^j)] \right| \geq \left| \mathcal{L}_{k^i \chi_{[a^i, b^i]}^{\eta^*}g}V[\mu(t^j)] \right| \geq \frac{C^*}{2I},$$

where we have used that  $k^i \leq 1$  and that the number  $I$  does not depend on the parameter  $\eta^*$ . Observe now that, for a sufficiently large  $T$ , we have  $(a^i, b^i, \eta^*) \in \Omega'_t$  for every  $t \geq T$ . Similarly, taking a larger  $T$  if necessary, we have

$s_{t^j}(a^i, b^i, \eta^*) \geq \frac{C^*}{2I} \geq t_j^{-1}$ . As a consequence, the control algorithm provides a maximizer  $(a^j, b^j, \eta^j) \in \Omega'_{t^j}$  of  $s_{t^j}$ , for which

$$s_{t^j}(a^j, b^j, \eta^j) \geq (1 - h)s_{t^j}(a^i, b^i, \eta^*) \geq (1 - h)\frac{C^*}{2I}.$$

The corresponding derivative satisfies

$$\dot{V}(t^j) = \mathcal{L}_{f+U(a^j, b^j, \eta^j)g}V[\mu(t^j)] \leq 0 - (1 - h)\frac{C^*}{2I}.$$

In particular, this is in contradiction with the fact that  $\lim_{t \rightarrow \infty} \dot{V}(t) = 0$ . The theorem is proved.  $\square$

### 5. Generalization to Several Controls

The control algorithm and the convergence result of Theorem 1.1 can be easily adapted to the case in which there are several controlled vector fields and several control gains. Consider  $m + 1$  vector fields  $f, g_1, \dots, g_m$ , for some integer  $m \geq 1$ , and

$$\partial_t \mu + \nabla \cdot \left( \left( f[\mu] + \sum_{i=1}^m \chi_{\omega_i} u_i g_i[\mu] \right) \mu \right) = 0. \tag{5.1}$$

We say that the vector fields  $f, g_1, \dots, g_m$  and the Lyapunov function  $V$  satisfy Assumption **(H)** if every triple of  $f, g_i$  with Lyapunov function  $V$  satisfy Assumptions **(H)**, for every  $i = 1, \dots, m$ . In this setting admissible controls are  $\chi_{\omega_i} u_i$  satisfying the control constraint **(U)** with the following additional constraint.

**Componentwise Sparsity Constraint (C)**

For every  $t \in [0, +\infty)$ , there exists at most one index  $i$  such that  $u_i(t, \cdot)$  is not identically zero.

This sparsity constraint was first considered in the finite-dimensional setting for multi-agent models in Refs. 11 and 12. We recently generalized the Jurdjevic–Quinn stabilization method with this additional constraint in Ref. 13. In this framework we will prove convergence to

$$\mathcal{Z} = \{ \mu \in \mathcal{P}_c(\mathbb{R}^d) \mid \mathcal{L}_f V[\mu] = \mathcal{L}_{u g_i} V[\mu] = 0 \ \forall i = 1, \dots, m, \text{ and } \forall u \in \mathcal{U} \}$$

and, more precisely, Theorem 1.1 reads as follows.

**Corollary 5.1.** *Let  $f, g_1 : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \text{Lip}(\mathbb{R}^d, \mathbb{R}^d)$  and  $V : \mathcal{P}_c(\mathbb{R}^d) \rightarrow \mathbb{R}$  satisfy Assumptions **(H)**. Consider the controlled transport equation with non-local velocity (5.1) with initial data  $\mu_0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$  such that  $\text{supp}(\mu_0) \subset \overline{B(0, R)}$ . Fix the hysteresis parameter  $h \in (0, 1)$ . Fix the initial parameters  $n = 0$  and  $t_0 = 0$ . Define the following algorithm step:*

**Step n:** At time  $t_n$ , choose the maximizer  $(a^*, b^*, \eta^*, i^*)$  of  $|\mathcal{L}_{U(a,b,\eta)g_i} V[\mu(t_n)]|$  in the space  $\Omega_{t_n} \times \{1, \dots, m\}$ .

Then, we have two cases:

- If  $|\mathcal{L}_{U(a^*,b^*,\eta^*)g_{i^*}} V[\mu(t_n)]| < t_n^{-1}$  or if  $\Omega_{t_n}$  is empty, then choose the zero control

$$\chi_\omega u(t, x) \equiv 0$$

and let the measure  $\mu(t)$ , starting at  $\mu(t_n)$ , evolve according to (5.1) over the time interval  $[t_n, t_{n+1}]$ , where  $t_{n+1}$  is the smallest time greater than  $t_n$  such that there exists  $(\bar{a}, \bar{b}, \bar{\eta}, \bar{i}) \in \Omega'_{t_{n+1}} \times \{1, \dots, m\}$  for which  $|\mathcal{L}_{U(\bar{a},\bar{b},\bar{\eta})g_{\bar{i}}} V[\mu(t_{n+1})]| \geq 2t^{-1}$ .

- If  $|\mathcal{L}_{U(a^*,b^*,\eta^*)g_{i^*}} V[\mu(t_n)]| \geq t_n^{-1}$ , then choose the control

$$\chi_\omega u(t, \cdot) = -U(a^*, b^*, \eta^*) \operatorname{sign}(\mathcal{L}_{U(a^*,b^*,\eta^*)g_{i^*}}[\mu(t)] V[\mu(t)])$$

and let the measure  $\mu(t)$ , starting at  $\mu(t_n)$ , evolve according to (5.1) over the time interval  $[t_n, t_{n+1}]$ , where  $t_{n+1}$  is the smallest time greater than  $t_n$  satisfying at least one of the following conditions:

- either  $|\mathcal{L}_{U(a^*,b^*,\eta^*)g_{i^*}}[\mu(t_{n+1})]| \leq \frac{t_{n+1}^{-1}}{2}$ ;
- or there exists  $(\bar{a}, \bar{b}, \bar{\eta}, \bar{i}) \in \Omega'_{t_{n+1}} \times \{1, \dots, m\}$  such that

$$|\mathcal{L}_{U(a^*,b^*,\eta^*)g_{i^*}}[\mu(t_{n+1})]| \leq (1 - h) |\mathcal{L}_{U(\bar{a},\bar{b},\bar{\eta})g_{\bar{i}}}[\mu(t_{n+1})]|.$$

For this control strategy, the control satisfies the control constraints **(U)** componentwise and **(C)**, the unique solution  $\mu(t)$  of (5.1) is such that  $\mu(t) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$  for any  $t \in [0, +\infty)$ , and  $\mu(t)$  converges to  $\mathcal{Z} \cap \mathcal{P}_c(\overline{B(0, R)})$ , i.e.:

- $\lim_{t \rightarrow +\infty} \inf_{\nu \in \mathcal{Z} \cap \mathcal{P}_c(\overline{B(0, R)})} W_p(\mu(t), \nu) = 0$ ,
- or equivalently, there exists a choice  $\nu(t) \in \mathcal{Z} \cap \mathcal{P}_c(\overline{B(0, R)})$  for each  $t \geq 0$  such that  $\mu(t) \rightharpoonup \nu(t)$ , i.e. for all  $\phi \in C_c^\infty(\mathbb{R}^d)$  it holds  $\lim_{t \rightarrow \infty} \int \phi d(\mu(t) - \nu(t)) = 0$ .

The proof of convergence of  $\mu(t)$  to  $\mathcal{Z}$  is a simple combination of the proof of Theorem 1.1 in Sec. 4 with the proof of the finite-dimensional sparse Jurdjevic–Quinn stabilization method with hysteresis given in Ref. 13. Note that the control strategy is the same as in Theorem 1.1 and we only added the index  $i$  in the maximization process.

### 6. Application to Kinetic Multi-Agent Models

In this section, we give some relevant models to which Theorem 1.1 can be applied. Control problems for equations of the form (1.17) arise naturally when studying large crowds of interacting agents. Consider a system of  $N$  interacting agents in

which the dynamics of the state  $x_i \in \mathbb{R}^d$  of the  $i$ th agent are influenced by the state of the other  $N - 1$  agents, according to the time evolution

$$\dot{x}_i = \frac{1}{N} \sum_{j \neq i} F(x_i, x_j), \quad i = 1, \dots, N, \tag{6.1}$$

where  $F \in \text{Lip}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$  represents interaction rules, that are the same for any pair of agents. When the number  $N$  of agents is large, it is often convenient to describe the evolution of the system as a mean-field equation. In the mean-field limit, when  $N \rightarrow +\infty$ , the evolution of the mass of the agents  $\mu \in \mathcal{P}_c(\mathbb{R}^d)$  is described by (1.1) with

$$f[\mu](x) = \int F(x, y) d\mu(y). \tag{6.2}$$

Indeed, to derive the mean-field model (1.1) from the finite-dimensional multi-agent models (6.1), it suffices to consider the empirical measure  $\mu(t) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)}$ .

We consider then the controlled version of the multi-agent system (6.1), given by

$$\dot{x}_i = \frac{1}{N} \sum_{j \neq i} F(x_i, x_j) + u_i g_i(x_1, \dots, x_N), \quad i = 1, \dots, N, \tag{6.3}$$

for some Lipschitz vector field  $(g_1, \dots, g_N)$  on  $(\mathbb{R}^d)^N$  and controls  $(u_1, \dots, u_N)$  in some subset of  $(\mathbb{R}^d)^N$ . In the case in which the control vector field is defined, for every agent  $i$ , only via the interaction between the other agents and the action of the control is the same on any agent, namely if there exists an interaction kernel  $G \in \text{Lip}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$  such that

$$g_i(x_1, \dots, x_N) = \frac{1}{N} \sum_{j \neq i} G(x_i, x_j),$$

for every  $i = 1, \dots, N$ , and if  $u_i = u_j$  for all  $i, j \in \{1, \dots, N\}$ , then we can consider the limit of (6.3) as  $N \rightarrow +\infty$ , which gives the mean-field equation (1.2) with (6.2) and

$$g[\mu](x) = \int G(x, y) d\mu(y). \tag{6.4}$$

The controllability problem is then the following: given an initial measure  $\mu_0$  and a final measure  $\mu_1$ , find a suitable control function  $(t, x) \mapsto u(t, x)$  steering the system (1.2) from  $\mu_0$  to  $\mu_1$ . We refer to Ref. 31 for a first result on the control of a mean-field equation of the form (1.2) with constraint **(U)** and **(U')**. In particular the paper focuses on the controlled version of the kinetic Cucker–Smale system introduced in Ref. 22 with constant  $g$ , and the existence of a control steering the system to a neighborhood of a Dirac measure is proved.

Existence and uniqueness for the mean-field equation (1.2) when the vector fields are given by (6.2) and (6.4) are ensured by Theorem 2.1 provided that  $F(x, y)$  and  $G(x, y)$  have compact support. Indeed if  $F(x, y)$  and  $G(x, y)$  have compact support, then the vector fields  $f[\mu]$  and  $g[\mu]$  satisfy Assumptions **(H)**.

Multi-agent models with a compactly supported interaction potential are sometimes called “bounded confidence” or homophilous models. The idea is that the agents interact only with the ones having closer states. This kind of interaction is used, for instance, to model opinion formation in first-order systems. One of the most influential models in opinion formation is, indeed, the Bounded Confidence Model by Hegselmann and Krause<sup>23</sup> (see also Ref. 7). The main feature of this model is that the interaction is zero when the distance between two opinions is larger than a certain threshold:

$$F(x_i, x_j) = \begin{cases} (x_j - x_i) & \text{if } |x_i - x_j| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It has been proved<sup>7</sup> that, for almost every initial configuration, the opinions converge asymptotically to clusters. In particular, the system does not reach global consensus in general. Since the right-hand side is discontinuous with respect to the state variable, for some configurations, the system has no unique solution, hence we consider the more general first-order consensus model

$$\dot{x}_i = \frac{1}{N} \sum_{j \neq i} \phi(x_j - x_i)(x_j - x_i), \quad i = 1, \dots, N, \tag{6.5}$$

where the function  $\phi$  is defined by

$$\phi(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ -\frac{|x|}{\varepsilon} + 1 + \frac{1}{\varepsilon} & \text{if } |x| \in [1, 1 + \varepsilon], \\ 0 & \text{if } |x| > 1 + \varepsilon, \end{cases} \tag{6.6}$$

for some small  $\varepsilon > 0$ . This is a variant of the Hegselmann–Krause model, in which the Lipschitz property of  $\phi$  ensures existence and uniqueness of solutions of (6.5). Therefore the associated vector field for the mean-field equation (1.2) is

$$f[\mu](x) = \int \phi(y - x)(y - x)d\mu(y). \tag{6.7}$$

The kinetic version of the Hegselmann–Krause model has been first studied in Ref. 10 for discrete-time dynamics. Existence of solutions has been first proved in Ref. 6 for a general bounded decreasing  $\phi(x)$  such that  $|x\phi'(x)| \leq \phi(x)$ . Moreover, if  $\phi(x)$  is everywhere nonzero, then the system converges unconditionally to consensus, meaning that for every  $\mu_0$  the solution  $\mu(t)$  converges asymptotically to a Dirac mass. If  $\phi(x)$  is compactly supported, as in our case, however, then the large time behavior of the dynamics is not yet completely understood and a precise description of the asymptotic dynamics is, in general, a hard task. As in the finite-dimensional analogue, generically the solution  $\mu(t)$  converges to a finite sum of Dirac deltas, representing the clusters of opinion, but sufficient conditions for global consensus are still unknown. Theorem 1.1 provides then a useful tool to establish convergence to global consensus.

Here, we consider the controlled kinetic Hegselmann–Krause model in dimension  $d = 1$  with drift vector field given by (6.7), control vector field  $g = 1$ . We aim to enforce consensus of the whole group toward a unique value; thus, we choose the variance as Lyapunov function

$$V[\mu] = \int x^2 d\mu(x) - \left( \int x d\mu(x) \right)^2.$$

It is easy to prove that  $\mathcal{L}_{ug}V[\mu] = 0$  for every  $u \in \mathcal{U} \Leftrightarrow \mu = \delta_x$  for some  $x \in \mathbb{R}$ . Since  $\mathcal{L}_f[\delta_x] = 0$ , it follows that  $\mathcal{Z} = \{\delta_x \mid x \in \mathbb{R}\}$ .

We are now left to prove that  $L_fV[\mu] \leq 0$ . For the sake of readability, we prove that, if  $\mu = \frac{1}{2}(\delta_x + \delta_y)$  for some  $x, y \in \mathbb{R}$ , then  $\mathcal{L}_fV[\mu] \leq 0$ . Then, by standard properties of the variance, the same statement keeps being valid for the Lie derivative of  $V$  along any combination of Dirac  $\frac{1}{N} \sum_{i=1}^N \delta_{x_i}$  for some  $x_i \in \mathbb{R}$ ; finally, by passing to the limit in  $N$ , this holds for any measure  $\mu$ . Observe that it holds

$$f \left[ \frac{\delta_x + \delta_y}{2} \right] (z) = \frac{1}{2}\phi(x - z)(x - z) + \frac{1}{2}\phi(y - z)(y - z),$$

thus

$$\begin{aligned} \mathcal{L}_fV[\mu] &= \frac{1}{2} \frac{d}{dt} \Big|_{t=0} V(e^{tf}[\delta_x]) + \frac{1}{2} \frac{d}{dt} \Big|_{t=0} V(e^{tf}[\delta_y]) \\ &= xf \left[ \frac{\delta_x + \delta_y}{2} \right] (x) + yf \left[ \frac{\delta_x + \delta_y}{2} \right] (y) \\ &= \frac{1}{2}\phi(y - x)(y - x)x + \frac{1}{2}\phi(x - y)(x - y)y \\ &= -\frac{\phi(x - y)}{2} \|x - y\|^2 \leq 0. \end{aligned}$$

Then, using the continuity conditions (1.12) and the density of the sum of Dirac deltas in  $\mathcal{P}_c(\mathbb{R})$ , one can extend the estimate on  $\mathcal{P}_c(\mathbb{R})$ .

In particular this system fits into the framework of Theorem 1.1, which thus provides the existence of a control strategy concentrating the mass at 0, in other words, steering the system to global consensus.

Theorem 1.1 also gives an explicit construction of a control achieving consensus. Assume that at a certain switching time, say  $t$ , the solution is  $\mu(t) = \frac{1}{2}\chi_{[-1,1]}$ . In this case, we can write explicitly the slope function

$$\begin{aligned} s_t(a, b, \eta) &= |\mathcal{L}_{U(a,b,\eta)g[\mu]}V[\mu]| \\ &= \left| \frac{d}{dt} \Big|_{t=0} V(e^{tU(a,b,\eta)}\mu) \right| \\ &= 2 \left| \int_{\mathbb{R}} xU(a, b, \eta)(x) d\mu(x) \right| \end{aligned}$$

$$= 2 \left| \frac{1}{\eta} \int_{a-\eta}^a x(x-a+\eta) d\mu(x) + \int_a^b x d\mu(x) + \frac{1}{\eta} \int_b^{b+\eta} x(-x+b+\eta) d\mu(x) \right|,$$

where  $b - a + 2\eta \leq c$ . For  $\eta$  large, the biggest contribution is given by the second integral term

$$\int_a^b x d\mu(x) = \frac{1}{4}(\min(b, 1)^2 - \max(a, -1)^2).$$

If  $c < 2$ , then the control set  $\omega$  cannot cover the whole support of  $\mu$  and it will be close to  $-1$  or  $1$ . The action of the control steers the mass in the region  $\omega$  toward  $0$  breaking the symmetry of the measure  $\mu$ . It may happen therefore that the region  $\omega$  will lose the optimality of the slope function and the control will switch to another region on the opposite side. In general, if the measure  $\mu$  is symmetric with respect to the origin, then the control may chatter. This is the rationale for the introduction of an hysteresis parameter  $h$ : the control acts on a set and holds it also sometimes after losing optimality in order to prevent high-oscillating controls.

Here we present numerical simulations for this system. We consider an initial data  $\mu_0$  randomly distributed on the interval  $[0, 10]$  and we apply the control given in Theorem 1.1 with three different choice of the hysteresis parameter  $h$ . In Fig. 2, we represent the free evolution of the system, i.e. with  $u = 0$ . Each blue line describes the trajectory of a percentile  $x$  between  $x_0$  and  $x_{100}$ , i.e. of the value  $x_n$  such that

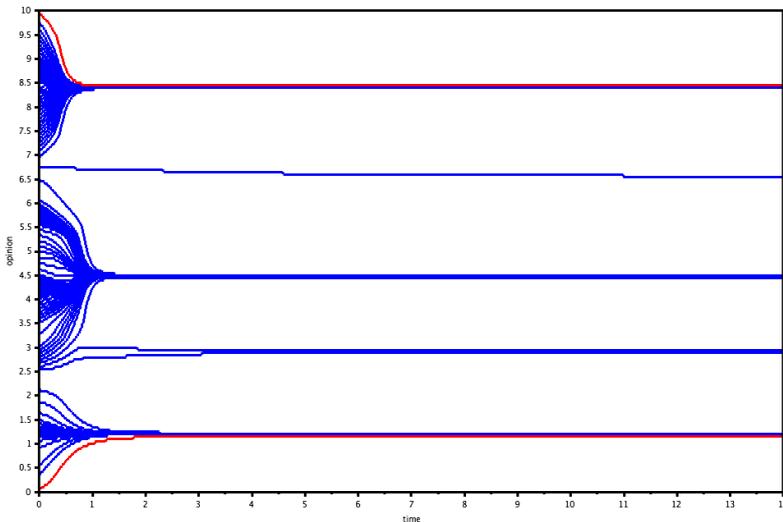


Fig. 2. (Color online) Uncontrolled kinetic Hegselmann–Krause model. Blue lines represent the evolution  $\mu(t)$ . The evolution of max and min of the support of  $\mu(t)$  is represented with red lines.

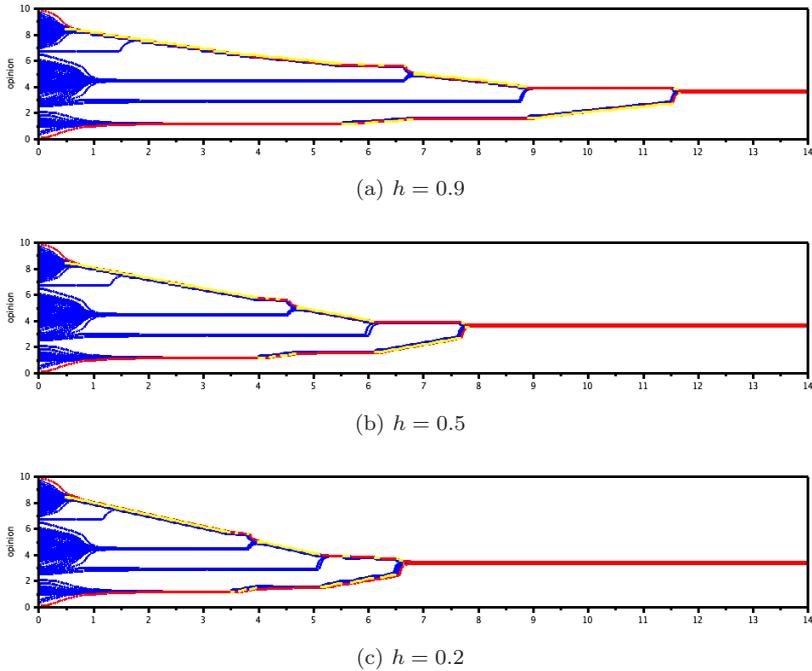


Fig. 3. (Color online) Application of the control strategy of Theorem 1.1 for different values of the hysteresis parameter  $h$ . The yellow region is the controlled area.

$\mu((-\infty, x_n)) = \frac{n}{100}$  and  $\mu((x_n, +\infty)) = 1 - \frac{n}{100}$ . Notice that the solution tends to a finite combination of Dirac deltas representing clusters. Red lines represent the trajectory of the minimal and maximal value.

The action of Theorem 1.1, with the variance as Lyapunov function, is represented in Fig. 3. The control set is represented by a yellow interval. We choose  $c = 0.2$  as the size of the control set. In this case the whole mass tends to a single Dirac delta, representing consensus. One can observe that the control set is piecewise constant with respect to time, as a consequence of the hysteresis.

Chattering may usually happen when dealing with *sparse* controls designed with optimality criteria. Sparsity of the control in finite dimension is usually coded in terms of control acting on the smallest number of components/agents and the term sparsity comes from the fact that the control operator  $g_i$  in (6.3) is a sparse vector. This notion has been introduced in Refs. 11 and 12 for second-order alignment systems (see also Ref. 35 for the controllability via leader of the Hegselmann–Krause finite-dimensional model).

In the infinite-dimensional framework, the assumption that agents are indistinguishable is crucial for defining mean-field limits; therefore, the notion of componentwise sparsity loses its sense. The infinite-dimensional analogue of componentwise sparsity is the sparsity population constraint ( $U'$ ). In Sec. 3, we have described the issues arising from this definition.

The sparsity space constraint given in **(U)** is, on the other hand, the natural definition of sparsity for mean-field equation of the form (1.2). The constraint is in some sense geographical, since the control can act only on a region of the space. The finite-dimensional analogue of such a control is the so-called *decentralized control*. A decentralized control acts based on partial information on the agents inside a certain neighborhood of the controlled ones. The decentralized control for multi-agent systems is a well-established topic, we refer for instance to Refs. 17, 33 and 36 for decentralized consensus algorithms, see also Ref. 8 for a recent result with  $L^\infty$ -constraints on the control of the form (1.4).

## 7. Conclusion and Open Questions

In this paper, we have generalized the classical Jurdjevic–Quinn stabilization method to infinite-dimensional control systems described by transport partial differential equations with non-local terms. Such equations arise in crowd models that are mean-field limits of particle systems for a finite number of agents: for this reason, it is natural to require some sparsity constraint to the control.

We established a mean-field Jurdjevic–Quinn stabilization method under the sparsity constraint **(U)**: the control acts on a small set of the configuration space, with a bounded strength.

Improving Theorem 1.1 in the original spirit of Jurdjevic and Quinn may be done in several ways: on the one side, by reducing the target goal to the largest subset of  $\mathcal{Z}$  that is invariant under the uncontrolled dynamics  $f[\mu]$ ; on the other side, by reducing the target by imposing zero higher-order derivatives, i.e. when  $\mathcal{Z}$  is the set of all  $\mu \in \mathcal{P}_c(\overline{B(0, R)})$  such that

$$\mathcal{L}_f V[\mu] = \mathcal{L}_f^k \mathcal{L}_{u_{g_i}} V[\mu] = 0,$$

for every  $u \in \text{Lip}(\mathbb{R}^d, \mathbb{R})$ ,  $i = 1, \dots, m$ , and  $k \in \mathbb{N}$ .

Dealing with iterated Lie derivatives, that is, with Lie brackets, in the kinetic setting is an open perspective.

Addressing more general systems than those presented in Sec. 5 is also of great interest. In particular, it would be interesting to develop similar approaches to enforce stabilization of a transport equation to a specific set  $\mathcal{Z}$  of configurations, such as steady-states or periodic trajectories. In this spirit, a remarkable result for describing cell migrations is given in Ref. 18, where steady-states are “rosettes”, that are symmetric configurations of cells leading to emergence of specific macroscopic structures.

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