Sparse feedback stabilization of multi-agent dynamics*

Marco Caponigro[1], Benedetto Piccoli[2], Francesco Rossi[3], and Emmanuel Trélat[4]

***Abstract*— In this paper we focus on practical feedback stabilization strategies for dissipative systems. We design control strategies that are sparse, in the sense that they require a minimal number of active components. The result applies to multi-agent systems and it allows consensus arising via external intervention.**

I. INTRODUCTION

Multi-agent systems have attracted the attention of researchers from many disciplines since the pioneering works [1], [2]. The reason is that examples of networks of agents are ubiquitous: biological networks (genetic regulation, ecosystems); technological networks (internet); economical networks (production, distribution, and financial networks) and social networks (scientific collaboration networks, Facebook). The problem of reaching a consensus in a group of autonomous agents has been the object of study in a number of situations ranging from linguistics to distributed computing and from physics to animal behavior. Common examples of singular behavior in a network is the emergence of a common belief in a price system when activity takes place in a given market, the distribution of wealth in modern society, or the emergence of common languages in primitive societies.

A common feature to many Multi-agent systems with their self-organized emergent behaviors is that they represent a natural example of dissipative systems. The intrinsic relation between dissipative systems and self-organization has been pointed out in thermodynamics by the seminal works [3], [4] introducing the scientific community to the analysis of self-organized dynamics and more in general to multi-agent systems.

In this paper, see Section III below, we focus on two kinds of mathematical models for multiagent systems with a dissipative nature: first-order consensus dynamics and second-order alignment models. In first-order models N agents each with a vector of positions $x_i \in \mathbb{R}^d$ interact with each other according to

$$\dot{x}_i = \sum_{j\neq i} a_{ij}(x_j - x_i) \qquad \text{for } i = 1,\dots,N, \tag{1}$$

for some coefficients $a_{ij} \geq 0$. First-order consensus dynamics are sometimes called opinion formation models since they have been used to model the evolution of the opinions x_i. For instance, one of the most influential models in opinion formation is the Bounded Confidence Model by Hegselmann and Krause [5] (see also [6]). The main feature of this model is that the interaction is zero when the distance between two opinions is larger than a certain threshold,

$$a_{ij} = \begin{cases} 1 & \text{if } |x_i - x_j| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In second-order models the state of each agent of the N interacting agents is described by a pair (x_i, v_i) of vectors of the Euclidean space $\mathbb{R}^d$, where x_i represents the *main state* of agent i and the v_i its *consensus parameter*. The time evolution of the state (x_i, v_i) of the i^{th} agent is given by

$$\begin{cases} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \displaystyle\sum_{j=1}^{N} a_{ij}(x(t))(v_j(t) - v_i(t)), \end{cases} \tag{2}$$

for every $i = 1,\dots,N$. One of the most widely studied alignment model is the Cucker-Smale model [7] who attracted a great attention (see for instance [8], [9]) and boasts several extensions, for instance [10], [11], [12], [13].

Despite the existence of a global Lyapunov function both systems present solutions not converging to the global consensus. In (1) opinions converge asymptotically to clusters hence, in general, consensus is not achieved (see for instance [6]). For system (2) there exist initial conditions for which the system does not tend to the alignment, that is $v_1 = \cdots = v_N$ (see[7, Remark 4]).

When consensus is not achieved by self-organization, it is then natural to wonder whether it is possible to steer the group to consensus by means of an external action. In our analysis we consider the problem of the *organization via intervention*. We set the problem in the more general framework of control-affine systems on $\mathbb{R}^n$ when the free evolution of the system is dissipative, i.e. when there exists a nonstrict Lyapunov function, as in the two above-mentioned cases. Since consensus represents in some sense a steady configuration for the system, enforcing self-organization can be seen as an asymptotic stabilization problem, which is classical in control theory and usually relies on Lyapunov design (see [14]). For instance, the well-known Jurdjevic–Quinn

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[1]Marco Caponigro is with Conservatoire National des Arts et Métiers, Équipe M2N, 292 rue Saint-Martin, 75003, Paris, France marco.caponigro@cnam.fr

[2]Benedetto Piccoli is with the Department of Mathematical Sciences and Center for Computational and Integrative Biology, Rutgers University, Camden, NJ 08102, USA piccoli@camden.rutgers.edu

[3]Francesco Rossi is with Aix Marseille Université, CNRS, ENSAM, Université de Toulon, LSIS, Marseille, France. francesco.rossi@lsis.org

[4]Emmanuel Trélat is with Sorbonne Universités, UPMC Univ. Paris 6, CNRS UMR 7598, Laboratoire Jacques-Louis Lions and Institut Universitaire de France, 4 place Jussieu, 75005, Paris, France emmanuel.trelat@upmc.fr

Theorem [15] gives an explicit expression for a continuous global stabilizer. However the feedback stabilizers provided by these methods may have, in general, several nonzero components and in terms of multiagent dynamics acting on every agent may be unfeasible in practice, particularly when dealing with large groups of agents. In this paper we focus on feedback stabilization strategies for control-affine systems requiring a minimal amount of active components for the control. This is the problem of the sparse feedback stabilization. The sparse stabilization and controllability for alignment systems has been introduced in [16] and [17]. With similar techniques in [18] they prove a non-global sparse stabilization for a system submitted to repulsion and attraction forces [19]. Beside the sparse controllability we mention also the controllability via leadership which deals with single input control-affine systems (or when $m \ll n$), see [20], [21], [22], [23].

In this paper, in Section II we prove a general result of sparse stabilization for control-affine systems admitting a control Lyapunov function and in which the origin is locally asymptotically stable. The result applies to the sparse stabilization and the stabilization via leadership of first-order systems, in Section III-A, and second-order alignment models, in Section III-B.

II. MAIN RESULTS

Let n and m be two positive integers. Let $\mathbb{U} \subset \mathbb{R}^m$ be a compact set of nonempty interior such that 0 belongs to the interior of $\mathbb{U}$. Consider the control-affine system on $\mathbb{R}^n$

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t)), \tag{3}$$

with $u(t) \in \mathbb{U} \subset \mathbb{R}^m$, where f and g_i, $i = 1, \ldots, m$ are smooth vector fields on $\mathbb{R}^n$. We assume the existence of a *proper Lyapunov function* for the uncontrolled dynamics $\dot{x} = f(x)$. Namely, a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that :

- $(V1)$ V is radially unbounded (or proper), i.e., $V^{-1}((-\infty, L])$ is compact for every $L \in \mathbb{R}$
- $(V2)$ for every $x \in \mathbb{R}^n$,

$$L_f V(x) \leq 0.$$

Here $L_f V(x) = \langle \nabla V(x), f(x) \rangle$ denotes the Lie derivative of V along f. A system admitting a proper Lyapunov function is sometimes called *dissipative*.

Our aim is to find feedback controls stabilizing the system to the origin which are *sparse* in the sense that at most one nonzero component, that is for every $x \in \mathbb{R}^n$ there exists at most one $k \in \{1, \ldots, m\}$ such that $u_k(x) \neq 0$. Sparse controls are not continuous in general since no assumptions are made on the regularity of index k of the active component with respect to the state x. Discontinuous sparse stabilizers arise naturally, see for instance [16], [17], [18]. The definition of solution of an ODE with discontinuous right-hand side is a classical matter in control, see for instance [24]. In this paper we deal with the notion of sampling solution as introduced in [25].

Definition 1 (Sampling solution): Let $U \subset \mathbb{R}^m$, $F : \mathbb{R}^n \times U \to \mathbb{R}^n$ be continuous and locally Lipschitz in x uniformly on compact subset of $\mathbb{R}^n \times U$. Given a feedback $u : \mathbb{R}^n \to U$, $\tau > 0$, and $x_0 \in \mathbb{R}^n$ we define the *sampling solution* of the differential system

$$\dot{x} = F(x, u(x)), \quad x(0) = x_0,$$

as the continuous, piecewise C^1, function $x : [0, T] \to \mathbb{R}^n$ solving recursively for $k \geq 0$

$$\dot{x}(t) = F(x(t), u(x(k\tau))), \quad t \in [k\tau, (k+1)\tau]$$

using as initial value $x(k\tau)$, the endpoint of the solution on the preceding interval, and starting with $x(0) = x_0$. We call τ the *sampling time*.

This definition of solution is of particular interest for applications in which a minimal interval of time between two switchings of the control law is demanded.

Our main result provides an explicit sparse stabilizing control under the assumption that the origin is locally asymptotically stable and it is the only critical point for the Lie derivatives of V. Here σ denotes the *saturation* operator

$$\sigma(u) = \begin{cases} u & \text{if } u \in \mathbb{U}, \\ \sup\{\lambda > 0 \mid \lambda u \in \mathbb{U}\}\, u & \text{otherwise.} \end{cases}$$

Theorem 1: Assume $f(0) = 0$ and that there exists a proper Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$. If

- (i) $\{x \in \mathbb{R}^n \mid L_f V(x) = 0, L_{g_i} V(x) = 0, \text{ for } i = 1, \ldots, m\} = \{0\}$,
- (ii) there exists $r > 0$ such that if $x_0 \in B_r(0)$ then $x(t; x_0) \to 0$ as $t \to +\infty$.

Then there exists a sparse control u and a sampling time τ such that every sampling solution of (3) associated with the control u and the sampling time τ converges to 0.

More precisely, the control defined by

$$u_i(x) = -\sigma\left(L_{g_i} V(x)\right), \quad \text{and} \quad u_j = 0, \text{ for } j \neq i, \tag{4}$$

where i is the smallest integer such that

$$|L_{g_i} V(x)| \geq |L_{g_j} V(x)|, \quad \forall j \neq i, \tag{5}$$

is a sparse feedback control globally asymptotically stabilizing the system to 0.

Proof: Consider $x_0 \in \mathbb{R}^n \setminus B_r(0)$ and let $K = V^{-1}((-\infty, V(x_0)])$, which is compact by assumption. The function V has a minimum in the interior of K which is realized in 0. Without loss of generality we can assume for simplicity that $V(0) = 0$. Let $\varepsilon > 0$ such that $V^{-1}([0, \varepsilon)) \subset B_r(0)$. Let μ, ν be positive constants such that

$$\left(|L_f L_{g_i} V(x)| + \sum_{j=1}^{m} u_j |L_{g_j} L_{g_i} V(x)| \right) \leq \nu,$$

for every $x, y \in K$, $i \in \{1, \ldots, m\}$, $u = (u_1, \ldots, u_m)$ admissible, and

$$\mu = \min_{x \in K \setminus V^{-1}([0,\varepsilon))} \max_{i \in \{1,\ldots,m\}} |L_{g_i} V(x)|. \tag{6}$$

Note that $\mu > 0$ by Assumption (i). Consider $0 < \tau_0 < \mu/(2\nu)$.

Now for any $\tau \le \tau_0$, for any $y \in K$ consider the sampling solution $x(t), t \in [0,\tau]$ associated with y, τ_0 and the control (4), namely the solution of

$$\begin{cases} \dot{x} &= f(x) - L_{g_i}(y) g_i(x), \\ x(0) &= y \end{cases}$$

where the index $i \in \{1,\ldots,m\}$ is given by (5). Again, if $y \in V^{-1}([0,\varepsilon)$ there is nothing to prove. On the other hand if $V(y) \ge \varepsilon$ note that, for every $t \in [0,\tau]$,

$$\begin{aligned} |L_{g_i}V(x(t))| &\ge |L_{g_i}V(y)| - \tau \sup_{t\in[0,\tau]} |\frac{d}{dt} L_{g_i}V(x(t))| \\ &\ge \mu - \tau \sup_{x,u} \left(|L_f L_{g_i} V(x)| + \sum_j u_j |L_{g_j} L_{g_i} V(x)| \right) \\ &\ge \mu - \tau\nu \\ &> \mu/2. \end{aligned} \tag{7}$$

In particular $L_{g_i}V(x(t))$ is uniformly bounded away from 0 for every $t \in [0,\tau]$ and so

$$L_{g_i}V(x(t)) L_{g_i}V(y) > 0.$$

Therefore

$$L_{g_i}V(x(t)) L_{g_i}V(y) = |L_{g_i}V(x(t)) L_{g_i}V(y)| \ge \frac{\mu^2}{2},$$

thanks to (7). This gives a uniform estimate on the decay rate of the Lyapunov function along a sampling period when the initial node y satifies $V(y) \ge \varepsilon$, indeed

$$\frac{d}{dt} V(x(t)) = L_f V(x(t)) - L_{g_i}V(y) L_{g_i}V(x(t)) < -\frac{\mu^2}{2}, \tag{8}$$

for every $t \in [0,\tau]$. In conclusion if we consider the sampling solution $x(t)$of (3) associated with x_0, $\tau < \tau_0$ and the control defined by (4)-(5) we have that while $x(t) \notin V^{-1}([0,\varepsilon))$ the Lyapunov function is strictly decreasing at a rate at most $-mu^2/2$ and after a time

$$T \le (V(x_0) - \varepsilon/2) 2/\mu^2$$

reaches the sublevel basin of attraction of the origin $B_r(0)$. ∎

Remark 1: A proper positive definite function V verifying (i) is called *Control Lyapunov Function* (CLF). A sufficient condition for the existence of a CLF is the so-called *W*eak Jurdjevic–Quinn Condition: there exists $l \ge 0$ such that $\{x \in \mathbb{R}^n \mid L_f V(x) = 0 \text{ and } L_f^k L_{g_i} V(x) = 0, \text{ for } i = 1,\ldots,m,\ k \le l\} = \{0\}$,see for instance (see for instance [26, Proposition 4.1 and Theorem 4.1]). Theorem 1 says that if 0 is a locally asymptotically stable equilibrium and the systems verifies the Weak Jurdjevic–Quinn condition then 0 can be globally asymptotically stabilized by means of a sparse feedback.

Remark 2: Assumption $(V1)$ that V is proper can be dropped provided that there is a strictly positive lower bound for (6). For instance if

$$\inf_{x \notin B_\varepsilon(0)} \max_{i\in\{1,\ldots,m\}} |L_{g_i}V(x)| > 0.$$

This is the case for exemple of alignment models, see Section III-B below, in which the energy of the system is not a proper function.

Remark 3: If there is a uniform lower bound, say $\mu > 0$, on $\max_i |L_{g_i}V(x)|$ and if the sampling time τ is sufficiently small, then by (8),

$$V(x(t)) \le V(x_0) - \frac{\mu\delta}{2} t.$$

In other words the Lyapunov function decays at least linearly whenever the trajectory is sufficiently far from the manifold $\{L_{g_i}V(x) = 0, \text{ for } i = 1,\ldots,m\}$. This estimate is however quite conservative and, for some particular case, may be improved. For instance in [17] it has been proved that for the controlled Cucker–Smale system, see (12) below, the decay rate is

$$V(x(t)) \le \left(\sqrt{V(x_0)} - \frac{\delta}{2N} t \right)^2.$$

for sufficiently small sampling times and whenever the solution stays out the basin of attraction of the consensus manifold $\{v_1 = \cdots = v_N\}$.

III. APPLICATION TO CONSENSUS EMERGENCE

Theorem 1 provides a powerful tool in the framework of multiagent systems. Indeed it is possible to change the asymptotic behavior of the system of several interacting agents by acting only on at most one agent at each instant of time. The only controlled agent wears the role of instantaneous leader of the group. By choosing adequately the leader we can enforce consensus. The leader is choosen as the one maximizing the decay rate of the Lyapunov function. Here we present applications of Theorem 1 to general first order and second order consensus models.

A. Opinion formation models

1) Sparse stabilization: Consider the controlled first order consensus model

$$\dot{x}_i = \sum_{j\ne i} a_{ij}(x)(x_j - x_i) + u_i \qquad \text{for } i = 1,\ldots,N,$$

for nonnegative smooth real functions $a_{ij} : (\mathbb{R}^d)^N \to [0,+\infty)$ and for controls $u \in \mathbb{U} := \{(u_1,\ldots,u_N) \in (\mathbb{R}^d)^N \mid \sum_{i=1}^N \|u_i\| \le \delta\}$ for some given $\delta > 0$.

Proposition 1: For every $\delta > 0$, let $u = (u_1,\ldots,u_N)$ be the feedback defined by

$$u_i(x) = -\sigma((x_i - x_j)), \quad \text{and} \quad u_k(x) = 0 \text{ for } k \ne i,$$

where $i \in \{1,\ldots,N\}$ is the smallest integer for which there exists $j > i$ such that

$$\|x_i - x_j\| \ge \|x_k - x_l\|$$

for all $1 \le k < l \le N$. There exists a sampling time $\tau > 0$ such that for every initial condition $x_0 \in (\mathbb{R}^d)^N$ the sampling solution associated with x_0, τ, and u verifies

$$\lim_{t\to\infty} x_i(t) = x^*, \qquad \text{for every } i = 1,\ldots,N,$$

for some $x^* \in \mathbb{R}^d$.

Proof: Since the dynamics is invariant by translation, we consider the equivalence relation on $(\mathbb{R}^d)^N$

$$x \sim y \iff x_1 - x_k = y_1 - y_k, \quad \text{for every } k = 2, \dots, N,$$

and we consider as state space the quotient $\mathbb{R}^{dN}/_\sim \simeq \mathbb{R}^{d(N-1)}$. We identify the (so-called) consensus manifold $\{x \in (\mathbb{R}^d)^N \mid x_1 = \cdots = x_N\}$ with the equivalence class of 0. The function defined by

$$V(t) = \frac{1}{2} \max_{i,j} \|x_i(t) - x_j(t)\|^2 \tag{9}$$

is a proper Lyapunov function for the system for any adjacency matrix $(a_{ij}(x))_{ij}$. Indeed, in the uncontrolled case, by [12, Theorem 2.3], we have $L_f V(x) = \frac{d}{dt} V(t) \leq 0$. Moreover,

$$L_{g_i} V(x) = \frac{\partial}{\partial x_i} V(x) = \begin{cases} x_i - x_j & \text{if } \|x_i - x_j\| \text{ is maximal,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular $L_{g_i} V(x) = 0$ for every $i = 1, \dots, N$ if and only if $x_1 = x_2 = \cdots = x_N$.

The consensus region, that is the basin of attraction of the equivalence class of 0, is given by [12, Theorem 2.3 and (2.8)]. The result then follows from Theorem 1. ∎

2) Stabilization via leadership: Theorem 1 guarantees the existence of feedback stabilizers also in the case of underactuated systems, in which only a subset of agents can be influenced by the control. Here we discuss the particular case of systems with leadership. Consider an opinion formation model in which only the dynamics of the agent 0, the leader, can be controlled. The system of N interacting agents with one leader with opinions in $\mathbb{R}^d$ is given by

$$\begin{cases} \dot{x}_0(t) = & u(t), \\ \dot{x}_i(t) = & \displaystyle\sum_{j=0}^{N} a_{ij}(x)(x_j(t) - x_i(t)), \qquad i = 1,\dots,N, \end{cases} \tag{10}$$

with given initial positions $x_j(0) \in \mathbb{R}^d$, for $j = 0, 1, \dots, N$. We assume that there exists a neighborhood $\mathcal{N}_0$ of x_0 such that

$$a_{i0}(x) + a_{0i}(x) \neq 0, \quad \text{for every } x_i \in \mathcal{N}_0,\ i = 1, \dots, N. \tag{11}$$

The meaning of Assumption (11) is that the leader x_0 can interact, at least locally, with any possible agent.

This opinion formation model with leader has been introduced and studied in [21] for a particular class of interaction functions a_{ij}. Hereafter, we extend their stability results to the case of general interactions.

Proposition 2: For every $\delta > 0$, consider the feedback defined by

$$u(x) = -\delta \frac{(x_i - x_0)}{\|x_i - x_0\|},$$

where $i \in \{1, \dots, N\}$ is the smallest integer such that $\|x_i - x_0\| \geq \|x_j - x_0\|$ for every $j \in \{1, \dots, N\}$. There exists a sampling time $\tau > 0$ such that for every initial condition $x_0 \in (\mathbb{R}^d)^N$ the sampling solution associated with x_0, τ, and u verifies,

$$\lim_{t \to \infty} x_i(t) = x^*, \quad \text{for every } i = 1, \dots, N,$$

for some $x^* \in \mathbb{R}^d$.

Proof: The function $V : (\mathbb{R}^d)^{N+1} \to \mathbb{R}$ defined by

$$V(x) = \frac{1}{2} \max_i \|x_i - x_0\|^2,$$

is a proper Lyapunov function for the system (10) (see [12, Proposition 2.1]). The explicit formula for the feedback control is given by Theorem 1. Finally the fact that $\{x \in (\mathbb{R}^d)^{N+1} \mid x_0 = x_1 = \cdots = x_N\}$ is locally attractive is a consequence of condition (11) (see [21, Lemma 1]). ∎

B. Alignment models

Consider a system of N interacting agents in which the state of each agent is described by a pair (x_i, v_i) of vectors of $\mathbb{R}^d$, where x_i represents the *main state* of agent i and the v_i its *consensus parameter*. The time evolution of the state (x_i, v_i) of the i^{th} agent is given by

$$\begin{cases} \dot{x}_i(t) & = v_i(t), \\ \dot{v}_i(t) & = \displaystyle\sum_{j=1}^{N} a_{ij}(x(t))(v_j(t) - v_i(t)) + u_i(t), \end{cases} \tag{12}$$

for every $i = 1, \dots, N$. The control $u(t) = (u_1(t), \dots, u_N(t))$ models the external force acting on the acceleration of the agents and satisfy for every t the $\ell^1 - \ell^2$ constraint $\sum_{i=1}^{N} \|u_i\| \leq \delta$ for some given $\delta > 0$. There are several models of this kind depending on the choice of the coefficients a_{ij}. In the famous Cucker–Smale model, introduced in the seminal paper [7], the interaction functions are

$$a_{ij}(x) = \frac{1}{N(1 + \|x_j - x_i\|)^\beta}.$$

In [27] this model has been studied in the more general case in which

$$a_{ij} = a(\|x_j - x_i\|), \tag{13}$$

where $a \in C^1([0, +\infty))$ is a nonincreasing positive function. In [28] the authors proposed a non symmetrical model with

$$a_{ij} = \frac{a(\|x_j - x_i\|)}{\sum_{j \neq i} a(\|x_j - x_i\|)}.$$

For other extensions of this model we refer for instance to [10], [11], [29], [13].

A general result of explicit sparse stabilization for the alignment model (12) with interactions (13) has been firstly obtained in [16] and in [17]. Hereafter, we extend these results to the case of general interaction coefficients $a_{ij}(x(t)) \geq 0$. The proof follows the same lines of the proof of Proposition 1 and it is omitted.

Proposition 3: For every $\delta > 0$, let $u = (u_1, \dots, u_N)$ be the feedback control defined by

$$u_i(v) = -\sigma(v_i - v_j), \quad \text{and} \quad u_k(v) = 0 \text{ for } k \neq i, \tag{14}$$

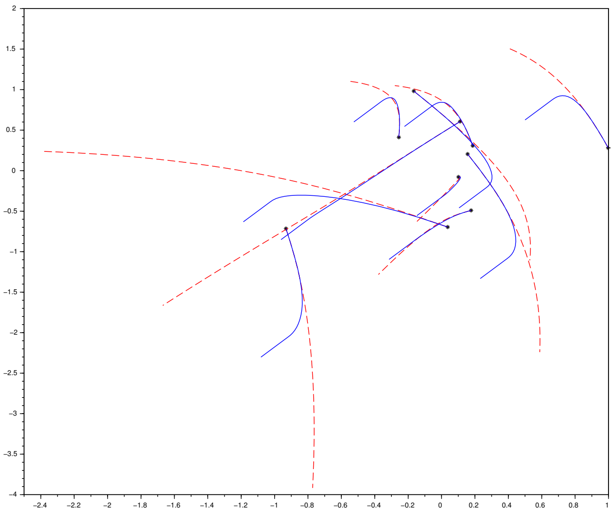

Fig. 1. Trajectories of the agents for $0 \le t \le 6$. The black stars represent the initial positions. In solid blue the controlled trajectories. In dashed red the uncontrolled ones.

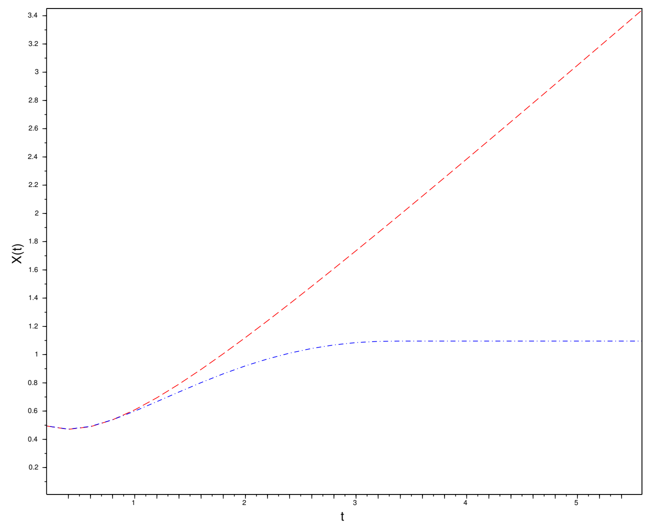

Fig. 2. Comparison of the evolution of the dispersion $X(t)$ in the controlled case, dot-dashed blue line, and in free evolution, dashed red line.

where $i \in \{1,\dots,N\}$ is the smallest integer for which there exists $j > i$ such that

$$\|v_i - v_j\| \ge \|v_k - v_l\|$$

for all $1 \le k < l \le N$. There exists a sampling time $\tau > 0$ such that every sampling solution associated with u and with sampling time τ tends to the consensus manifold

$$(\mathbb{R}^d)^N \times \{v \in (\mathbb{R}^d)^N \mid v_1 = \dots = v_N\}.$$

IV. Numerical results

We present here numerical simulations showing the effectivness of the stabilizing feedback given by Theorem 1. We consider the Cucker–Smale system (12) with interaction coefficients

$$a_{ij}(x) = (1 + \|x_i - x_j\|)^{-1},$$

with $N = 10$ agents in dimension $d = 2$. The initial positions and velocities of each agent are randomly choosen in $[-1,1]^2 \times [-1,1]^2$. We compare the free evolution of the system with the evolution under the action of the stabilizing feedback given by Theorem 1. In Figure 1 the trajectories of the system in free evolution, i.e. with $u = 0$, diverge (dashed red lines). On the other hand, the trajectories under the action of the sparse feedback (14), in solid blue, tend to align. In order to analyze the convergence to alignment we define the *dispersion*

$$X(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\|^2.$$

and the *disagreement*

$$V(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|v_i(t) - v_j(t)\|^2.$$

of the group. The evolution of X is depicted in Figure 2. In the uncontrolled case the dispersion increases meaning that the agents are spreading. On the contrary, under the action of the control the agents stay close and the dispersion tends to a constant.

The disagreement V plays the role of Lyapunov function and gives a measure of the distance from consensus. Indeed $V(t) = 0$ if and only if a solution $(x(t), v(t))$ is in the consensus manifold $(\mathbb{R}^d)^N \times \{v \in (\mathbb{R}^d)^N \mid v_1 = \dots = v_N\}$. Let

$$\gamma(t) = \int_{\sqrt{X_t}}^{\infty} a(\sqrt{2N}r)dr.$$

The threshold γ gives an estimate on the basin of attraction of the consensus manifold since the solution of (12) associated with initial conditions (x_0, v_0) satifying $\sqrt{V(0)} \le \gamma(0)$, tends asymptotically to consensus, as proved in [27].

As the dispersion X increases the threshold γ decreases. Fitting this sufficient condition for consensus is more and more difficult as the group spreads. In Figure 3 in the controlled cases (dot-dashed lines) the disagreement V goes below the threshold γ in time smaller than 1.8. The uncontrolled dynamics however do not enter the consensus region given by the threshold γ until $t \sim 10$ (not represented in the picture).

For further numerical examples, refined time estimates, and an analysis of the optimality of the sparse feedback as well as an analysis on the sparsity of the optimal control in the case of the Cucker–Smale systems, we refer to [16], [17].

V. Conclusions

We provided an explicit construction of a global sparse feedback stabilizers. The main assumptions are the local asymptotic stability of the equilibrium and the existence

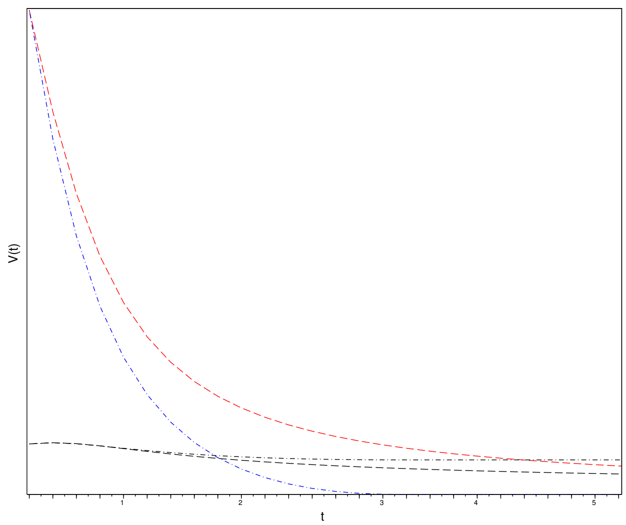

Fig. 3. Comparison of the evolution of the disagreement $V(t)$ in the controlled case, dot-dashed blue line, and in free evolution, dashed red line. The evolution of the threshold $\gamma(t)$ is in black. The dot-dashed line for the controlled case and the dashed line for the free evolution.

of a control Lyapunov function. The result is applied to consensus emergence in first-order dynamics and in second-order alignment systems. The practical sparse stabilizer is discontinuous and possible future developments include the analysis of the existence of continuous sparse stabilizers in order to compare the result with classical stabilization results such as, for instance, the Artstein–Sontag Theorem. Further extensions to system with attraction and repulsion features are also very interesting, the main difficulty being to find a global Lyapunov function since the deviation is not dissipative in general. Local results in this direction have already been studied for some particular system. Finally it is interesting to study the sparse stabilization of dissipative systems without the assumption of local asymptotic stability of the origin or the existence of a control Lyapunov function, in the spirit of the Jurdjevic–Quinn Theorem.

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