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UNIFIED RICCATI THEORY FOR OPTIMAL PERMANENT AND SAMPLED-DATA CONTROL PROBLEMS IN FINITE AND INFINITE TIME HORIZONS*

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Abstract. We revisit and extend the Riccati theory, unifying continuous-time linear-quadratic optimal permanent and sampled-data control problems in finite and infinite time horizons. In a nutshell, we prove that the following diagram commutes:

$$\begin{array}{ccccc} \text{(SD-DRE)} & E^{T,\Delta} & \xrightarrow{T\to+\infty} & E^{\infty,\Delta} & \text{(SD-ARE)} \\ & \big\downarrow{\scriptstyle \|\Delta\|\to 0} & & \big\downarrow{\scriptstyle \|\Delta\|\to 0} & \\ \text{(P-DRE)} & E^{T} & \xrightarrow[T\to+\infty]{} & E^{\infty} & \text{(P-ARE)} \end{array}$$

i.e., that (i) when the time horizon T tends to $+\infty$, one passes from the Sampled-Data Difference Riccati Equation (SD-DRE) to the Sampled-Data Algebraic Riccati Equation (SD-ARE), and from the Permanent Differential Riccati Equation (P-DRE) to the Permanent Algebraic Riccati Equation (P-ARE); (ii) when the maximal step $\|\Delta\|$ of the time partition Δ tends to 0, one passes from (SD-DRE) to (P-DRE), and from (SD-ARE) to (P-ARE). The notation E in the above diagram (with various superscripts) refers to the solution of each of the Riccati equations listed above. Our notation and analysis provide a unified framework in order to settle all corresponding results.

Key words. optimal control, sampled-data control, linear-quadratic problems, Riccati theory, feedback control, convergence

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1. Introduction.

Optimal control theory is concerned with acting on controlled dynamical systems by minimizing a given criterion. We speak of a *Linear-Quadratic* (LQ) optimal control problem when the control system is a linear differential equation and the cost is given by a quadratic integral (see, e.g., [26]). One of the main results in that context is that the optimal control can be expressed as a linear state feedback called *Linear-Quadratic Regulator* (LQR). The linear state feedback is described by using the *Riccati matrix*, which is the solution to a nonlinear backward matrix Cauchy problem in finite time horizon (DRE: Differential Riccati Equation), and to a nonlinear algebraic matrix equation in infinite time horizon (ARE: Algebraic Riccati Equation).

The LQR problem is a fundamental issue in LQ optimal control theory. Since the pioneering works by Maxwell, Lyapunov, and Kalman (see the textbooks [26, 28, 40]), it has been extended to many contexts, including discrete-time [25], stochastic [49], infinite-dimensional [15], and fractional [31]. One of these concerns the case where controls must be piecewise constant. We speak here of *sampled-data controls* (or

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digital controls), in contrast to *permanent controls*. Recall that a control is said to be permanent when it is authorized to be modified at any time. In many problems, achieving the corresponding optimal trajectory requires a permanent modification of the control. However, such a requirement is not conceivable in practice for human beings, even for mechanical or numerical devices. Therefore, sampled-data controls, for which only a finite number of modifications is authorized over any compact time interval, are usually considered for engineering issues. The corresponding set of *sampling times* (at which the control value can be modified) is called a *time partition*. A vast literature deals with sampled-data control systems, as evidenced by numerous references and books (see, e.g., [1, 2, 4, 5, 6, 14, 18, 20, 22, 27, 35, 38, 41, 43, 44] and references therein). One of the first contributions on LQ optimal sampled-data control problems can be found in [23]. This field has significantly grown since the '70s, motivated by the electrical and mechanical engineering issues with applications, for example, to strings of vehicles (see [3, 17, 29, 30, 33, 34, 39]). Sampled-data versions of feedback controls and of Riccati equations have been derived and, like in the fully discrete-time case (see [32, Remark 2]), these two concepts in the sampled-data control case have various equivalent formulations in the literature, due to different developed approaches: in most of the references, LQ optimal sampled-data control problems are recast as fully discrete-time problems, and then the feedback control and the Riccati equation are obtained by applying the discrete-time dynamical programming principle (see [7, 17, 23]) or by applying a discrete-time version of the Pontryagin maximum principle (see [3, 17, 24]).

In the present paper, our objective is to provide a mathematical framework in which LQ optimal control theories, in the permanent case and in the sampled-data case, can be settled in a unified way. We build upon our recent article [11], in which we have developed a novel approach, keeping the initial continuous-time formulation of the LQ optimal sampled-data control problem, based on a sampled-data version of the Pontryagin maximum principle (see [9, 10]). Analogies between LQ optimal permanent and sampled-data control problems have already been noticed in several works (see, e.g., [39] or [47, Remark 5.4]). In this article, we gather in a unified setting the main results of LQ optimal control theory in the following four situations: permanent/sampled-data control, finite/infinite time horizon. To this aim, an important tool is the map $\mathcal{F}$ defined in section 2.1, thanks to which we formulate, in the above-mentioned four situations, feedback controls and Riccati equations in Propositions 2.3, 2.4, 2.6, and 2.7 (stated in sections 2.2 and 2.3). Moreover, exploiting the continuity of $\mathcal{F}$, we establish convergence results between the involved Riccati matrices, either as the length of the time partition goes to zero or as the finite time horizon goes to infinity. Four convergence results are summarized in the diagram presented in the abstract, and we refer the reader to our main result (Theorem 3.2 in section 3) for the complete mathematical statement. Some of the convergence results are already known, while others are new. Hence Theorem 3.2 fills some gaps in the existing literature and, in some sense, it closes the loop, which is the meaning of the commutative diagram that conveys the main message of this article.

Theorem 3.2 is proved in Appendix A. An important role in the proof is played by the *optimizability* property (or *finite cost* property), which is well known in infinite time horizon LQ optimal permanent control problems and is related to various notions of controllability and of stabilizability (see [16, 42, 46]). For sampled-data controls, when rewriting the original problem as a fully discrete-time problem, optimizability is formulated on the corresponding discrete-time problem (see [17, Theorem 3] or [30, p. 348]). Here, we prove in the instrumental Lemma 3.1 that if the permanent optimizability property is satisfied, then the sampled-data optimizability property is satisfied for all time partitions of sufficiently small length (moreover, a bound of the minimal sampled-data cost is given, uniform with respect to the length of the time partition). This lemma plays a key role in proving the convergence of the sampled-data Riccati matrix to the permanent one in infinite time horizon when the length of the time partition goes to zero.

2. Preliminaries on LQ optimal control problems.

Throughout the paper, we denote $\mathbb{R}_+ := [0,+\infty)$ and $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$. Given any $p \in \mathbb{N}^*$, we denote by $\mathcal{S}^p_+$ (resp., $\mathcal{S}^p_{++}$) the set of all symmetric positive semidefinite (resp., positive definite) matrices of $\mathbb{R}^{p\times p}$, and by Id_p the identity matrix of $\mathbb{R}^{p\times p}$.

Let n, $m \in \mathbb{N}^*$, let $P \in \mathcal{S}^n_+$, and, for every $t \in \mathbb{R}$, let $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $Q(t) \in \mathcal{S}^n_+$, and $R(t) \in \mathcal{S}^m_{++}$ be matrices depending continuously on t. Let $\Phi(\cdot,\cdot)$ be the *state-transition matrix* associated to $A(\cdot)$ (see [40, Appendix C.4]); that is, $\Phi(\cdot,\cdot)$ is the unique matrix function which satisfies $\frac{\partial \Phi}{\partial t_1}(t_1,t_2) = A(t_1)\Phi(t_1,t_2)$, with $\Phi(t_2,t_2) = \mathrm{Id}_n$ for all $(t_1,t_2) \in \mathbb{R}\times\mathbb{R}$.

DEFINITION 2.1. *We speak of an autonomous setting when $A(t) \equiv A \in \mathbb{R}^{n\times n}$, $B(t) \equiv B \in \mathbb{R}^{n\times m}$, $Q(t) \equiv Q \in \mathcal{S}^n_+$, and $R(t) \equiv R \in \mathcal{S}^m_{++}$ are constant in t.*

2.1. Notation for a unified setting.

In this paper, we consider four different LQ optimal control problems: permanent control versus sampled-data control, and finite time horizon versus infinite time horizon. To provide a unified presentation of our results (see Propositions 2.3, 2.4, 2.6, and 2.7), we define the map

$$\begin{array}{rccl} \mathcal{F}: & \mathbb{R}\times\mathcal{S}^n_+\times\mathbb{R}_+ & \longrightarrow & \mathbb{R}^{n\times n} \\ & (t,E,h) & \longmapsto & \mathcal{M}(t,E,h)\mathcal{N}(t,E,h)^{-1}\mathcal{M}(t,E,h)^\top - \mathcal{G}(t,E,h), \end{array}$$

where $\mathcal{M}(t,E,h) := \mathcal{M}_1(t,E,h) + \mathcal{M}_2(t,E,h)$, where $\mathcal{N}(t,E,h) := \mathcal{N}_1(t,E,h) + \mathcal{N}_2(t,E,h) + \mathcal{N}_3(t,E,h)$, and where $\mathcal{G}(t,E,h) := \mathcal{G}_1(t,E,h) + \mathcal{G}_2(t,E,h)$, with

	if $h>0$	if $h=0$
$\mathcal{M}_1(t,E,h) :=$	$\Phi(t,t-h)^\top E\left(\frac{1}{h}\int_{t-h}^t \Phi(t,\tau)B(\tau)\,d\tau\right)$	$EB(t)$
$\mathcal{M}_2(t,E,h) :=$	$\frac{1}{h}\int_{t-h}^t \Phi(\tau,t-h)^\top Q(\tau)\left(\int_{t-h}^\tau \Phi(\tau,\xi)B(\xi)\,d\xi\right)\,d\tau$	$0_{\mathbb{R}^{n\times m}}$
$\mathcal{N}_1(t,E,h) :=$	$\frac{1}{h}\int_{t-h}^t R(\tau)\,d\tau$	$R(t)$
$\mathcal{N}_2(t,E,h) :=$	$\frac{1}{h}\int_{t-h}^t \left(\int_{t-h}^\tau B(\xi)^\top\Phi(\tau,\xi)^\top\,d\xi\right)Q(\tau)\left(\int_{t-h}^\tau \Phi(\tau,\xi)B(\xi)\,d\xi\right)\,d\tau$	$0_{\mathbb{R}^{m\times m}}$
$\mathcal{N}_3(t,E,h) :=$	$\frac{1}{h}\left(\int_{t-h}^t B(\tau)^\top\Phi(t,\tau)^\top\,d\tau\right)E\left(\int_{t-h}^t \Phi(t,\tau)B(\tau)\,d\tau\right)$	$0_{\mathbb{R}^{m\times m}}$
$\mathcal{G}_1(t,E,h) :=$	$\frac{1}{h}\int_{t-h}^t \Phi(\tau,t-h)^\top Q(\tau)\Phi(\tau,t-h)\,d\tau$	$Q(t)$
$\mathcal{G}_2(t,E,h) :=$	$\frac{1}{h}\left(\Phi(t,t-h)^\top E\Phi(t,t-h) - E\right)$	$A(t)^\top E + EA(t)$

The map $\mathcal{F}$ is well-defined and continuous (see Lemma A.3 in Appendix A.1). Moreover, for $h = 0$, we have

$$\mathcal{F}(t,E,0) = EB(t)R(t)^{-1}B(t)^\top E - Q(t) - A(t)^\top E - EA(t) \qquad \forall (t,E)\in\mathbb{R}\times\mathcal{S}^n_+.$$

One recognizes here the second member of the Permanent Differential Riccati Equation (see Proposition 2.3 and Remark 2.5). The map $\mathcal{F}$ is designed to provide a unified notation for the permanent and sampled-data control settings.

Remark 2.2. In the *autonomous setting* (see Definition 2.1), the state-transition matrix is $\Phi(t,\tau) = e^{(t-\tau)A}$ for all $(t,\tau) \in \mathbb{R} \times \mathbb{R}$ (see, e.g., [40, Lemma C.4.1]), and thus, in this case, the map $\mathcal{F}$ does not depend on t, and

$$\mathcal{F}(E,h) = \mathcal{M}(E,h)\mathcal{N}(E,h)^{-1}\mathcal{M}(E,h)^\top - \mathcal{G}(E,h) \qquad \forall (E,h) \in \mathcal{S}^n_+ \times \mathbb{R}_+,$$

where $\mathcal{M}(E,h) := \mathcal{M}_1(E,h)+\mathcal{M}_2(E,h)$, $\mathcal{N}(E,h) := \mathcal{N}_1(E,h)+\mathcal{N}_2(E,h)+\mathcal{N}_3(E,h)$, and $\mathcal{G}(E,h) := \mathcal{G}_1(E,h) + \mathcal{G}_2(E,h)$, with

	if $h > 0$	if $h = 0$
$\mathcal{M}_1(E,h) :=$	$e^{hA^\top} E \left(\frac{1}{h} \int_0^h e^{\tau A} \, d\tau \right) B$	EB
$\mathcal{M}_2(E,h) :=$	$\frac{1}{h} \left(\int_0^h e^{\tau A^\top} Q \left(\int_0^\tau e^{\xi A} \, d\xi \right) \, d\tau \right) B$	$0_{\mathbb{R}^{n\times m}}$
$\mathcal{N}_1(E,h) :=$	R	R
$\mathcal{N}_2(E,h) :=$	$B^\top \left(\frac{1}{h} \int_0^h \left(\int_0^\tau e^{\xi A^\top} \, d\xi \right) Q \left(\int_0^\tau e^{\xi A} \, d\xi \right) \, d\tau \right) B$	$0_{\mathbb{R}^{m\times m}}$
$\mathcal{N}_3(E,h) :=$	$B^\top \left(\frac{1}{h} \left(\int_0^h e^{\tau A^\top} \, d\tau \right) E \left(\int_0^h e^{\tau A} \, d\tau \right) \right) B$	$0_{\mathbb{R}^{m\times m}}$
$\mathcal{G}_1(E,h) :=$	$\frac{1}{h} \int_0^h e^{\tau A^\top} Q e^{\tau A} \, d\tau$	Q
$\mathcal{G}_2(E,h) :=$	$\frac{1}{h} \left(e^{hA^\top} E e^{hA} - E \right)$	$A^\top E + EA$

In particular, in the autonomous setting and for $h = 0$, we have

$$\mathcal{F}(E,0) = EBR^{-1}B^\top E - Q - A^\top E - EA \qquad \forall E \in \mathcal{S}^n_+.$$

2.2. Finite time horizon: Permanent/sampled-data control. Given any $T > 0$, we denote by $\mathrm{AC}([0,T],\mathbb{R}^n)$ the space of absolutely continuous functions defined on $[0,T]$ with values in $\mathbb{R}^n$, and by $\mathrm{L}^2([0,T],\mathbb{R}^m)$ the Lebesgue space of square-integrable functions defined almost everywhere on $[0,T]$ with values in $\mathbb{R}^m$. In what follows, $\mathrm{L}^2([0,T],\mathbb{R}^m)$ is the set of *permanent controls*.

A *time partition* of the interval $[0,T]$ is a finite set $\Delta = \{t_i\}_{i=0,\dots,N}$, with $N \in \mathbb{N}^*$, such that $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. We denote by $\mathrm{PC}^\Delta([0,T],\mathbb{R}^m)$ the space of functions defined on $[0,T]$ with values in $\mathbb{R}^m$ that are piecewise constant according to the time partition Δ, that is,

$$\begin{aligned}&\mathrm{PC}^\Delta([0,T],\mathbb{R}^m)\\ &\quad := \{u : [0,T] \to \mathbb{R}^m \mid \forall i \in \{0,\dots,N-1\},\ \exists u_i \in \mathbb{R}^m,\ \forall t \in [t_i,t_{i+1}),\ u(t) = u_i\}.\end{aligned}$$

In what follows $\mathrm{PC}^\Delta([0,T],\mathbb{R}^m)$ is the set of *sampled-data controls* according to the time partition Δ. We denote $\|\Delta\| := \max_{i=1,\dots,N} h_i > 0$, where $h_i := t_i - t_{i-1} > 0$ for all $i \in \{1,\dots,N\}$. When $h_i = h$ for some $h > 0$ for every $i \in \{1,\dots,N\}$, the time partition Δ is said to be *h-uniform* (which corresponds to *periodic sampling*; see [8, section II.A]).

In this section we consider two LQ optimal control problems in finite time horizon: permanent control $u \in \mathrm{L}^2([0,T],\mathbb{R}^m)$ (Proposition 2.3) and sampled-data control $u \in \mathrm{PC}^\Delta([0,T],\mathbb{R}^m)$ (Proposition 2.4).

PROPOSITION 2.3 (permanent control in finite time horizon). *Let $T>0$ and let $x_0 \in \mathbb{R}^n$. The LQ optimal permanent control problem in finite time horizon T*

$$(\mathrm{OCP}^T_{x_0}) \qquad \begin{array}{ll} \textit{minimize} & \langle Px(T),x(T)\rangle_{\mathbb{R}^n} \\ & + \displaystyle\int_0^T \Big(\langle Q(\tau)x(\tau),x(\tau)\rangle_{\mathbb{R}^n} + \langle R(\tau)u(\tau),u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau \\ \textit{subject to} & \left\{ \begin{array}{ll} x \in \mathrm{AC}([0,T],\mathbb{R}^n), & u \in \mathrm{L}^2([0,T],\mathbb{R}^m), \\ \dot{x}(t) = A(t)x(t)+B(t)u(t) & \textit{for a.e. } t \in [0,T], \\ x(0)=x_0 & \end{array} \right. \end{array}$$

has a unique optimal solution (x^*,u^*). Moreover u^* is the time-varying state feedback

$$u^*(t) = -\mathcal{N}(t,E^T(t),0)^{-1}\mathcal{M}(t,E^T(t),0)^\top x^*(t) \qquad \textit{for a.e. } t \in [0,T],$$

where $E^T : [0,T] \to \mathcal{S}^n_+$ is the unique solution to the Permanent Differential Riccati Equation

$$(\text{P-DRE}) \qquad \dot{E^T}(t) = \mathcal{F}(t,E^T(t),0) \quad \forall t \in [0,T], \qquad E^T(T)=P.$$

Furthermore, the minimal cost of $(\mathrm{OCP}^T_{x_0})$ is equal to $\langle E^T(0)x_0,x_0\rangle_{\mathbb{R}^n}$.

PROPOSITION 2.4 (sampled-data control in finite time horizon). *Let $T>0$, let $\Delta = \{t_i\}_{i=0,\dots,N}$ be a time partition of the interval $[0,T]$, and let $x_0 \in \mathbb{R}^n$. The LQ optimal sampled-data control problem in finite time horizon T given by*

$$(\mathrm{OCP}^{T,\Delta}_{x_0}) \qquad \begin{array}{ll} \textit{minimize} & \langle Px(T),x(T)\rangle_{\mathbb{R}^n} \\ & + \displaystyle\int_0^T \Big(\langle Q(\tau)x(\tau),x(\tau)\rangle_{\mathbb{R}^n} + \langle R(\tau)u(\tau),u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau \\ \textit{subject to} & \left\{ \begin{array}{ll} x \in \mathrm{AC}([0,T],\mathbb{R}^n), & u \in \mathrm{PC}^\Delta([0,T],\mathbb{R}^m), \\ \dot{x}(t) = A(t)x(t)+B(t)u(t) & \textit{for a.e. } t \in [0,T], \\ x(0)=x_0 & \end{array} \right. \end{array}$$

has a unique optimal solution (x^*,u^*). Moreover u^* is the time-varying state feedback

$$u^*_i = -\mathcal{N}(t_{i+1},E^{T,\Delta}_{i+1},h_{i+1})^{-1}\mathcal{M}(t_{i+1},E^{T,\Delta}_{i+1},h_{i+1})^\top x^*(t_i) \qquad \forall i \in \{0,\dots,N-1\},$$

where $E^{T,\Delta} = (E^{T,\Delta}_i)_{i=0,\dots,N} \subset \mathcal{S}^n_+$ is the unique solution to the Sampled-Data Difference Riccati Equation

$$(\text{SD-DRE}) \qquad \left\{ \begin{array}{ll} E^{T,\Delta}_{i+1} - E^{T,\Delta}_i = h_{i+1}\mathcal{F}(t_{i+1},E^{T,\Delta}_{i+1},h_{i+1}) & \forall i \in \{0,\dots,N-1\}, \\ E^{T,\Delta}_N = P. & \end{array} \right.$$

Furthermore, the minimal cost of $(\mathrm{OCP}^{T,\Delta}_{x_0})$ is equal to $\langle E^{T,\Delta}_0 x_0,x_0\rangle_{\mathbb{R}^n}$.

Remark 2.5. The mathematical contents of Propositions 2.3 and 2.4 are not new. The time-varying state feedback u^* in Proposition 2.3 is usually written (we skip the dependence on t for readability) as $u^* = -R^{-1}B^\top E^T x^*$ on $[0,T]$, and (P-DRE) is usually written as $\dot{E^T} = E^T B R^{-1} B^\top E^T - Q - A^\top E^T - E^T A$ on $[0,T]$, $E^T(T) = P$ (see [12, 26, 28, 40, 45]). As in the fully discrete-time case [32, Remark 2], the analogous results in the sampled-data control case have various equivalent formulations in the literature. Using the Duhamel formula, problem $(\mathrm{OCP}^{T,\Delta}_{x_0})$ can be recast as a fully discrete-time LQ optimal control problem. In this way, the time-varying state feedback control u^* in Proposition 2.4 and (SD-DRE) were first obtained in [23] by applying the discrete-time dynamical programming principle (method revisited in [17, p. 616], or more recently in [7, Theorem 4.1]), while they are derived in [3, Appendix B] or in [17, p. 618] by applying a discrete-time version of the Pontryagin maximum principle (see [24]). In Theorem 3.2 below, we are going to prove convergence of $E^{T,\Delta}$ to E^T when $\|\Delta\| \to 0$.

2.3. Infinite time horizon: Permanent/sampled-data control (autonomous setting and uniform time partition). This section is dedicated to the infinite time horizon case. We denote by $\mathrm{AC}([0,+\infty),\mathbb{R}^n)$ the space of functions defined on $[0,+\infty)$ with values in $\mathbb{R}^n$ which are absolutely continuous over all intervals $[0,T]$ with $T>0$, and by $\mathrm{L}^2([0,+\infty),\mathbb{R}^m)$ the Lebesgue space of square-integrable functions defined almost everywhere on $[0,+\infty)$ with values in $\mathbb{R}^m$. Assume that we are in the autonomous setting (see Definition 2.1). We consider the following assumptions:

(H_1) $Q \in \mathcal{S}^n_{++}$.

(H_2) For every $x_0 \in \mathbb{R}^n$, there exists $(x,u) \in \mathrm{AC}([0,+\infty),\mathbb{R}^n) \times \mathrm{L}^2([0,+\infty),\mathbb{R}^m)$ such that $\dot{x}(t) = Ax(t) + Bu(t)$ for a.e. $t \geq 0$ and $x(0) = x_0$, satisfying

$$\int_0^{+\infty} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau < +\infty.$$

Assumption (H_2) is known in the literature as an *optimizability* assumption (or *finite cost* assumption) and is related to various notions of *stabilizability* of linear permanent control systems (see [46]). An extensive literature is dedicated to this topic (see [42] and references mentioned in [16, section 10.10]). Recall that if the pair (A,B) satisfies the Kalman condition (see [48, Theorem 1.2]) or only the weaker Popov–Belevitch–Hautus test condition (see [42, Theorem 6.2]), then (H_2) is satisfied.

Let $h>0$. The h-uniform time partition of the interval $[0,+\infty)$ is the sequence $\Delta = \{t_i\}_{i\in\mathbb{N}}$, where $t_i := ih$ for every $i \in \mathbb{N}$. We denote by $\|\Delta\| = h$ and by $\mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$ the space of functions defined on $[0,+\infty)$ with values in $\mathbb{R}^m$ that are piecewise constant according to the time partition Δ, that is,

$$\mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m) := \{u : [0,+\infty) \to \mathbb{R}^m \mid \forall i \in \mathbb{N},\ \exists u_i \in \mathbb{R}^m,\ \forall t \in [t_i,t_{i+1}),\ u(t) = u_i\}.$$

We also consider the following assumption, which we call the *h-optimizability* assumption:

(H^h_2) For every $x_0 \in \mathbb{R}^n$, there exists $(x,u) \in \mathrm{AC}([0,+\infty),\mathbb{R}^n) \times \mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$ such that $\dot{x}(t) = Ax(t) + Bu(t)$ for almost every $t \geq 0$ and $x(0) = x_0$, satisfying

$$\int_0^{+\infty} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau < +\infty.$$

Obviously, if (H_2^h) is satisfied for some $h > 0$, then (H_2) is satisfied. In other words, (H_2^h) for a given $h > 0$ is stronger than (H_2). Conversely, we will prove in Lemma 3.1 further that if (H_1) and (H_2) are satisfied, then there exists $\overline{h} > 0$ such that (H_2^h) is satisfied for every $h \in (0, \overline{h}]$.

In this section, in the autonomous setting (Definition 2.1), we consider two infinite time horizon LQ optimal control problems: permanent control $u \in \mathrm{L}^2([0,+\infty), \mathbb{R}^m)$ (Proposition 2.6) and sampled-data control $u \in \mathrm{PC}^\Delta([0,+\infty), \mathbb{R}^m)$ with h-uniform time partition Δ (Proposition 2.7).

Proposition 2.6 (permanent control in infinite time horizon). *Assume that we are in the autonomous setting (see Definition* 2.1*). Let* $x_0 \in \mathbb{R}^n$*. Under assumptions* (H_1) *and* (H_2)*, the LQ optimal permanent control problem in infinite time horizon*

$$(\mathrm{OCP}^\infty_{x_0}) \qquad \begin{array}{ll} \textit{minimize} & \displaystyle\int_0^{+\infty} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau \\ \textit{subject to} & \begin{cases} x \in \mathrm{AC}([0,+\infty), \mathbb{R}^n), \quad u \in \mathrm{L}^2([0,+\infty), \mathbb{R}^m), \\ \dot{x}(t) = Ax(t) + Bu(t) \qquad \textit{for a.e. } t \geq 0, \\ x(0) = x_0 \end{cases} \end{array}$$

has a unique optimal solution (x^*, u^*)*. Moreover* u^* *is the state feedback*

$$u^*(t) = -\mathcal{N}(E^\infty, 0)^{-1}\mathcal{M}(E^\infty, 0)^\top x^*(t) \qquad \textit{for a.e. } t \geq 0,$$

where $E^\infty \in \mathcal{S}^n_{++}$ *is the unique solution to the Permanent Algebraic Riccati Equation*

$$(\text{P-ARE}) \qquad \mathcal{F}(E^\infty, 0) = 0_{\mathbb{R}^{n\times n}}, \qquad E^\infty \in \mathcal{S}^n_+.$$

Furthermore, the minimal cost of $(\mathrm{OCP}^\infty_{x_0})$ *is equal to* $\langle E^\infty x_0, x_0\rangle_{\mathbb{R}^n}$*.*

Proposition 2.7 (sampled-data control in infinite time horizon). *Assume that we are in the autonomous setting (see Definition* 2.1*). Let* $\Delta = \{t_i\}_{i\in\mathbb{N}}$ *be an* h*-uniform time partition of the interval* $[0,+\infty)$*, and let* $x_0 \in \mathbb{R}^n$*. Under assumptions* (H_1) *and* (H_2^h)*, the LQ optimal sampled-data control problem in infinite time horizon*

$$(\mathrm{OCP}^{\infty,\Delta}_{x_0}) \qquad \begin{array}{ll} \textit{minimize} & \displaystyle\int_0^{+\infty} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau \\ \textit{subject to} & \begin{cases} x \in \mathrm{AC}([0,+\infty), \mathbb{R}^n), \quad u \in \mathrm{PC}^\Delta([0,+\infty), \mathbb{R}^m), \\ \dot{x}(t) = Ax(t) + Bu(t) \qquad \textit{for a.e. } t \geq 0, \\ x(0) = x_0 \end{cases} \end{array}$$

has a unique optimal solution (x^*, u^*)*. Moreover* u^* *is the state feedback*

$$u^*_i = -\mathcal{N}(E^{\infty,\Delta}, h)^{-1}\mathcal{M}(E^{\infty,\Delta}, h)^\top x^*(t_i) \qquad \forall i \in \mathbb{N},$$

where $E^{\infty,\Delta} \in \mathcal{S}^n_{++}$ *is the unique solution to the Sampled-Data Algebraic Riccati Equation*

$$(\text{SD-ARE}) \qquad \mathcal{F}(E^{\infty,\Delta}, h) = 0_{\mathbb{R}^{n\times n}}, \qquad E^{\infty,\Delta} \in \mathcal{S}^n_+.$$

Furthermore, the minimal cost of $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ *is equal to* $\langle E^{\infty,\Delta} x_0, x_0\rangle_{\mathbb{R}^n}$*.*

Remark 2.8. The mathematical content of Proposition 2.6 is well known in the literature (see [12, 26, 28, 40, 45]). The state feedback control u^* in Proposition 2.6 is usually written as $u^*(t) = -R^{-1}B^\top E^\infty x^*(t)$ for a.e. $t \geq 0$, and (P-ARE) is usually written as $E^\infty BR^{-1}B^\top E^\infty - Q - A^\top E^\infty - E^\infty A = 0_{\mathbb{R}^{n\times n}}$, $E^\infty \in \mathcal{S}^n_+$. As said in Remark 2.5, our formulation of Proposition 2.6, using the continuous map $\mathcal{F}$ defined in section 2.1, provides a unified presentation for the permanent and sampled-data cases. In Theorem 3.2, we prove the convergence of $E^{\infty,\Delta}$ to E^∞ when $h = \|\Delta\| \to 0$.

Remark 2.9. Similarly to the finite time horizon case (see Remark 2.5), the state feedback control in Proposition 2.7 and (SD-ARE) have various equivalent formulations in the literature (see [8, 29, 30, 33, 34]). In most of these references, Problem $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ is recast as a fully discrete-time LQ optimal control problem with infinite time horizon. In particular, the optimizability property for problem $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ is equivalent to the optimizability of the corresponding fully discrete-time problem (see [17, Theorem 3] or [30, p. 348]). In the present work, we will prove that if (H_1) and (H_2) are satisfied, then there exists $\overline{h} > 0$ such that the h-optimizability assumption (H^h_2) is satisfied for every $h \in (0, \overline{h}]$ (see Lemma 3.1 further). Moreover, in that context, a uniform bound of the minimal cost of problem $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ (independently of $h \in (0, \overline{h}]$) is obtained. This plays a key role in proving the convergence of $E^{\infty,\Delta}$ to E^∞ when $h = \|\Delta\| \to 0$. We provide in Appendix A.2 a proof of Proposition 2.7 based on the h-optimizability assumption (H^h_2) by keeping the initial continuous-time formulation of problem $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ as in our recent paper [11]. This proof is an adaptation to the sampled-data control case of the proof of Proposition 2.6 (see [12, p. 153], [28, Theorem 7, p. 198] or [45, Theorem 4.13]). Moreover it contains in particular the proof of the convergence of $E^{T,\Delta}$ to $E^{\infty,\Delta}$ when $T \to +\infty$.

Remark 2.10. Contrary to the previous section (finite time horizon case), the above results, stated in infinite time horizon, are both restricted to the autonomous setting and, additionally, Proposition 2.7 is restricted to an h-uniform time partition $\Delta = \{t_i\}_{i\in\mathbb{N}}$. Our goal in this remark is to provide details on these restrictions. According to the proofs of the two previous propositions, by considering $P = 0_{\mathbb{R}^{n\times n}}$, the Riccati matrices E^∞ and $E^{\infty,\Delta}$ are obtained as limits of $E^T(0)$ when $T \to +\infty$ and of E^{t_N,Δ_N}_0, where $\Delta_N := \{t_i\}_{i=0,\dots,N}$, when $N \to +\infty$, respectively. The main difficulty lies in the characterization of the Riccati matrices E^∞ and $E^{\infty,\Delta}$ as solutions to algebraic matrix equations. By using a simple change of time variable, these limits can be seen as equilibria of a matrix differential equation and of a matrix difference equation, respectively. Toward this goal, in the case of permanent controls, it is required that the involved matrix differential equation is autonomous and independent of T, which is satisfied in an autonomous setting since then the map $\mathcal{F}$ does not depend on t (see Remark 2.2). Similarly, in the case of sampled-data controls, it is required that the involved matrix difference equation is autonomous and independent of N, which is satisfied in an autonomous setting when the time partition Δ is h-uniform.

3. Main result.

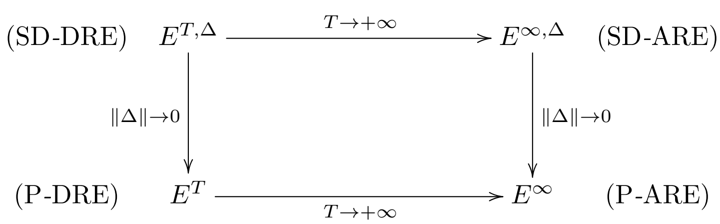
Propositions 2.3, 2.4, 2.6, and 2.7 in section 2 give state feedback optimal controls for permanent and sampled-data LQ problems in finite and infinite time horizons. In each case, the optimal control is expressed thanks to a Riccati matrix: E^T, $E^{T,\Delta}$, E^∞, and $E^{\infty,\Delta}$, respectively. Our main result (Theorem 3.2 below) establishes convergence properties between these four matrices. Let us first state the following lemma (proved in Appendix A.3).

Lemma 3.1. *In the autonomous setting (Definition* 2.1*), under assumptions* (H_1) *and* (H_2)*, there exist* $\overline{h} > 0$ *and* $\overline{c} \geq 0$ *such that, for all h-uniform time partitions* Δ *of the interval* $[0, +\infty)$*, with* $0 < h \leq \overline{h}$*, and for every* $x_0 \in \mathbb{R}^n$*, there exists a pair* $(x, u) \in \mathrm{AC}([0, +\infty), \mathbb{R}^n) \times \mathrm{PC}^{\Delta}([0, +\infty), \mathbb{R}^m)$ *such that* $\dot{x}(t) = Ax(t) + Bu(t)$ *for a.e.* $t \geq 0$ *and* $x(0) = x_0$*, satisfying*

$$\int_0^{+\infty} \Big(\langle Qx(\tau), x(\tau) \rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau) \rangle_{\mathbb{R}^m} \Big) \, d\tau \leq \overline{c} \langle E^{\infty} x_0, x_0 \rangle_{\mathbb{R}^n} < +\infty.$$

Not only does Lemma 3.1 assert that if (H_1) and (H_2) are satisfied, then there exists $\overline{h} > 0$ such that (H_2^h) is satisfied for every $h \in (0, \overline{h}]$, but it also provides a *uniform* h-optimizability for all $0 < h \leq \overline{h}$ (in the sense that the finite right-hand term is independent of h). This uniform bound plays a crucial role in order to derive the convergence of $E^{\infty,\Delta}$ to E^{∞} when $h = \|\Delta\| \to 0$ (which corresponds to the fourth item of Theorem 3.2 below). On the other hand, from the proof of Lemma 3.1 in Appendix A.3, note that a lower bound of the threshold $\overline{h} > 0$ can be expressed in function of the norms of A, B, Q, R, and E^{∞}.

Theorem 3.2 (commutative diagram). *The following diagram commutes:*



(i) Left arrow of the diagram: *Given any* $T > 0$*, we have*

$$\lim_{\|\Delta\| \to 0} \max_{i=0,\ldots,N} \|E^T(t_i) - E_i^{T,\Delta}\|_{\mathbb{R}^{n \times n}} = 0$$

for all time partitions $\Delta = \{t_i\}_{i=0,\ldots,N}$ *of the interval* $[0, T]$*.*

(ii) Bottom arrow of the diagram: *Assume that* $P = 0_{\mathbb{R}^{n\times n}}$ *and that we are in the autonomous setting (see Definition* 2.1*). Under assumptions* (H_1) *and* (H_2)*, we have*

$$\lim_{T \to +\infty} E^T(t) = E^{\infty} \qquad \forall t \geq 0.$$

(iii) Top arrow of the diagram: *Assume that* $P = 0_{\mathbb{R}^{n\times n}}$ *and that we are in the autonomous setting (see Definition* 2.1*). Let* $\Delta = \{t_i\}_{i \in \mathbb{N}}$ *be an h-uniform time partition of the interval* $[0, +\infty)$*. For all* $N \in \mathbb{N}^*$*, we denote by* $\Delta_N := \Delta \cap [0, t_N]$ *the h-uniform time partition of the interval* $[0, t_N]$*. Under assumptions* (H_1) *and* (H_2^h)*, we have*

$$\lim_{N \to +\infty} E_i^{t_N, \Delta_N} = E^{\infty,\Delta} \qquad \forall i \in \mathbb{N}.$$

(iv) Right arrow of the diagram: *In the autonomous setting (see Definition* 2.1*), under assumptions* (H_1) *and* (H_2)*, we have*

$$\lim_{h \to 0} E^{\infty,\Delta} = E^{\infty}$$

for all h-uniform time partitions $\Delta = \{t_i\}_{i \in \mathbb{N}}$ *of the interval* $[0, +\infty)$ *with* $0 < h \leq \overline{h}$ *(where* $\overline{h} > 0$ *is given by Lemma* 3.1*).*

Remark 3.3. The proof of Theorem 3.2 is done in Appendix A.4. Some results similar to the four items of Theorem 3.2 have already been discussed and can be found in the literature. For example, in the autonomous case and with h-uniform time partitions, the first item of Theorem 3.2 was proved in [3, Corollary 2.3] (a second-order convergence was even derived). The second item of Theorem 3.2 is a well-known fact and follows from the proof of Proposition 2.6 (see [12, p. 153], [28, Theorem 7], or [45, Theorem 4.13]). The third item of Theorem 3.2 follows from the proof of Proposition 2.7 given in Appendix A.2 by keeping the initial continuous-time writing of problem ($\mathrm{OCP}^{\infty,\Delta}_{x_0}$). As evoked in Remarks 2.5 and 2.9, in the literature, the LQ optimal sampled-data control problems are usually rewritten as fully discrete-time LQ optimal control problems. As a consequence, the result of the third item of Theorem 3.2 is usually reduced in the literature to the corresponding result at the discrete level (see [17, Theorem 3] or [30, p. 348]). The last item of Theorem 3.2 is proved in Appendix A.4 by using the uniform h-optimizability obtained in Lemma 3.1. Finally, the contribution of the present work is to provide a framework allowing us to gather Propositions 2.3, 2.4, 2.6, and 2.7 in a unified setting, based on the continuous map $\mathcal{F}$, which moreover allows us to prove several convergence results for Riccati matrices and to summarize them in a single diagram.

Remark 3.4. Note that sensitivity analysis of (SD-ARE) with respect to h has been explored in [19, 29, 30, 33] by computing its derivative algebraically in view of optimization of the sampling period h. Let us observe that the map $\mathcal{F}$ introduced in section 2.1 is a suitable candidate in order to invoke an implicit function argument and justify the differentiability of $E^{\infty,\Delta}$ with respect to h. We mention the recent work [5] in which, in the context of Model Predictive Control (MPC), the authors prove the continuity of the MPC value function associated to (SD-ARE) (with an h-uniform partition) with respect to the couple (h,T); they also establish the differentiability of the value function when the floor of $\frac{h}{T}$ is fixed.

Appendix A. Proofs. Preliminaries and reminders are provided in section A.1. We prove Proposition 2.7 in section A.2, Lemma 3.1 in section A.3, and Theorem 3.2 in section A.4.

A.1. Preliminaries.

LEMMA A.1 (backward discrete Grönwall lemma). *Let $N \in \mathbb{N}^*$, and let $(w_i)_{i=0,\dots,N}$, $(z_i)_{i=1,\dots,N}$, and $(\mu_i)_{i=1,\dots,N}$ be three nonnegative real finite sequences which satisfy $w_N = 0$ and $w_i \leq (1+\mu_{i+1})w_{i+1} + z_{i+1}$ for every $i \in \{0,\dots,N-1\}$. Then*

$$w_i \leq \sum_{j=i+1}^{N} \prod_{q=i+1}^{j-1} (1+\mu_q) z_j \leq \sum_{j=i+1}^{N} e^{\sum_{q=i+1}^{j-1} \mu_q} z_j \qquad \forall i \in \{0,\dots,N-1\}.$$

Proof. The first inequality follows from a backward induction. The second inequality comes from the inequality $1+\mu \leq e^{\mu}$ for all $\mu \geq 0$. ∎

LEMMA A.2 (some reminders on symmetric matrices). *Let $p \in \mathbb{N}^*$. The following properties are satisfied:*

(i) *Let $E \in \mathcal{S}^p_+$ (resp., $E \in \mathcal{S}^p_{++}$). Then all eigenvalues of E are nonnegative (resp., positive) real numbers.*

(ii) *Let $E \in \mathcal{S}^p_+$. Then $\rho_{\min}(E)\|y\|^2_{\mathbb{R}^p} \leq \langle Ey, y\rangle_{\mathbb{R}^p} \leq \rho_{\max}(E)\|y\|^2_{\mathbb{R}^p}$ for all $y \in \mathbb{R}^p$, where $\rho_{\min}(E)$ and $\rho_{\max}(E)$ stand, respectively, for the smallest and the largest nonnegative eigenvalues of E.*

(iii) *Let* $E \in \mathcal{S}^p_{++}$. *Then* E *is invertible and* $E^{-1} \in \mathcal{S}^p_{++}$. *Moreover* $\rho_{\min}(E^{-1}) = 1/\rho_{\max}(E)$ *and* $\rho_{\max}(E^{-1}) = 1/\rho_{\min}(E)$.

(iv) *Let* $E \in \mathcal{S}^p_{+}$. *It holds that* $\|E\|_{\mathbb{R}^{p\times p}} = \rho_{\max}(E)$.

(v) *Let* $E \in \mathcal{S}^p_{+}$. *If there exists* $c \geq 0$ *such that* $\langle Ey, y\rangle_{\mathbb{R}^p} \leq c\|y\|^2_{\mathbb{R}^p}$ *for all* $y \in \mathbb{R}^p$, *then* $\|E\|_{\mathbb{R}^{p\times p}} \leq c$.

(vi) *Let* $E_1, E_2 \in \mathcal{S}^p_{+}$. *If* $\langle E_1 y, y\rangle_{\mathbb{R}^p} = \langle E_2 y, y\rangle_{\mathbb{R}^p}$ *for all* $y \in \mathbb{R}^p$, *then* $E_1 = E_2$.

(vii) *Let* $(E_k)_{k\in\mathbb{N}}$ *be a sequence of matrices in* $\mathcal{S}^p_{+}$. *If* $\langle E_k y, y\rangle_{\mathbb{R}^p}$ *converges when* $k \to +\infty$ *for all* $y \in \mathbb{R}^p$, *then* $(E_k)_{k\in\mathbb{N}}$ *has a limit* $E \in \mathcal{S}^p_{+}$.

Proof. The first four items are classical results (see, e.g., [21]). The fifth item follows from the fourth one. The last two items follow from the following fact: if $E \in \mathcal{S}^p_{+}$, with $E = (e_{ij})_{i,j=1,\dots,p}$, then

$$e_{ij} = \langle Eb_j, b_i\rangle_{\mathbb{R}^p} = \frac{1}{2}\Big(\langle E(b_i + b_j), b_i + b_j\rangle_{\mathbb{R}^p} - \langle Eb_i, b_i\rangle_{\mathbb{R}^p} - \langle Eb_j, b_j\rangle_{\mathbb{R}^p}\Big)$$

for all $i, j \in \{1, \dots, p\}$, where $\{b_i\}_{i=1,\dots,p}$ stands for the canonical basis of $\mathbb{R}^p$. ∎

LEMMA A.3 (properties of the map $\mathcal{F}$).

(i) *The map* $\mathcal{F}$ *is well-defined on* $\mathbb{R} \times \mathcal{S}^n_+ \times \mathbb{R}_+$.

(ii) *The map* $\mathcal{F}$ *is continuous on* $\mathbb{R} \times \mathcal{S}^n_+ \times \mathbb{R}_+$.

(iii) *If* $\mathcal{K}$ *is a compact subset of* $\mathbb{R} \times \mathcal{S}^n_+ \times \mathbb{R}_+$, *then there exists* $c \geq 0$ *such that*

$$\|\mathcal{F}(t, E_2, h) - \mathcal{F}(t, E_1, h)\|_{\mathbb{R}^{n\times n}} \leq c\|E_2 - E_1\|_{\mathbb{R}^{n\times n}}$$

for all (t, E_1, E_2, h) *such that* $(t, E_1, h) \in \mathcal{K}$ *and* $(t, E_2, h) \in \mathcal{K}$.

Proof. (i) For $(t, E, h) \in \mathbb{R} \times \mathcal{S}^n_+ \times \mathbb{R}_+$, note that $\mathcal{N}_1(t, E, h) \in \mathcal{S}^m_{++}$, $\mathcal{N}_2(t, E, h) \in \mathcal{S}^m_+$, and $\mathcal{N}_3(t, E, h) \in \mathcal{S}^m_+$. Hence the sum $\mathcal{N}(t, E, h)$ belongs to $\mathcal{S}^m_{++}$ and thus is invertible from (iii) of Lemma A.2.

(ii) Since taking the inverse of a matrix is a continuous operation, we only need to prove that $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{G}$ are continuous over $\mathbb{R} \times \mathcal{S}^n_+ \times \mathbb{R}_+$. Let $(t_k, E_k, h_k)_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R} \times \mathcal{S}^n_+ \times \mathbb{R}_+$ which converges to some $(t, E, h) \in \mathbb{R} \times \mathcal{S}^n_+ \times \mathbb{R}_+$. We need to prove that $\mathcal{M}(t_k, E_k, h_k)$, $\mathcal{N}(t_k, E_k, h_k)$, and $\mathcal{G}(t_k, E_k, h_k)$ converge, respectively, to $\mathcal{M}(t, E, h)$, $\mathcal{N}(t, E, h)$, and $\mathcal{G}(t, E, h)$ when $k \to +\infty$. The case $h \neq 0$ can be treated using, for instance, the Lebesgue dominated convergence theorem. Let us discuss the case $h = 0$ and assume, without loss of generality (since A, B, Q, and R are continuous matrices), that $h_k > 0$ for every $k \in \mathbb{N}$. In that situation, we conclude by using in particular the fact that t is a Lebesgue point of all integrands involved in the definitions of the functions $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{G}$.

(iii) It is clear that $\mathcal{F}$ is continuously differentiable over $\mathcal{S}^n_+$ with respect to its second variable. Similarly to the previous item, we can moreover prove that the map $(t, E, h) \mapsto \mathcal{D}_2\mathcal{F}(t, E, h)$ is continuous over $\mathbb{R} \times \mathcal{S}^n_+ \times \mathbb{R}_+$. Thus the third item follows by applying the Taylor expansion formula with integral remainder. ∎

LEMMA A.4 (uniform bound for E^T and $E^{T,\Delta}$). *Let* $T > 0$. *We have*

$$\|E^T(t)\|_{\mathbb{R}^{n\times n}} \leq \Big(\|P\|_{\mathbb{R}^{n\times n}} + (T - t)\|Q_{|[t,T]}\|_\infty\Big)e^{2\|A_{|[t,T]}\|_\infty (T-t)} \qquad \forall t \in [0, T].$$

If $\Delta = \{t_i\}_{i=0,\dots,N}$ *is a time partition of the interval* $[0, T]$, *then*

$$\|E^{T,\Delta}_i\|_{\mathbb{R}^{n\times n}} \leq \Big(\|P\|_{\mathbb{R}^{n\times n}} + (T - t_i)\|Q_{|[t_i,T]}\|_\infty\Big)e^{2\|A_{|[t_i,T]}\|_\infty (T-t_i)} \qquad \forall i \in \{0, \dots, N\}.$$

Proof. Let us prove the first part of Lemma A.4. We first deal with the case $t = 0$. Taking the null control in problem (OCP^T_y) and using the Duhamel formula, we deduce that its minimal cost satisfies

$$\langle E^T(0)y, y\rangle_{\mathbb{R}^n} \leq \Big(\|P\|_{\mathbb{R}^{n\times n}} + T\|Q_{|[0,T]}\|_\infty \Big) e^{2T\|A_{|[0,T]}\|_\infty} \|y\|^2_{\mathbb{R}^n} \qquad \forall y \in \mathbb{R}^n.$$

The result at $t = 0$ then follows from (v) in Lemma A.2. The case $0 < t < T$ can be treated similarly by considering the restriction of problem (OCP^T_y) to the time interval $[t, T]$ (instead of $[0, T]$). Finally the case $t = T$ is obvious since $E^T(T) = P$. The second part of Lemma A.4 is derived in a similar way. ∎

Lemma A.5 (zero limit of finite cost trajectories at infinite time horizon). *In the autonomous setting (see Definition* 2.1*), under assumption* (H_1)*, for every* $(x, u) \in \mathrm{AC}([0, +\infty), \mathbb{R}^n) \times \mathrm{L}^2([0, +\infty), \mathbb{R}^m)$ *such that* $\dot{x}(t) = Ax(t) + Bu(t)$ *for almost every* $t \geq 0$ *and satisfying*

$$\int_0^{+\infty} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau < +\infty,$$

we have $\lim_{t\to+\infty} x(t) = 0_{\mathbb{R}^n}$*.*

Proof. Since $Q \in \mathcal{S}^n_{++}$, we have $\|x(t)\|^2_{\mathbb{R}^n} \leq \frac{1}{\rho_{\min}(Q)} \langle Qx(t), x(t)\rangle_{\mathbb{R}^n}$ for all $t \geq 0$. Using the assumptions we deduce that $x \in \mathrm{L}^2([0, +\infty), \mathbb{R}^m)$. Let us introduce $X \in \mathrm{AC}([0, +\infty), \mathbb{R})$ defined by $X(t) := \|x(t)\|^2_{\mathbb{R}^n} \geq 0$ for all $t \geq 0$. Since $\dot{X}(t) = 2\langle Ax(t) + Bu(t), x(t)\rangle_{\mathbb{R}^n}$ for almost every $t \geq 0$, we deduce that $\dot{X} \in \mathrm{L}^1([0, +\infty), \mathbb{R})$ and thus $X(t)$ admits a limit $\ell \geq 0$ when $t \to +\infty$. By contradiction let us assume that $\ell > 0$. Then there exists $s \geq 0$ such that $X(t) \geq \frac{\ell}{2} > 0$ for all $t \geq s$. We get that

$$\int_0^{\bar{t}} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau \geq \rho_{\min}(Q) \left(\int_0^{\bar{t}} X(\tau)\, d\tau \right)$$
$$= \rho_{\min}(Q) \int_0^s X(\tau)\, d\tau + \int_s^{\bar{t}} X(\tau)\, d\tau \geq \rho_{\min}(Q) \left(\int_0^s X(\tau)\, d\tau + (\bar{t} - s)\frac{\ell}{2} \right)$$

for every $\bar{t} \geq s$. A contradiction is obtained by letting $\bar{t} \to +\infty$. ∎

A.2. Proof of Proposition 2.7. This proof is inspired by the proof of Proposition 2.6 (see [12, p. 153], [28, Theorem 7, p. 198], or [45, Theorem 4.13]) and is an adaptation to the sampled-data control case. We denote by $\Delta_N := \Delta \cap [0, t_N]$ the h-uniform time partition of the interval $[0, t_N]$ for every $N \in \mathbb{N}^*$.

Existence and uniqueness of the optimal solution. Let $x_0 \in \mathbb{R}^n$. For every $u \in \mathrm{L}^2([0, +\infty), \mathbb{R}^m)$, we denote by $x(\cdot, u) \in \mathrm{AC}([0, +\infty), \mathbb{R}^n)$ the unique solution to the Cauchy problem $\dot{x}(t) = Ax(t) + Bu(t)$ for a.e. $t \geq 0$, $x(0) = x_0$. We define the cost

$$\begin{array}{rccl} \mathcal{C}: & \mathrm{L}^2([0, +\infty), \mathbb{R}^m) & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & u & \longmapsto & \displaystyle\int_0^{+\infty} \Big(\langle Qx(\tau, u), x(\tau, u)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau. \end{array}$$

Problem $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ can be recast as $\min\{\mathcal{C}(u) \mid u \in \mathrm{PC}^\Delta([0, +\infty), \mathbb{R}^m)\}$. Since (H^h_2) is satisfied, we have $\mathcal{C}^* := \inf\{\mathcal{C}(u) \mid u \in \mathrm{PC}^\Delta([0, +\infty), \mathbb{R}^m)\} < +\infty$. Let us consider a minimizing sequence $(u_k)_{k\in\mathbb{N}} \subset \mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$ and, without loss of generality, we assume that $\mathcal{C}(u_k) < +\infty$ for every $k \in \mathbb{N}$. Since $R \in \mathcal{S}^n_{++}$, we deduce that the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^2([0,+\infty),\mathbb{R}^m)$ and thus, up to a subsequence (that we do not relabel), converges weakly to some $u^* \in \mathrm{L}^2([0,+\infty),\mathbb{R}^m)$. Since $\mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$ is a weakly closed subspace of $\mathrm{L}^2([0,+\infty),\mathbb{R}^m)$, it follows that $u^* \in \mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$. Moreover, denoting $x_k := x(\cdot,u_k)$ for every $k \in \mathbb{N}$, the Duhamel formula gives $x_k(t) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu_k(\tau)\,d\tau$ for every $t \geq 0$ and every $k \in \mathbb{N}$. By weak convergence, for every $t \geq 0$, the sequence $(x_k(t))_{k\in\mathbb{N}}$ converges to $x^*(t) := e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu^*(\tau)\,d\tau$. Then, obviously, $x^* = x(\cdot,u^*)$. Moreover, from the Fatou lemma (see, e.g., [13, Lemma 4.1]) and by weak convergence, we get that

$$\begin{aligned}
\mathcal{C}^* = \lim_{k\to+\infty}\mathcal{C}(u_k) &= \liminf_{k\to+\infty}\mathcal{C}(u_k)\\
&= \liminf_{k\to+\infty}\int_0^{+\infty}\Big(\langle Qx_k(\tau),x_k(\tau)\rangle_{\mathbb{R}^n} + \langle Ru_k(\tau),u_k(\tau)\rangle_{\mathbb{R}^m}\Big)\,d\tau\\
&\geq \liminf_{k\to+\infty}\int_0^{+\infty}\langle Qx_k(\tau),x_k(\tau)\rangle_{\mathbb{R}^n}\,d\tau + \liminf_{k\to+\infty}\|u_k\|^2_{\mathrm{L}^2_R}\\
&\geq \int_0^{+\infty}\langle Qx^*(\tau),x^*(\tau)\rangle_{\mathbb{R}^n}\,d\tau + \|u^*\|^2_{\mathrm{L}^2_R}\\
&= \int_0^{+\infty}\left(\langle Qx^*(\tau),x^*(\tau)\rangle_{\mathbb{R}^n} + \langle Ru^*(\tau),u^*(\tau)\rangle_{\mathbb{R}^m}\right)d\tau = \mathcal{C}(u^*),
\end{aligned}$$

where the norm defined by $\|u\|_{\mathrm{L}^2_R} := (\int_0^{+\infty}\langle Ru(\tau),u(\tau)\rangle_{\mathbb{R}^m}\ d\tau)^{1/2}$ for every $u \in \mathrm{L}^2([0,+\infty),\mathbb{R}^m)$ is equivalent to the usual one since $R \in \mathcal{S}^m_{++}$. We conclude that (x^*,u^*) is an optimal solution to $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$.

Let us prove uniqueness. Note that $x(\cdot,\lambda u + (1-\lambda)v) = \lambda x(\cdot,u) + (1-\lambda)x(\cdot,v)$ for all u, $v \in \mathrm{L}^2([0,+\infty),\mathbb{R}^m)$ and all $\lambda \in [0,1]$. Hence, since moreover $Q \in \mathcal{S}^n_{++}$ and $R \in \mathcal{S}^m_{++}$, the cost function $\mathcal{C}$ is strictly convex, and thus the optimal solution to $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ is unique.

Existence of a solution to (SD-ARE). Let us introduce the sequence $(D_i)_{i\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$ being the solution to the forward matrix induction given by $D_{i+1} - D_i = -h\mathcal{F}(D_i,h)$ for $i \in \mathbb{N}$, $D_0 = 0_{\mathbb{R}^{n\times n}}$. Taking $P = 0_{\mathbb{R}^{n\times n}}$, one has $D_i = E^{t_N,\Delta_N}_{N-i}$ for every $i \in \{0,\dots,N\}$ and every $N \in \mathbb{N}^*$. Hence $(D_i)_{i\in\mathbb{N}}$ is well-defined and is in $\mathcal{S}^n_+$.

Let us prove that the sequence $(D_i)_{i\in\mathbb{N}}$ converges. Let $x_0 \in \mathbb{R}^n$. We set

$$M := \int_0^{+\infty}\Big(\langle Qx(\tau),x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau),u(\tau)\rangle_{\mathbb{R}^m}\Big)\,d\tau < +\infty,$$

where $(x,u) \in \mathrm{AC}([0,+\infty),\mathbb{R}^n) \times \mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$ is the pair provided in (H^h_2). Since the minimal cost of $(\mathrm{OCP}^{t_N,\Delta_N}_{x_0})$ (with $P = 0_{\mathbb{R}^{n\times n}}$) is $\langle E^{t_N,\Delta_N}_0 x_0,x_0\rangle_{\mathbb{R}^n} = \langle D_N x_0,x_0\rangle_{\mathbb{R}^n}$ and is increasing with respect to N, we deduce that $\langle D_N x_0,x_0\rangle_{\mathbb{R}^n}$ is increasing with respect to N. Since it is also bounded by M, we deduce that it converges when $N \to +\infty$. By (vii) of Lemma A.2, we conclude that the sequence $(D_i)_{i\in\mathbb{N}}$ in $\mathcal{S}^n_+$ converges to some $D \in \mathcal{S}^n_+$ which satisfies $\mathcal{F}(D,h) = 0_{\mathbb{R}^{n\times n}}$ by continuity of $\mathcal{F}$ (see Lemma A.3).

Positive definiteness of *D*. Let $x_0 \in \mathbb{R}^n\backslash\{0\}$. Since $Q \in \mathcal{S}^n_{++}$, the minimal cost of $(\mathrm{OCP}^{t_N,\Delta_N}_{x_0})$ (with $P = 0_{\mathbb{R}^{n\times n}}$) given by $\langle E^{t_N,\Delta_N}_0 x_0,x_0\rangle_{\mathbb{R}^n} = \langle D_N x_0,x_0\rangle_{\mathbb{R}^n}$

for every $N \in \mathbb{N}^*$ is positive. Since $\langle D_N x_0, x_0\rangle_{\mathbb{R}^n}$ is increasing with respect to N and converges to $\langle D x_0, x_0\rangle_{\mathbb{R}^n}$, we deduce that $\langle D x_0, x_0\rangle_{\mathbb{R}^n} > 0$ and thus $D \in \mathcal{S}^n_{++}$.

Lower bound of the minimal cost of $(\mathbf{OCP}^{\infty,\Delta}_{x_0})$. Our aim in this subsection is to prove that if $Z \in \mathcal{S}^n_+$ satisfies $\mathcal{F}(Z,h) = 0_{\mathbb{R}^{n\times n}}$, then $\langle Z x_0, x_0\rangle_{\mathbb{R}^n}$ is a lower bound of the minimal cost of $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ for every $x_0 \in \mathbb{R}^n$.

Let $x_0 \in \mathbb{R}^n$. Let $(x,u) \in \mathrm{AC}([0,+\infty),\mathbb{R}^n) \times \mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$ be a pair such that $\dot{x}(t) = Ax(t) + Bu(t)$ for a.e. $t \geq 0$ and $x(0) = x_0$. Let us prove that

$$\langle Z x_0, x_0\rangle_{\mathbb{R}^n} \leq \int_0^{+\infty} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau.$$

If the integral on the right-hand side is infinite, the result is obvious. Let us assume that the integral is finite. By Lemma A.5, $x(t)$ tends to $0_{\mathbb{R}^n}$ when $t \to +\infty$. By Proposition 2.4, the minimal cost of $(\mathrm{OCP}^{t_N,\Delta_N}_{x_0})$ with $P = Z$ is given by $\langle E^{t_N,\Delta_N}_0 x_0, x_0\rangle_{\mathbb{R}^n}$ for every $N \in \mathbb{N}^*$. Since $E^{t_N,\Delta_N}_N = Z$ and $\mathcal{F}(Z,h) = 0_{\mathbb{R}^{n\times n}}$, from the backward matrix induction, we get that $E^{t_N,\Delta_N}_i = Z$ for every $i \in \{0,\dots,N\}$ and every $N \in \mathbb{N}^*$. In particular the minimal cost of $(\mathrm{OCP}^{t_N,\Delta_N}_{x_0})$ with $P = Z$ is given by $\langle Z x_0, x_0\rangle_{\mathbb{R}^n}$ for every $N \in \mathbb{N}^*$. Hence

$$\langle Z x_0, x_0\rangle_{\mathbb{R}^n} \leq \langle Zx(t_N), x(t_N)\rangle_{\mathbb{R}^n} + \int_0^{t_N} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau.$$

Taking the limit $N \to +\infty$, the proof is complete.

Upper bound of the minimal cost of $(\mathbf{OCP}^{\infty,\Delta}_{x_0})$. Our aim in this subsection is to prove that if $Z \in \mathcal{S}^n_+$ satisfies $\mathcal{F}(Z,h) = 0_{\mathbb{R}^{n\times n}}$, then $\langle Z x_0, x_0\rangle_{\mathbb{R}^n}$ is an upper bound of the minimal cost of $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ for every $x_0 \in \mathbb{R}^n$. Denote $\mathcal{M} := \mathcal{M}(Z,h)$, $\mathcal{N} := \mathcal{N}(Z,h)$, and $\mathcal{G} := \mathcal{G}(Z,h)$. We similarly use the notation $\mathcal{M}_i$, $\mathcal{N}_i$, and $\mathcal{G}_i$ for $i = 1,2,3$ (see section 2.1 for details).

Let $x_0 \in \mathbb{R}^n$. Let $x \in \mathrm{AC}([0,+\infty),\mathbb{R}^n)$ be the unique solution to

$$\dot{x}(t) = Ax(t) - B\mathcal{N}^{-1}\mathcal{M}^\top x(t_i) \quad \text{for a.e. } t \in [t_i,t_{i+1}) \quad \forall i \in \mathbb{N}, \qquad x(0) = x_0,$$

and let $u \in \mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$ defined by $u_i := -\mathcal{N}^{-1}\mathcal{M}^\top x(t_i)$ for every $i \in \mathbb{N}$. In particular $\dot{x}(t) = Ax(t) + Bu(t)$ for a.e. $t \geq 0$ and $x(0) = x_0$.

By the Duhamel formula, we have $x(t) = (\alpha_i(t) - \beta_i(t))x(t_i)$ for all $t \in [t_i,t_{i+1})$ and every $i \in \mathbb{N}$, where

$$\alpha_i(t) := e^{(t-t_i)A} \quad \text{and} \quad \beta_i(t) := \left(\int_{t_i}^t e^{\xi A}\, d\xi \right) B\mathcal{N}^{-1}\mathcal{M}^\top \quad \forall t \in [t_i,t_{i+1}) \quad \forall i \in \mathbb{N}.$$

Using the above expressions of α_i and β_i, and after some computations, we get that

$$\int_{t_i}^{t_{i+1}} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau = h\langle W_1 x(t_i), x(t_i)\rangle_{\mathbb{R}^n} \qquad \forall i \in \mathbb{N},$$

where $W_1 := \mathcal{G}_1 + \mathcal{M}\mathcal{N}^{-1}\mathcal{N}_2\mathcal{N}^{-1}\mathcal{M}^\top - 2\mathcal{M}_2\mathcal{N}^{-1}\mathcal{M}^\top + \mathcal{M}\mathcal{N}^{-1}\mathcal{N}_1\mathcal{N}^{-1}\mathcal{M}^\top$. On the other hand, using again the above expressions of α_i and β_i, we compute

$$\langle Zx(t_i), x(t_i)\rangle_{\mathbb{R}^n} - \langle Zx(t_{i+1}), x(t_{i+1})\rangle_{\mathbb{R}^n} = h\langle W_2 x(t_i), x(t_i)\rangle_{\mathbb{R}^n} \qquad \forall i \in \mathbb{N},$$

where $W_2 := -\mathcal{G}_2 + 2\mathcal{M}_1\mathcal{N}^{-1}\mathcal{M}^\top - \mathcal{M}\mathcal{N}^{-1}\mathcal{N}_3\mathcal{N}^{-1}\mathcal{M}$. Using the equality $\mathcal{F}(Z,h) = \mathcal{M}\mathcal{N}^{-1}\mathcal{M}^\top - \mathcal{G} = 0_{\mathbb{R}^{n\times n}}$, we obtain $W_2 - W_1 = 0_{\mathbb{R}^{n\times n}}$ and thus $W_2 = W_1$. Hence

$$\int_{t_i}^{t_{i+1}} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau = \langle Zx(t_i), x(t_i)\rangle_{\mathbb{R}^n} - \langle Zx(t_{i+1}), x(t_{i+1})\rangle_{\mathbb{R}^n} \qquad \forall i \in \mathbb{N}.$$

Summing these equalities and using that $Z \in \mathcal{S}^n_+$, we get

$$\int_0^{t_N} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau \leq \langle Zx_0, x_0\rangle_{\mathbb{R}^n} \qquad \forall N \in \mathbb{N}^*.$$

Passing to the limit $N \to +\infty$, we finally obtain

$$\int_0^{+\infty} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau \leq \langle Zx_0, x_0\rangle_{\mathbb{R}^n},$$

i.e., $\langle Zx_0, x_0\rangle_{\mathbb{R}^n}$ is an upper bound of the minimal cost of $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ for every $x_0 \in \mathbb{R}^n$.

Minimal cost of $(\mathbf{OCP}^{\infty,\Delta}_{x_0})$ and state feedback control. Let $x_0 \in \mathbb{R}^n$. By the previous subsections, since $D \in \mathcal{S}^n_{++} \subset \mathcal{S}^n_+$ satisfies $\mathcal{F}(D,h) = 0_{\mathbb{R}^{n\times n}}$, the minimal cost of $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ is equal to $\langle Dx_0, x_0\rangle_{\mathbb{R}^n}$. Moreover, by the previous subsection, denoting by $x \in \mathrm{AC}([0,+\infty),\mathbb{R}^n)$ the unique solution to

$$\dot{x}(t) = Ax(t) - B\mathcal{N}(D,h)^{-1}\mathcal{M}(D,h)^\top x(t_i) \quad \text{on } [t_i, t_{i+1}) \quad \forall i \in \mathbb{N}, \qquad x(0) = x_0,$$

and by $u \in \mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$ the control given by $u_i := -\mathcal{N}(D,h)^{-1}\mathcal{M}(D,h)^\top x(t_i)$ for every $i \in \mathbb{N}$, we get that $\dot{x}(t) = Ax(t) + Bu(t)$ for a.e. $t \geq 0$ and $x(0) = x_0$, and

$$\int_0^{+\infty} \Big(\langle Qx(\tau), x(\tau)\rangle_{\mathbb{R}^n} + \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^m} \Big)\, d\tau \leq \langle Dx_0, x_0\rangle_{\mathbb{R}^n}.$$

Since $\langle Dx_0, x_0\rangle_{\mathbb{R}^n}$ is the minimal cost of $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$, the above inequality is actually an equality. By uniqueness of the optimal solution (x^*, u^*), we get that $(x,u) = (x^*,u^*)$, and thus the optimal sampled-data control u^* is given by $u^*_i = -\mathcal{N}(D,h)^{-1}\mathcal{M}(D,h)^\top x^*(t_i)$ for every $i \in \mathbb{N}$.

Uniqueness of the solution to (SD-ARE). Assume that there exist $Z_1, Z_2 \in \mathcal{S}^n_+$ satisfying $\mathcal{F}(Z_1,h) = \mathcal{F}(Z_2,h) = 0_{\mathbb{R}^{n\times n}}$. By the previous subsections, the minimal cost of $(\mathrm{OCP}^{\infty,\Delta}_{x_0})$ is equal to $\langle Z_1x_0, x_0\rangle_{\mathbb{R}^n} = \langle Z_2x_0, x_0\rangle_{\mathbb{R}^n}$ for every $x_0 \in \mathbb{R}^n$. By (vi) of Lemma A.2, we conclude that $Z_1 = Z_2$.

End of the proof. Defining $E^{\infty,\Delta} := D \in \mathcal{S}^n_{++}$, Proposition 2.7 is proved.

A.3. Proof of Lemma 3.1. This proof is inspired from the techniques developed in [36] for preserving the stabilizing property of controls of nonlinear systems under sampling. We set $W := BR^{-1}B^\top E^\infty \in \mathbb{R}^{n\times n}$, where E^∞ is given by Proposition 2.6. Note that $E^\infty W \in \mathcal{S}^n_+$. Using (P-ARE), we obtain

$$2\langle E^\infty y, (A-W)y\rangle_{\mathbb{R}^n} = -\langle Qy, y\rangle_{\mathbb{R}^n} - \langle E^\infty W y, y\rangle_{\mathbb{R}^m} \leq -\rho_{\min}(Q)\|y\|^2_{\mathbb{R}^n} \qquad \forall y \in \mathbb{R}^n,$$

where $\rho_{\min}(Q) > 0$ since $Q \in \mathcal{S}^n_{++}$. Let $\overline{h} > 0$ be such that, for every $h \in (0,\overline{h}]$, $h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}} < 1$ and

$$2\rho_{\max}(E^\infty W)\frac{h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}}}{1 - h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}}} \leq \frac{\rho_{\min}(Q)}{2}.$$

Let $x_0 \in \mathbb{R}^n$, and let $\Delta = \{t_i\}_{i\in\mathbb{N}}$ be an h-uniform time partition of $[0,+\infty)$ satisfying $h \in (0,\overline{h}]$. Let $x \in \mathrm{AC}([0,+\infty),\mathbb{R}^n)$ be the unique solution to

$$\dot{x}(t) = Ax(t) - Wx(t_i) \quad \text{for a.e. } t \in [t_i,t_{i+1}) \quad \forall i \in \mathbb{N}, \qquad x(0) = x_0,$$

and let $u \in \mathrm{PC}^\Delta([0,+\infty),\mathbb{R}^m)$ be defined by $u_i := -R^{-1}B^\top E^\infty x(t_i)$ for every $i \in \mathbb{N}$. In particular $\dot{x}(t) = Ax(t) + Bu(t)$ for a.e. $t \geq 0$ and $x(0) = x_0$. First, we have

$$\begin{aligned}\|x(t)-x(t_i)\|_{\mathbb{R}^n} &= \left\| \int_{t_i}^t \Big(Ax(\tau) - Wx(t_i) \Big)\, d\tau \right\|_{\mathbb{R}^n} \\ &= \left\| \int_{t_i}^t \Big(A(x(\tau)-x(t_i)) + (A-W)x(t_i) \Big)\, d\tau \right\|_{\mathbb{R}^n} \\ &\leq h\|A-W\|_{\mathbb{R}^{n\times n}}\|x(t_i)\|_{\mathbb{R}^n} + \|A\|_{\mathbb{R}^{n\times n}} \int_{t_i}^t \|x(\tau)-x(t_i)\|_{\mathbb{R}^n}\, d\tau,\end{aligned}$$

and, by the Grönwall lemma (see [40, Appendix C.3]), we get that

$$\|x(t)-x(t_i)\|_{\mathbb{R}^n} \leq h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}} \|x(t_i)\|_{\mathbb{R}^n} \qquad \forall t \in [t_i,t_{i+1}), \qquad \forall i \in \mathbb{N}.$$

Since $\|x(t_i)\|_{\mathbb{R}^{n\times n}} \leq \|x(t)-x(t_i)\|_{\mathbb{R}^{n\times n}} + \|x(t)\|_{\mathbb{R}^{n\times n}}$ and $h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}} < 1$, we get that $\|x(t_i)\|_{\mathbb{R}^{n\times n}} \leq \|x(t)\|_{\mathbb{R}^{n\times n}}/(1-h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}})$ and thus

$$\|x(t)-x(t_i)\|_{\mathbb{R}^{n\times n}} \leq \frac{h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}}}{1-h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}}} \|x(t)\|_{\mathbb{R}^{n\times n}}$$

for all $t \in [t_i,t_{i+1})$ and $i \in \mathbb{N}$. Second, we have

$$\begin{aligned}\frac{d}{dt}\langle E^\infty x(t),x(t)\rangle_{\mathbb{R}^n} &= 2\langle E^\infty x(t),\dot{x}(t)\rangle_{\mathbb{R}^n} = 2\langle E^\infty x(t), Ax(t)-Wx(t_i)\rangle_{\mathbb{R}^n} \\ &= 2\langle E^\infty x(t),(A-W)x(t)\rangle_{\mathbb{R}^n} + 2\langle E^\infty x(t), W(x(t)-x(t_i))\rangle_{\mathbb{R}^n}\end{aligned}$$

for a.e. $t \in [t_i,t_{i+1})$ and for every $i \in \mathbb{N}$. We deduce that

$$\begin{aligned}&\frac{d}{dt}\langle E^\infty x(t),x(t)\rangle_{\mathbb{R}^n} \\ &\quad \leq \left(-\rho_{\min}(Q) + 2\rho_{\max}(E^\infty W) \frac{h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}}}{1-h\|A-W\|_{\mathbb{R}^{n\times n}} e^{h\|A\|_{\mathbb{R}^{n\times n}}}} \right) \|x(t)\|^2_{\mathbb{R}^n} \\ &\quad \leq -\frac{\rho_{\min}(Q)}{2}\|x(t)\|^2_{\mathbb{R}^n} \leq -\frac{\rho_{\min}(Q)}{2\rho_{\max}(E^\infty)}\langle E^\infty x(t),x(t)\rangle_{\mathbb{R}^n}\end{aligned}$$

for a.e. $t \geq 0$. We deduce from the Grönwall lemma that

$$\|x(t)\|^2_{\mathbb{R}^n} \leq \frac{1}{\rho_{\min}(E^\infty)}\langle E^\infty x(t),x(t)\rangle_{\mathbb{R}^n} \leq \frac{1}{\rho_{\min}(E^\infty)}\langle E^\infty x_0,x_0\rangle_{\mathbb{R}^n} e^{-\frac{\rho_{\min}(Q)}{2\rho_{\max}(E^\infty)}t}$$

for every $t \geq 0$. Hence

$$\begin{aligned}\int_0^{+\infty} \langle Qx(\tau),x(\tau)\rangle_{\mathbb{R}^n}\, d\tau &\leq \frac{\rho_{\max}(Q)}{\rho_{\min}(E^\infty)}\langle E^\infty x_0,x_0\rangle_{\mathbb{R}^n} \int_0^{+\infty} e^{-\frac{\rho_{\min}(Q)}{2\rho_{\max}(E^\infty)}\tau}\, d\tau \\ &= \frac{2\rho_{\max}(Q)\rho_{\max}(E^\infty)}{\rho_{\min}(Q)\rho_{\min}(E^\infty)}\langle E^\infty x_0,x_0\rangle_{\mathbb{R}^n} < +\infty.\end{aligned}$$

Moreover, using that $t_i = ih$ for every $i \in \mathbb{N}$, we have

$$\begin{aligned}\int_0^{+\infty} \langle Ru(\tau), u(\tau)\rangle_{\mathbb{R}^n}\, d\tau &\leq h\rho_{\max}(R) \sum_{i\in\mathbb{N}} \|u_i\|^2_{\mathbb{R}^m} \\ &\leq h\rho_{\max}(R)\|R^{-1}B^\top E^\infty\|^2_{\mathbb{R}^{m\times n}} \sum_{i\in\mathbb{N}} \|x(t_i)\|^2_{\mathbb{R}^m} \\ &\leq h\frac{\rho_{\max}(R)}{\rho_{\min}(E^\infty)}\|R^{-1}B^\top E^\infty\|^2_{\mathbb{R}^{m\times n}}\langle E^\infty x_0, x_0\rangle_{\mathbb{R}^n} \sum_{i\in\mathbb{N}} \left(e^{-\frac{\rho_{\min}(Q)}{2\rho_{\max}(E^\infty)}h}\right)^i \\ &= h\frac{\rho_{\max}(R)}{\rho_{\min}(E^\infty)}\|R^{-1}B^\top E^\infty\|^2_{\mathbb{R}^{m\times n}}\langle E^\infty x_0, x_0\rangle_{\mathbb{R}^n} \frac{1}{1-e^{-\frac{\rho_{\min}(Q)}{2\rho_{\max}(E^\infty)}h}} \\ &\leq \frac{2\rho_{\max}(R)\rho_{\max}(E^\infty)}{\rho_{\min}(Q)\rho_{\min}(E^\infty)}\|R^{-1}B^\top E^\infty\|^2_{\mathbb{R}^{m\times n}}\langle E^\infty x_0, x_0\rangle_{\mathbb{R}^n} e^{\frac{\rho_{\min}(Q)}{2\rho_{\max}(E^\infty)}\overline{h}} < +\infty.\end{aligned}$$

Taking $\bar{c} := \frac{2\rho_{\max}(E^\infty)}{\rho_{\min}(Q)\rho_{\min}(E^\infty)}(\rho_{\max}(Q) + \rho_{\max}(R)\|R^{-1}B^\top E^\infty\|^2_{\mathbb{R}^{m\times n}} e^{\frac{\rho_{\min}(Q)}{2\rho_{\max}(E^\infty)}\overline{h}}) \geq 0$, the proof is complete. ∎

A.4. Proof of Theorem 3.2.

First item. This proof is inspired by the classical Lax theorem (see [37, p. 73]). Let $\varepsilon > 0$. By continuity of E^T on $[0, T]$ and by Lemma A.3, the map

$$\begin{array}{rccl} \varphi : & [0,T]\times[0,T] & \longrightarrow & \mathbb{R}^{n\times n} \\ & (t,h) & \longmapsto & \varphi(t,h) := \mathcal{F}(t, E^T(t), h) \end{array}$$

is uniformly continuous on the compact set $[0, T] \times [0, T]$. Hence there exists $\delta > 0$ such that $\|\varphi(t_2, h_2) - \varphi(t_1, h_1)\|_{\mathbb{R}^{n\times n}} \leq \frac{\varepsilon}{2Te^{cT}}$ for all (t_1, h_1), $(t_2, h_2) \in [0, T] \times [0, T]$ satisfying $|t_2 - t_1| + |h_2 - h_1| \leq \delta$, where $c \geq 0$ is the constant given in Lemma A.3 associated to the compact set $\mathcal{K} := [0, T] \times \mathrm{K} \times [0, T]$, where

$$\mathrm{K} := \left\{ E \in \mathcal{S}^n_+ \mid \|E\|_{\mathbb{R}^{n\times n}} \leq \left(\|P\|_{\mathbb{R}^{n\times n}} + T\|Q_{|[0,T]}\|_\infty\right) e^{2T\|A_{|[0,T]}\|_\infty} \right\}.$$

Let $\Delta = \{t_i\}_{i=0,\dots,N}$ be a time partition of $[0, T]$ such that $0 < \|\Delta\| \leq \delta$. Note that

$$\begin{aligned}&E^{T,\Delta}_i = E^{T,\Delta}_{i+1} - h_{i+1}\mathcal{F}(t_{i+1}, E^{T,\Delta}_{i+1}, h_{i+1}) \quad \text{and} \\ &E^T(t_i) = E^T(t_{i+1}) - h_{i+1}\mathcal{F}(t_{i+1}, E^T(t_{i+1}), h_{i+1}) + \eta_{i+1} \qquad \forall i \in \{0, \dots, N-1\},\end{aligned}$$

where $\eta_{i+1} := E^T(t_i) - E^T(t_{i+1}) + h_{i+1}\mathcal{F}(t_{i+1}, E^T(t_{i+1}), h_{i+1})$. By Lemmas A.3 and A.4, $\|E^T(t_i) - E^{T,\Delta}_i\|_{\mathbb{R}^{n\times n}} \leq (1 + ch_{i+1})\|E^T(t_{i+1}) - E^{T,\Delta}_{i+1}\|_{\mathbb{R}^{n\times n}} + \|\eta_{i+1}\|_{\mathbb{R}^{n\times n}}$, and it follows from the backward discrete Grönwall lemma (see Lemma A.1) that

$$\|E^T(t_i) - E^{T,\Delta}_i\|_{\mathbb{R}^{n\times n}} \leq \sum_{j=i+1}^N e^{c\sum_{q=i+1}^{j-1} h_q}\|\eta_j\|_{\mathbb{R}^{n\times n}} \leq e^{cT}\sum_{j=1}^N \|\eta_j\|_{\mathbb{R}^{n\times n}}$$

for every $i \in \{0, \dots, N-1\}$. Since

$$\begin{aligned}\eta_j &= h_j\Big(\mathcal{F}(t_j, E^T(t_j), h_j) - \mathcal{F}(t_j, E^T(t_j), 0)\Big) \\ &\qquad + \int_{t_{j-1}}^{t_j} \big(\mathcal{F}(t_j, E^T(t_j), 0) - \mathcal{F}(\tau, E^T(\tau), 0)\big)\, d\tau \\ &= h_j\Big(\varphi(t_j, h_j) - \varphi(t_j, 0)\Big) + \int_{t_{j-1}}^{t_j} (\varphi(t_j, 0) - \varphi(\tau, 0))\, d\tau \qquad \forall j \in \{1, \dots, N\}\end{aligned}$$

we obtain, by uniform continuity of φ and using that $0 < \|\Delta\| \leq \delta$, that $\|\eta_j\|_{\mathbb{R}^{n\times n}} \leq 2h_j \frac{\varepsilon}{2Te^{cT}} = h_j \frac{\varepsilon}{Te^{cT}}$ for every $j \in \{1,\dots,N\}$. We conclude that

$$\|E^T(t_i) - E_i^{T,\Delta}\|_{\mathbb{R}^{n\times n}} \leq e^{cT} \sum_{j=1}^{N} \|\eta_j\|_{\mathbb{R}^{n\times n}} \leq e^{cT} \sum_{j=1}^{N} h_j \frac{\varepsilon}{Te^{cT}} = \frac{\varepsilon}{T} \sum_{j=1}^{N} h_j = \varepsilon$$

for every $i \in \{0,\dots,N-1\}$. The proof is complete.

Second item. The second item of Theorem 3.2 is well known and follows from the proof of Proposition 2.6 (see [12, p. 153], [28, Theorem 7], or [45, Theorem 4.13]).

Third item. This result follows from the proof of Proposition 2.7. Indeed, using the notation from Appendix A.2, we have $\lim_{N\to+\infty} E_i^{t_N,\Delta_N} = \lim_{N\to+\infty} D_{N-i} = D = E^{\infty,\Delta}$ for every $i \in \mathbb{N}$.

Fourth item. By contradiction, let us assume that $E^{\infty,\Delta}$ does not converge to E^∞ when $h \to 0$. Then there exist $\varepsilon > 0$ and a positive sequence $(h_k)_{k\in\mathbb{N}}$ converging to 0 such that $\|E^{\infty,\Delta_k} - E^\infty\|_{\mathbb{R}^{n\times n}} \geq \varepsilon$ for every $k \in \mathbb{N}$, where Δ_k stands for the h_k-uniform time partition of the interval $[0,+\infty)$. Without loss of generality, we assume that $0 < h_k \leq \overline{h}$ for every $k \in \mathbb{N}$. It follows from Proposition 2.7 and from Lemma 3.1 that the minimal cost of $(\mathrm{OCP}_{x_0}^{\infty,\Delta_k})$ satisfies

$$\langle E^{\infty,\Delta_k} x_0, x_0\rangle_{\mathbb{R}^n} \leq \overline{c}\langle E^\infty x_0, x_0\rangle_{\mathbb{R}^n} \leq \overline{c}\|E^\infty\|_{\mathbb{R}^{n\times n}}\|x_0\|^2_{\mathbb{R}^n} \qquad \forall x_0 \in \mathbb{R}^n.$$

Hence $\|E^{\infty,\Delta_k}\|_{\mathbb{R}^{n\times n}} \leq \overline{c}\|E^\infty\|_{\mathbb{R}^{n\times n}}$ for every $k \in \mathbb{N}$ by (v) of Lemma A.2. Thus the sequence $(E^{\infty,\Delta_k})_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}^{n\times n}$ and, up to a subsequence (that we do not relabel), converges to some $L \in \mathbb{R}^{n\times n}$. In particular $\|L - E^\infty\|_{\mathbb{R}^{n\times n}} \geq \varepsilon$. Since $E^{\infty,\Delta_k} \in \mathcal{S}^n_{++} \subset \mathcal{S}^n_+$ for every $k \in \mathbb{N}$, it is clear that $L \in \mathcal{S}^n_+$. Moreover, by (SD-ARE) associated to h_k (see Proposition 2.7), we know that $\mathcal{F}(E^{\infty,\Delta_k}, h_k) = 0_{\mathbb{R}^{n\times n}}$ for all $k \in \mathbb{N}$. By continuity of $\mathcal{F}$ (see Lemma A.3), we conclude that $\mathcal{F}(L,0) = 0_{\mathbb{R}^{n\times n}}$. By uniqueness (see Proposition 2.6) we deduce that $L = E^\infty$, which raises a contradiction with the inequality $\|L - E^\infty\|_{\mathbb{R}^{n\times n}} \geq \varepsilon$. The proof is complete. ∎

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