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General Cauchy–Lipschitz theory for Δ-Cauchy problems with Carathéodory dynamics on time scales

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The aim of this paper was to complete some aspects of the classical Cauchy–Lipschitz (or Picard–Lindelöf) theory for general nonlinear systems posed on time scales. Despite a rich literature on Cauchy–Lipschitz type results on time scales, most of the existing results are concerned with rd-continuous dynamics (and $\mathscr{C}^1_{\mathrm{rd}}$-solutions) and do not cover the framework of general Carathéodory dynamics encountered for instance in control theory with measurable controls (which are not rd-continuous in general). In this paper, our main objective was to study Δ-Cauchy problems with general Carathéodory dynamics. We introduce the notion of absolutely continuous solution (weaker regularity than $\mathscr{C}^1_{\mathrm{rd}}$) and then the notion of maximal solution. We state and prove a Cauchy–Lipschitz theorem, providing existence and uniqueness of the maximal solution of a given Δ-Cauchy problem under suitable assumptions such as regressivity and local Lipschitz continuity. Three new related issues are also discussed in this paper: the boundary value is not necessarily an initial or a final condition, the solutions are constrained to take their values in a non-empty open subset and the behaviour of maximal solutions at terminal points is studied.

Keywords: time scale; Cauchy–Lipschitz (Picard–Lindelöf) theory; existence; uniqueness

AMS Subject Classification: 34N99; 34G20; 39A13; 39A12

1. Introduction

The *time scale* theory was introduced by S. Hilger in his PhD thesis [16] in 1988 in order to unify discrete and continuous analysis, with the general idea of extending classical theories on an arbitrary non-empty *closed* subset $\mathbb{T}$ of $\mathbb{R}$. Such a subset $\mathbb{T}$ is called a *time scale*. The objective is not only to establish the validity of some results both in the continuous case $\mathbb{T} = \mathbb{R}$ and in the purely discrete case $\mathbb{T} = \mathbb{N}$, but also to treat more general models of processes involving both continuous and discrete time elements. We refer the reader, e.g., to [14,28] where the authors study a seasonally breeding population whose generations do not overlap or to [4] for applications to economy. By considering $\mathbb{T} = \{0\} \cup \lambda^{\mathbb{N}}$ with $0 < \lambda < 1$, the time scale concept also allows to cover quantum calculus [22]. Since S. Hilger defined the Δ-derivative and the Δ-integral on a time scale, many authors have extended to time scales various results from the continuous or discrete standard calculus theory (see the surveys [1,2,6,7]). However some items of the basic nonlinear theory are still to be developed and refined.

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Cauchy–Lipschitz (Picard–Lindelöf) type results on time scales are provided in [6,12,17,24–26] where the authors prove the existence and uniqueness of solutions for Δ-Cauchy problems of the form

$$q^\Delta = f(q,t), \quad q(t_0) = q_0, \tag{1}$$

where $t_0 \in \mathbb{T}$. Note that papers are devoted to Δ-Cauchy problems with parameters in [20] and with time delays in [23]. Many authors are also interested in shifted Δ-Cauchy problems

$$q^\Delta = f(q^\sigma, t), \quad q(t_0) = q_0, \tag{2}$$

where $q^\sigma = q \circ \sigma$ (see further for the precise definition of σ). Such shifted problems are often used as models in the existing literature (see, e.g. [5,19,27], [18, Remark 3.9] and [20, Remark 3.6]), because they emerge in adjoint equations according to the Leibniz formula (see [6])

$$(q_1 q_2)^\Delta = q_1^\Delta q_2^\sigma + q_1 q_2^\Delta = q_1^\Delta q_2 + q_1^\sigma q_2^\Delta. \tag{3}$$

To the best of our knowledge, most of the existing results are concerned with rd-continuous dynamics f (and $\mathscr{C}^1_{\mathrm{rd}}$-solutions q) (see e.g. the first result on Δ-Cauchy problems due to Hilger [17, Section 5], or [6, Section 8.2], [24,26,29] and references therein). Nevertheless, they do not cover the framework of general Carathéodory dynamics f (not necessarily rd-continuous but only measurable in its variable t), encountered for instance in control theory with measurable controls (that are not rd-continuous in general). Our main objective in this paper is to treat such general Carathéodory dynamics (weaker regularity than rd-continuity) and to obtain existence and uniqueness results of absolutely continuous solutions (weaker regularity than $\mathscr{C}^1_{\mathrm{rd}}$).[2]

Note that the articles [12] and [25], respectively, deal with weak continuous and Carathéodory dynamics living in a general Banach space. Nevertheless, they only treat the non-shifted case (1) where q_0 is moreover an initial condition, that is solutions are only defined for $t \geq t_0$. In view of considering adjoint equations, it is of interest to study backward Δ-Cauchy problems where q_0 is a final condition, for which solutions are defined for $t \leq t_0$. As is well known in time scale calculus, the solvability of such backward non-shifted Δ-Cauchy problems requires a *regressivity* assumption on the dynamics (see e.g. [6,17] and [18, Remark 3.8]). This important issue is not addressed in [12,25]. Another issue that is not addressed in the literature is the fact that the classical Cauchy–Lipschitz theory (see, e.g. [13,21]) treats Cauchy problems constraining the solutions to take their values in a non-empty open subset Ω of $\mathbb{R}^n$. Finally, to the best of our knowledge, the notion of extension of a solution on time scales and the behaviour of the maximal solution at terminal points have not been studied in the literature.

This paper is devoted to fill an existing gap of the literature and to provide a general Cauchy–Lipschitz theory with Carathéodory dynamics f (only measurable in t) on time scales, generalizing the basic notions and results of the classical theory surveyed, e.g. in [13,21]. We first introduce the notion of *absolutely continuous solution* in the time scale context. Then we define the concept of *extension* of a solution and of *maximal* and *global* solutions. We establish a general version of the Cauchy–Lipschitz theorem (existence and uniqueness of the maximal solution, also referred to as Picard–Lindelöf theorem) under regressivity and local Lipschitz continuity assumptions for shifted and non-shifted general nonlinear Δ-Cauchy problems. Here we summarize the novelties of this paper:

- the dynamics f is a general Δ-Carathéodory function (where Δ-measure μ_Δ on a time scale $\mathbb{T}$ is defined in terms of Carathéodory extension in [7, chapter 5]), in contrast to the rd-continuity assumed in most of the existing literature. Consequently, the solutions obtained are absolutely continuous (more general than $\mathscr{C}^1_{\mathrm{rd}}$);
- the boundary value q_0 is not necessarily an initial or a final condition, i.e. t_0 is not necessarily equal to $\min \mathbb{T}$ or $\max \mathbb{T}$;
- the solutions are constrained to take their values in a non-empty open subset Ω of $\mathbb{R}^n$.
- the behaviour of maximal solutions that are not global is investigated: we prove that they leave any compact of Ω.

Remark 1. We stress that, in the absence of a regressivity assumption, a shifted problem cannot be reduced to an equivalent non-shifted problem in general.[3] Therefore our results are established first for general non-shifted Δ-Cauchy problems (1) and then, separately, for shifted Δ-Cauchy problems (2). Note that analogous results on ∇-Cauchy problems (ρ-shifted or not) can be derived in a similar way. We refer to the duality principle provided in [11].

Remark 2. Our study uses the work of Cabada and Vivero [9], who derived a criterion for absolutely continuous functions written as the Δ-integral of their Δ-derivatives. Their result allows us to give a Δ-integral characterization of the solutions of Δ-Cauchy problems with Carathéodory dynamics which is instrumental in our proofs.

The paper is structured as follows. Section 2 is devoted to recall basic notions of time scale calculus. In Section 3, we define the notions of a solution, of an extension of a solution, of a maximal and a global solution for general non-shifted Δ-Cauchy problems. Under suitable assumptions on the dynamics, we establish a Cauchy–Lipschitz theorem and then investigate the behaviour of the maximal solution at its terminal points. Section 4 is devoted to establish similar results for *shifted* Δ-Cauchy problems.

2. Preliminaries on time scale calculus

In this section, we recall basic results in time scale calculus. The first part concerns the structure of time scales and the notion of Δ-differentiability (see [6]). The second part concerns the Δ-Lebesgue measure defined in terms of Carathéodory extension (see [7,15]) and surveys results on Δ-integrability proved in [10]. The last part gathers the properties of absolutely continuous functions borrowed from [9].

Let $n \in \mathbb{N}^*$. Throughout, the notation $\|\cdot\|$ stands for the Euclidean norm of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ and every $R \geq 0$, the notation $\bar{B}(x, R)$ stands for the closed ball of $\mathbb{R}^n$ centred at x and with radius R.

2.1 Time scale and Δ-differentiability

Let $\mathbb{T}$ be a time scale, that is a non-empty closed subset of $\mathbb{R}$. We assume that $\mathrm{card}(\mathbb{T}) \geq 2$. For every $A \subset \mathbb{R}$, we denote $A_\mathbb{T} = A \cap \mathbb{T}$. An interval of $\mathbb{T}$ is defined by $I_\mathbb{T}$ where I is an interval of $\mathbb{R}$.

The backward and forward jump operators $\rho, \sigma : \mathbb{T} \to \mathbb{T}$ are, respectively, defined by

$$\rho(t) = \sup\{s \in \mathbb{T} | s < t\}, \quad \sigma(t) = \inf\{s \in \mathbb{T} | s > t\},$$

for every $t \in \mathbb{T}$, where $\rho(\min \mathbb{T}) = \min \mathbb{T}$ (respectively, $\sigma(\max \mathbb{T}) = \max \mathbb{T}$) whenever $\mathbb{T}$ admits a minimum (respectively, a maximum).

A point $t \in \mathbb{T}$ is said to be a left-scattered (respectively, right-scattered) point of $\mathbb{T}$ if $\rho(t) < t$ (respectively, $\sigma(t) > t$). A point $t \in \mathbb{T}$ is said to be a left-dense (respectively, right-dense) point of $\mathbb{T}$ if $t > \inf \mathbb{T}$ and $\rho(t) = t$ (respectively, $t < \sup \mathbb{T}$ and $\sigma(t) = t$). The graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\mu(t) = \sigma(t) - t$ for every $t \in \mathbb{T}$.

We set $\mathbb{T}^\kappa = \mathbb{T} \backslash \{\max \mathbb{T}\}$ whenever $\mathbb{T}$ admits a left-scattered maximum, and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise. A function $q : \mathbb{T} \to \mathbb{R}^n$ is said to be Δ-differentiable at $t \in \mathbb{T}^\kappa$ if the limit

$$q^\Delta(t) = \lim_{\substack{s \to t \\ s \in \mathbb{T}}} \frac{q^\sigma(t) - q(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^n$, where $q^\sigma = q \circ \sigma$. We recall the following well-known results (see [6]):

- if $t \in \mathbb{T}^\kappa$ is a right-dense point of $\mathbb{T}$, then q is Δ-differentiable at t if and only if the limit

$$q^\Delta(t) = \lim_{\substack{s \to t \\ s \in \mathbb{T}}} \frac{q(t) - q(s)}{t - s}$$

  exists in $\mathbb{R}^n$;
- if $t \in \mathbb{T}^\kappa$ is a right-scattered point of $\mathbb{T}$ and if q is continuous at t, then q is Δ-differentiable at t, and

$$q^\Delta(t) = \frac{q^\sigma(t) - q(t)}{\mu(t)}.$$

2.2 Lebesgue Δ-measure and Lebesgue Δ-integrability

Recall that the set of right-scattered points $\mathscr{R} \subset \mathbb{T}$ is at most countable (see [10, Lemma 3.1]).

Let μ_Δ be the Lebesgue Δ-measure on $\mathbb{T}$ defined in terms of Carathéodory extension in [7, Chapter 5]. We also refer the reader to [3,10,15] for more details on the μ_Δ-measure theory. In particular, for all elements a, b of $\mathbb{T}$ such that $a \leq b$, one has $\mu_\Delta([a, b)_\mathbb{T}) = b - a$. Recall that $A \subset \mathbb{T}$ is a μ_Δ-measurable set of $\mathbb{T}$ if and only if A is an usual μ_L-measurable set of $\mathbb{R}$, where μ_L denotes the usual Lebesgue measure (see [10, Proposition 3.1]). Moreover, if $A \subset \mathbb{T} \backslash \{\sup \mathbb{T}\}$, then

$$\mu_\Delta(A) = \mu_\mathrm{L}(A) + \sum_{r \in A \cap \mathscr{R}} \mu(r).$$

Let $A \subset \mathbb{T}$. A property is said to hold Δ-almost everywhere (shortly Δ-a.e.) on A if it holds for every $t \in A \backslash A_0$, where $A_0 \subset A$ is some μ_Δ-measurable subset of $\mathbb{T}$ satisfying $\mu_\Delta(A_0) = 0$. In particular, since $\mu_\Delta(\{r\}) = \mu(r) > 0$ for every $r \in \mathscr{R}$, we conclude that if a property holds Δ-a.e. on A, then it holds for every $r \in A \cap \mathscr{R}$.

Let $A \subset \mathbb{T} \backslash \{\sup \mathbb{T}\}$ be a μ_Δ-measurable set of $\mathbb{T}$. Consider a function q defined Δ-a.e. on A with values in $\mathbb{R}^n$. Let $\tilde{A} = A \cup (r, \sigma(r))_{r \in A \cap \mathscr{R}}$, and let $\tilde{q}$ be the extension of q defined

μ_{L}-a.e. on $\tilde{A}$ by

$$\tilde{q}(t) = \begin{cases} q(t) & \text{if } \ t \in A, \\ q(r) & \text{if } \ t \in (r, \sigma(r)) \text{ for every } \ r \in A \cap \mathscr{R}. \end{cases}$$

We recall that q is μ_Δ-measurable on A if and only if $\tilde{q}$ is μ_L-measurable on $\tilde{A}$ (see [10, Proposition 4.1]).

The functional space $\mathrm{L}^\infty_{\mathbb{T}}(A, \mathbb{R}^n)$ is the set of all functions q defined Δ-a.e. on A, with values in $\mathbb{R}^n$, which are μ_Δ-measurable on A and such that

$$\sup_{\tau \in A} \operatorname{ess} \|q(\tau)\| < +\infty.$$

Endowed with the norm $\|q\|_{\mathrm{L}^\infty_{\mathbb{T}}(A)} = \sup \operatorname{ess}_{\tau \in A} \|q(\tau)\|$, it is a Banach space (see [3, Theorem 2.5]). The functional space $\mathrm{L}^1_{\mathbb{T}}(A, \mathbb{R}^n)$ is the set of all functions q defined Δ-a.e. on A, with values in $\mathbb{R}^n$, which are μ_Δ-measurable on A and such that

$$\int_A \|q(\tau)\| \Delta\tau < +\infty.$$

Endowed with the norm $\|q\|_{\mathrm{L}^1_{\mathbb{T}}(A,\mathbb{R}^n)} = \int_A \|q(\tau)\|\Delta\tau$, it is a Banach space (see [3, Theorem 2.5]). Recall that if $q \in \mathrm{L}^1_{\mathbb{T}}(A, \mathbb{R}^n)$ then

$$\int_A q(\tau)\Delta\tau = \int_{\tilde{A}} \tilde{q}(\tau)\, \mathrm{d}\tau = \int_A q(\tau)\, \mathrm{d}\tau + \sum_{r \in A \cap \mathscr{R}} \mu(r) q(r)$$

(see [10, Theorems 5.1 and 5.2]). Note that if A is bounded then $\mathrm{L}^\infty_{\mathbb{T}}(A, \mathbb{R}^n) \subset \mathrm{L}^1_{\mathbb{T}}(A, \mathbb{R}^n)$.

2.3 Properties of absolutely continuous functions

Let a and b be two elements of $\mathbb{T}$ such that $a < b$. Let $\mathscr{C}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ denote the space of continuous functions defined on $[a,b]_{\mathbb{T}}$ with values in $\mathbb{R}^n$. Endowed with its usual norm $\|\cdot\|_\infty$, it is a Banach space. Let $\mathrm{AC}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ denote the subspace of absolutely continuous functions. We recall the two following results.

PROPOSITION 1. *Let $t_0 \in [a,b]_{\mathbb{T}}$ and $q : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$. Then $q \in \mathrm{AC}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ if and only if the two following conditions are satisfied:*

1. q *is Δ-differentiable Δ-a.e. on $[a,b)_{\mathbb{T}}$ and $q^\Delta \in \mathrm{L}^1_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R}^n)$;*
2. *For every $t \in [a,b]_{\mathbb{T}}$, there holds*

$$q(t) = q(t_0) + \int_{[t_0,t)_{\mathbb{T}}} q^\Delta(\tau)\Delta\tau$$

whenever $t \geq t_0$, and

$$q(t) = q(t_0) - \int_{[t,t_0)_{\mathbb{T}}} q^\Delta(\tau)\Delta\tau$$

whenever $t \leq t_0$.

This result can be easily derived from [9, Theorem 4.1]. By combining Proposition 1 and the usual Lebesgue's point theory in $\mathbb{R}$, we infer the following result (see also [30] for a similar result).

Proposition 2. *Let* $t_0 \in [a,b]_{\mathbb{T}}$ *and* $q \in \mathrm{L}^1_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R}^n)$. *Let* Q *be the function defined on* $[a,b]_{\mathbb{T}}$ *by*

$$Q(t) = \int_{[t_0,t)_{\mathbb{T}}} q(\tau)\Delta\tau$$

if $t \geq t_0$, *and by*

$$Q(t) = -\int_{[t,t_0)_{\mathbb{T}}} q(\tau)\Delta\tau$$

if $t \leq t_0$. *Then* $Q \in \mathrm{AC}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ *and* $Q^\Delta = q$ Δ*-a.e. on* $[a,b)_{\mathbb{T}}$.

Note that, in the results above, for $t \geq t_0$ the integral is taken over $[t_0,t)_{\mathbb{T}}$ (open at t), whereas for $t \leq t_0$, the integral is taken over $[t,t_0)_{\mathbb{T}}$ (closed at t). This will have some consequences in the forward or backward solvability of a Δ-Cauchy problem (see the remark following Lemma 1 further).

3. General non-shifted Δ-Cauchy problem

Throughout this section we consider the general non-shifted Δ-Cauchy problem

$$q^\Delta(t) = f(q(t),t), \quad q(t_0) = q_0, \qquad (\Delta\text{-CP})$$

where $t_0 \in \mathbb{T}$, $q_0 \in \Omega$, where Ω is a non-empty open subset of $\mathbb{R}^n$, and $f : \Omega \times \mathbb{T} \backslash \{\sup \mathbb{T}\} \to \mathbb{R}^n$ is a Δ-Carathéodory function. The notation $\mathcal{K}$ stands for the set of compact subsets of Ω.

3.1 Preliminaries

In what follows it will be important to distinguish between three cases:

1. $t_0 = \min \mathbb{T}$;
2. $t_0 = \max \mathbb{T}$;
3. $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$.

Indeed, the interval of definition of a solution of (Δ-CP) will depend on the specific case under consideration. If $t_0 = \min \mathbb{T}$, then a solution can only *go forward* since $(-\infty, t_0)_{\mathbb{T}} = \emptyset$. If $t_0 = \max \mathbb{T}$, then a solution can only *go backward*. If $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$, then a solution can go *backward* and *forward*.

Definition 1. *For all* $(a,b) \in \mathbb{T}^2$, *we say that* $a \trianglelefteq t_0 \trianglelefteq b$ *if*

- $a = t_0 < b$ *in the case* $t_0 = \min \mathbb{T}$;
- $a < t_0 = b$ *in the case* $t_0 = \max \mathbb{T}$;
- $a < t_0 < b$ *in the case* $t_0 \neq \inf \mathbb{T}$ *and* $t_0 \neq \sup \mathbb{T}$.

If $a \trianglelefteq t_0 \trianglelefteq b$ then $[a,b]_{\mathbb{T}}$ is a potential interval of definition for a solution of (Δ-CP). Due to this difference in intervals, it is required to make different assumptions on f accordingly, whence the following series of definitions.

DEFINITION 2. *The function* f *is said to be* locally bounded on $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$ *if, for every* $K \in \mathcal{K}$*, for all* $(a,b) \in \mathbb{T}^2$ *such that* $a < b$*, there exists* $M \geq 0$ *such that*

$$\|f(x,t)\| \leq M, \qquad (\mathrm{H}_\infty)$$

for every $x \in K$ *and for* Δ*-a.e.* $t \in [a,b)_{\mathbb{T}}$.

In what follows this property will be referred to as (H_∞).

DEFINITION 3. *The function* f *is said to be* locally Lipschitz continuous with respect to the first variable at right-dense points *if, for every* $\bar{x} \in \Omega$ *and every right-dense point* $\bar{t} \in \mathbb{T}\backslash\{\sup \mathbb{T}\}$*, there exist* $R > 0, \delta > 0$ *and* $L \geq 0$ *such that* $\bar{B}(\bar{x},R) \subset \Omega$ *and* $\bar{t} + \delta \in \mathbb{T}$*, and such that*

$$\|f(x_1,t) - f(x_2,t)\| \leq L\|x_1 - x_2\|, \qquad (\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$$

for all $x_1, x_2 \in \bar{B}(\bar{x},R)$ *and for* Δ*-a.e.* $t \in [\bar{t}, \bar{t}+\delta)_{\mathbb{T}}$.

In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$.

DEFINITION 4. *The function f is said to be* forward Ω-stable at right-scattered points *if the mapping*

$$G^+(t) : \Omega \longrightarrow \mathbb{R}^n \qquad (\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$$
$$x \longmapsto x + \mu(t)f(x,t)$$

takes its values in Ω*, for every* $t \in \mathscr{R}$.

In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$.

DEFINITION 5. *The function* f *is said to be* locally Lipschitz continuous with respect to the first variable at left-dense points *if, for every* $\bar{x} \in \Omega$ *and every left-dense point* $\bar{t} \in \mathbb{T}\backslash\{\inf \mathbb{T}\}$*, there exist* m $R > 0, \delta > 0$ *and* $L \geq 0$ *such that* $\bar{B}(\bar{x},R) \subset \Omega$ *and* $\bar{t} - \delta \in \mathbb{T}$ *and such that*

$$\|f(x_1,t) - f(x_2,t)\| \leq L\|x_1 - x_2\|, \qquad (\mathrm{H}^{\mathrm{ld}}_{\text{loc-Lip}})$$

for all $x_1, x_2 \in \bar{B}(\bar{x},R)$ *and for* Δ*-a.e.* $t \in [\bar{t}-\delta, \bar{t})_{\mathbb{T}}$.

In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{ld}}_{\text{loc-Lip}})$.

DEFINITION 6. *The function f is said to be* backward regressive at right-scattered points *if*

$$G^+(t) \text{ is invertible}, \qquad \left(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}}\right)$$

for every $t \in \mathscr{R}$.

In what follows this property will be referred to as (H^{back}_{regr}).

Assumption (H_∞) will be instrumental to provide a Δ-integral characterization of the solutions of (Δ-CP) (see Lemma 1 in Section 5.1). The other assumptions play a role in order to *go forward* or *backward* for a solution of a non-shifted Δ-Cauchy problem. More precisely, ($H^{rd}_{loc\text{-}Lip}$) and (H^{forw}_{stab}) allow to go forward, and ($H^{ld}_{loc\text{-}Lip}$) and (H^{back}_{regr}) allow to go backward (see the proofs of Propositions 3 and 4 in Section 5.1 for more details).

In view of investigating global solutions, the following definition will also be useful.

DEFINITION 7. *The function f is said to be* globally Lipschitz continuous in its first variable *if for all* $(a,b) \in \mathbb{T}^2$ *such that* $a < b$, *there exists* $L \geq 0$ *such that*

$$\|f(x_1,t) - f(x_2,t)\| \leq L\|x_1 - x_2\|. \qquad (H^{glob}_{Lip})$$

For all $x_1, x_2 \in \Omega$ *and for* Δ*-a.e.* $t \in [a,b)_{\mathbb{T}}$. *In what follows this property will be referred to as* (H^{glob}_{Lip}).

3.2 Definition of a maximal solution

Recall that an interval of $\mathbb{T}$ is defined by $I_{\mathbb{T}} = I \cap \mathbb{T}$ where I is an interval of $\mathbb{R}$. In view of defining the notion of a solution of (Δ-CP) on general intervals of $\mathbb{T}$, we set

$$\mathbb{I} = \{I_{\mathbb{T}} \mid \exists a, b \in I_{\mathbb{T}}, a \trianglelefteq t_0 \trianglelefteq b\}.$$

$\mathbb{I}$ is the set of potential intervals of $\mathbb{T}$ for a solution of (Δ-CP).

DEFINITION 8. *Let* $I_{\mathbb{T}} \in \mathbb{I}$ *and let* $q : I_{\mathbb{T}} \to \Omega$. *The couple* $(q, I_{\mathbb{T}})$ *is said to be a solution of* (Δ-CP) *if* $q(t_0) = q_0$ *and if, for all* $a, b \in I_{\mathbb{T}}$ *satisfying* $a \trianglelefteq t_0 \trianglelefteq b$, $q \in \mathrm{AC}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ *and* $q^\Delta(t) = f(q(t),t)$ *for* Δ*-a.e.* $t \in [a,b)_{\mathbb{T}}$.

Note that, if $(q, I^1_{\mathbb{T}})$ is a solution of (Δ-CP), then $(q, I_{\mathbb{T}})$ is as well a solution of (Δ-CP) for all $I_{\mathbb{T}} \in \mathbb{I}$ such that $I_{\mathbb{T}} \subset I^1_{\mathbb{T}}$. Finally, we define the notion of maximal solution.

DEFINITION 9. *Let* $(q, I_{\mathbb{T}})$ *and* $(q_1, I^1_{\mathbb{T}})$ *be two solutions of* (Δ-CP). *The solution* $(q_1, I^1_{\mathbb{T}})$ *is said to be an extension of the solution* $(q, I_{\mathbb{T}})$ *if* $I_{\mathbb{T}} \subset I^1_{\mathbb{T}}$ *and* $q_1 = q$ *on* $I_{\mathbb{T}}$. *A solution* $(q, I_{\mathbb{T}})$ *of* (Δ-CP) *is said to be maximal if, for every extension* $(q_1, I^1_{\mathbb{T}})$ *of* $(q, I_{\mathbb{T}})$, *there holds* $I^1_{\mathbb{T}} = I_{\mathbb{T}}$. *A solution* $(q, I_{\mathbb{T}})$ *of* (Δ-CP) *is said to be global if* $I_{\mathbb{T}} = \mathbb{T}$.

Note that, if $(q, I_{\mathbb{T}})$ is a global solution of (Δ-CP), then $(q, I_{\mathbb{T}})$ is a maximal solution of (Δ-CP).

3.3 Main results

Recall that we consider the general non-shifted Δ-Cauchy problem

$$q^\Delta(t) = f(q(t),t), \quad q(t_0) = q_0, \qquad (\Delta\text{-CP})$$

where $t_0 \in \mathbb{T}$, $q_0 \in \Omega$, where Ω is a non-empty open subset of $\mathbb{R}^n$, and $f : \Omega \times \mathbb{T} \backslash \{\sup \mathbb{T}\} \to \mathbb{R}^n$ is a Δ-Carathéodory function. We have the following general Cauchy–Lipschitz result.

THEOREM 1. *We make the following assumptions on the dynamics f, depending on* t_0.

1. *If* $t_0 = \min \mathbb{T}$*, then we assume that*
   - f *satisfies* (H_∞)*, that is* f *is locally bounded on* $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$*;*
   - f *satisfies* $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$*, that is* f *is locally Lipschitz continuous with respect to the first variable at right-dense points;*
   - f *satisfies* $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$*, that is* f *is forward* Ω*-stable at right-scattered points.*
2. *If* $t_0 = \max \mathbb{T}$*, then we assume that*
   - f *satisfies* (H_∞)*, that is* f *is locally bounded on* $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$*;*
   - f *satisfies* $(\mathrm{H}^{\mathrm{ld}}_{\text{loc-Lip}})$*, that is* f *is locally Lipschitz continuous with respect to the first variable at left-dense points;*
   - f *satisfies* $\left(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}}\right)$*, that is* f *is backward regressive in right-scattered points.*
3. *If* $t_0 \neq \inf \mathbb{T}$ *and* $t_0 \neq \sup \mathbb{T}$*, then we assume that*
   - f *satisfies* (H_∞)*, that is* f *is locally bounded on* $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$*;*
   - f *satisfies* $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$*, that is* f *is locally Lipschitz continuous with respect to the first variable at right-dense points;*
   - f *satisfies* $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$*, that is* f *is forward* Ω*-stable at right-scattered points;*
   - f *satisfies* $(\mathrm{H}^{\mathrm{ld}}_{\text{loc-Lip}})$*, that is* f *is locally Lipschitz continuous with respect to the first variable at left-dense points;*
   - f *satisfies* $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$*, that is* f *is backward regressive in right-scattered points.*

Then, the non-shifted Δ*-Cauchy problem* (Δ-CP) *has a unique maximal solution* $(q, I_\mathbb{T})$*. Moreover,* $(q, I_\mathbb{T})$ *is the maximal extension of any other solution of* (Δ-CP).

This theorem is proved in Section 5.1. The following result gives information on the behaviour of a maximal solution at its terminal points.

THEOREM 2. *Under the assumptions of Theorem* 1*, let* $(q, I_\mathbb{T})$ *be the maximal solution of the non-shifted* Δ*-Cauchy problem* (Δ-CP)*. Then either* $I_\mathbb{T} = \mathbb{T}$*, that is the maximal solution* $(q, I_\mathbb{T})$ *is global, or the maximal solution is not global and then*

1. *if* $t_0 = \min \mathbb{T}$ *then* $I_\mathbb{T} = [t_0, b)_\mathbb{T}$ *where* $b \in (t_0, +\infty)_\mathbb{T}$ *is a left-dense point of* $\mathbb{T}$*;*
2. *if* $t_0 = \max \mathbb{T}$ *then* $I_\mathbb{T} = (a, t_0]_\mathbb{T}$ *where* $a \in (-\infty, t_0)_\mathbb{T}$ *is a right-dense point of* $\mathbb{T}$*;*
3. *if* $t_0 \neq \inf \mathbb{T}$ *and* $t_0 \neq \sup \mathbb{T}$ *then* $I_\mathbb{T} = (a, +\infty)_\mathbb{T}$ *or* $I_\mathbb{T} = (-\infty, b)_\mathbb{T}$ *or* $I_\mathbb{T} = (a, b)_\mathbb{T}$*, where* $a \in (-\infty, t_0)_\mathbb{T}$ *is a right-dense point of* $\mathbb{T}$ *and* $b \in (t_0, +\infty)_\mathbb{T}$ *is a left-dense point of* $\mathbb{T}$*;*

and moreover, for every $K \in \mathcal{K}$ *there exists* $t \in I_\mathbb{T}$ (*close to a or b depending on the cases listed above*) *such that* $q(t) \in \Omega\backslash K$.

This theorem is proved in Section 5.2. It states that the maximal solution must go out of any compact of Ω near its terminal points whenever it is not global.

The following last result states that, under global Lipschitz assumption, the maximal solution is global.

THEOREM 3. *If* $t_0 = \min \mathbb{T}$*,* $\Omega = \mathbb{R}^n$*, if* f *satisfies* (H_∞)*, that is* f *is locally bounded on* $\mathbb{R}^n \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$*, and if* f *satisfies* $(\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}})$*, that is* f *is globally Lipschitz continuous in its first variable, then the non-shifted* Δ*-Cauchy problem* (Δ-CP) *has a unique maximal solution* $(q, I_\mathbb{T})$*, which is moreover global.*

The proof is done in Section 5.3.

Remark 3. As an application of Theorem 3, we recover the well-known fact that, in the linear case

$$q^{\Delta}(t) = h(t) \times q(t),$$

where $h : \mathbb{T}\backslash\{\sup \mathbb{T}\} \to \mathbb{R}^{n\times n}$ such that $h \in \mathrm{L}^{\infty}_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R}^{n\times n})$ for all $(a,b) \in \mathbb{T}^2$ with $a < b$, solutions are global.

3.4 Further comments

In this section, we provide simple examples (in the one-dimensional case) showing the sharpness of the assumptions made in Theorem 1. Indeed, if one of these assumptions is not satisfied, then the existence or the uniqueness of the maximal solution is no more guaranteed.

Example 1. (*Lack of Assumption* $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$ *in the first case*). Let $\mathbb{T} = [0, +\infty[$, $\Omega = \mathbb{R}$, $t_0 = 0$, $q_0 = 0$ and $f : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ be defined by $f(x,t) = 2\sqrt{|x|}$. The function f obviously satisfies $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$ since $\mathscr{R} = \emptyset$; however, it does not satisfy $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$. The corresponding Δ-Cauchy problem (Δ-CP) has two global solutions q_1 and q_2 given by $q_1(t) = 0$ and $q_2(t) = t^2$, for every $t \in \mathbb{T}$.

This example shows that, in the absence of Assumption $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$, the uniqueness of the maximal solution is not guaranteed.

Example 2. (*Lack of Assumption* $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$ *in the first case*). Let $\mathbb{T} = \{0, 1\}$, $\Omega = (-1, 1)$, $t_0 = 0$, $q_0 = 0$ and $f : \Omega \times \{0\} \to \mathbb{R}$ be defined by $f(x,t) = 1$. The function f obviously satisfies $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$ since $\mathbb{T}\backslash\{\sup\mathbb{T}\} = \{0\}$ does not admit any right-dense point of $\mathbb{T}$ however, it does not satisfy $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$ since $x + 1 \notin \Omega$ for $x \in [0, 1)$. Since $q(0) = 0$ and $q(1) = q(0) + \mu(0)f(q(0), 0)$ imply $q(1) = 1 \notin \Omega$, we conclude that (Δ-CP) does not admit any solution.

Therefore, in the absence of Assumption $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$, ($\Delta$-CP) may fail to have a solution.

Example 3. (*Lack of Assumption* $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$ *in the second case*). Let $\mathbb{T} = (-\infty, 0]$, $\Omega = \mathbb{R}$, $t_0 = 0$, $q_0 = 0$ and $f : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ be defined by $f(x,t) = -2\sqrt{|x|}$. The function f obviously satisfies $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$ since $\mathscr{R} = \emptyset$; however, it does not satisfy $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$. The corresponding Δ-Cauchy problem (Δ-CP) has two global solutions q_1 and q_2 given by $q_1(t) = 0$ and $q_2(t) = t^2$ for every $t \in \mathbb{T}$.

This example shows that, in the absence of Assumption $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$, the uniqueness of the maximal solution is not guaranteed.

Example 4. (*Lack of Assumption* $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$ *in the second case*). Let $\mathbb{T} = \{0, 1\}$, $\Omega = \mathbb{R}$, $t_0 = 1$, $q_0 \in \mathbb{R}$ and $f : \mathbb{R} \times \{0\} \to \mathbb{R}$ be defined by $f(x,t) = -x$. The function f obviously satisfies $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$ since $\mathbb{T}\backslash\{\inf \mathbb{T}\} = \{1\}$ does not admit any left-dense point of $\mathbb{T}$ however, it does not satisfy $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$ since $G^{+}(0) = 0$. As a consequence, if $q_0 \neq 0$, (Δ-CP) does not admit any solution. Indeed, $q(1) = q_0$ and $q(1) = q(0) + \mu(0)f(q(0), 0)$ imply $q(1) = 0$, which is a contradiction. If $q_0 = 0$, we obtain an infinite number of global solutions. Indeed, any function q defined on $\mathbb{T}$ with $q(1) = 0$ is then a global solution of (Δ-CP).

4. General shifted Δ-Cauchy problem

Throughout this section we consider the general *shifted* Δ-Cauchy problem

$$q^{\Delta}(t) = f(q^{\sigma}(t), t), \quad q(t_0) = q_0, \qquad (\Delta\text{-CP}^{\sigma})$$

where $t_0 \in \mathbb{T}$, $q_0 \in \Omega$, where Ω is a non-empty open subset of $\mathbb{R}^n$ and $f : \Omega \times \mathbb{T} \backslash \{\sup \mathbb{T}\} \longrightarrow \mathbb{R}^n$ is a Δ-Carathéodory function.

The results of the section follow the same lines as in the previous section. Therefore we do not give any proof nor counterexamples as above. Some comments are however done in Section 5.4.

4.1 Preliminaries

As in Section 3.1, it will be important to distinguish between three cases:

1. $t_0 = \min \mathbb{T}$;
2. $t_0 = \max \mathbb{T}$;
3. $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$.

With respect to Section 3.1, we introduce two additional concepts.

Definition 10. *The function* f *is said to be* backward Ω-stable at right-scattered points *if the mapping*

$$G^{-}(t) : \Omega \longrightarrow \mathbb{R}^n \qquad (\mathrm{H}^{\mathrm{back}}_{\mathrm{stab}})$$

$$x \longmapsto x - \mu(t) f(x, t)$$

takes its values in Ω*, for every* $t \in \mathscr{R}$.

In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{back}}_{\mathrm{stab}})$.

Definition 11. *The function* f *is said to be* forward regressive at right-scattered points *if*

$$G^{-}(t) : \Omega \longrightarrow \mathbb{R}^n \text{ is invertible}, \qquad (\mathrm{H}^{\mathrm{forw}}_{\mathrm{regr}})$$

For every $t \in \mathscr{R}$.

In what follows this property will be referred to as $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{regr}})$.

These above assumptions play a role in order to *go forward* or *backward* for a solution of a shifted Δ-Cauchy problem. Precisely, $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$ and $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{regr}})$ allow to *go forward*. Similarly, $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$ and $(\mathrm{H}^{\mathrm{back}}_{\mathrm{stab}})$ allow to *go backward*.

4.2 Definition of a maximal solution

Definition 12. *Let* $I_{\mathbb{T}} \in \mathbb{I}$ *and let* $q : I_{\mathbb{T}} \longrightarrow \Omega$. *The couple* $(q, I_{\mathbb{T}})$ *is said to be a solution of* $(\Delta\text{-CP}^{\sigma})$ *if* $q(t_0) = q_0$ *and if, for all* $a, b \in I_{\mathbb{T}}$ *satisfying* $a \trianglelefteq t_0 \trianglelefteq b$, $q \in \mathrm{AC}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ *and* $q^{\Delta}(t) = f(q^{\sigma}(t), t)$ *for Δ-a.e.* $t \in [a, b)_{\mathbb{T}}$.

Definition 13. *Let* $(q, I_{\mathbb{T}})$ *and* $(q_1, I^1_{\mathbb{T}})$ *be two solutions of* $(\Delta\text{-CP}^{\sigma})$. *The solution* $(q_1, I^1_{\mathbb{T}})$ *is said to be an extension of the solution* $(q, I_{\mathbb{T}})$ *if* $I_{\mathbb{T}} \subset I^1_{\mathbb{T}}$ *and* $q_1 = q$ *on* $I_{\mathbb{T}}$. *A solution* $(q, I_{\mathbb{T}})$ *of* $(\Delta\text{-CP}^{\sigma})$ *is said to be* maximal *if, for every extension* $(q_1, I^1_{\mathbb{T}})$ *of* $(q, I_{\mathbb{T}})$, *there holds* $I^1_{\mathbb{T}} = I_{\mathbb{T}}$. *A solution* $(q, I_{\mathbb{T}})$ *of* $(\Delta\text{-CP}^{\sigma})$ *is said to be* global *if* $I_{\mathbb{T}} = \mathbb{T}$.

4.3 Main results

Recall that we consider the general shifted Δ-Cauchy problem

$$q^{\Delta}(t) = f(q^{\sigma}(t), t), \quad q(t_0) = q_0, \qquad (\Delta\text{-CP}^{\sigma})$$

where $t_0 \in \mathbb{T}$, $q_0 \in \Omega$ where Ω is a non-empty open subset of $\mathbb{R}^n$ and $f : \Omega \times \mathbb{T} \backslash \{\sup \mathbb{T}\} \to \mathbb{R}^n$ is a Δ-Carathéodory function.

THEOREM 4. *We make the following assumptions on the dynamics* f*, depending on* t_0.

1. *If* $t_0 = \min \mathbb{T}$*, then we assume that*
   - f *satisfies* (H_∞)*, that is* f *is locally bounded on* $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$*;*
   - f *satisfies* $(H^{rd}_{\text{loc-Lip}})$*, that is* f *is locally Lipschitz continuous with respect to the first variable at right-dense points;*
   - f *satisfies* (H^{forw}_{regr})*, that is* f *is forward regressive in right-scattered points.*
2. *If* $t_0 = \max \mathbb{T}$*, then we assume that*
   - f *satisfies* (H_∞)*, that is* f *is locally bounded on* $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$*;*
   - f *satisfies* $(H^{rd}_{\text{loc-Lip}})$*, that is* f *is locally Lipschitz continuous with respect to the first variable at left-dense points;*
   - f *satisfies* (H^{back}_{stab})*, that is* f *is backward* Ω*-stable in right-scattered points.*
3. *If* $t_0 \neq \inf \mathbb{T}$ *and* $t_0 \neq \sup \mathbb{T}$*, then we assume that*
   - f *satisfies* (H_∞)*, that is* f *is locally bounded on* $\Omega \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$*;*
   - f *satisfies* $(H^{rd}_{\text{loc-Lip}})$*, that is* f *is locally Lipschitz continuous with respect to the first variable at right-dense points;*
   - f *satisfies* (H^{forw}_{regr})*, that is* f *is forward regressive at right-scattered points;*
   - f *satisfies* $(H^{rd}_{\text{loc-Lip}})$*, that is* f *is locally Lipschitz continuous with respect to the first variable at left-dense points;*
   - f *satisfies* (H^{back}_{stab})*, that is* f *is backward* Ω*-stable at right-scattered points.*

Then the shifted Δ-Cauchy problem $(\Delta\text{-CP}^{\sigma})$ *has a unique maximal solution* $(q, I_{\mathbb{T}})$*. Moreover,* $(q, I_{\mathbb{T}})$ *is the maximal extension of any other solution of* $(\Delta\text{-CP}^{\sigma})$.

THEOREM 5. *Under the assumptions of Theorem* 4*, let* $(q, I_{\mathbb{T}})$ *be the maximal solution of the shifted Δ-Cauchy problem* $(\Delta\text{-CP}^{\sigma})$*. Then either* $I_{\mathbb{T}} = \mathbb{T}$*, that is the maximal solution* $(q, I_{\mathbb{T}})$ *is global, or the maximal solution is not global and then*

1. *if* $t_0 = \min \mathbb{T}$ *then* $I_{\mathbb{T}} = [t_0, b)_{\mathbb{T}}$ *where* $b \in (t_0, +\infty)_{\mathbb{T}}$ *is a left-dense point of* $\mathbb{T}$*;*
2. *if* $t_0 = \max \mathbb{T}$ *then* $I_{\mathbb{T}} = (a, t_0]_{\mathbb{T}}$ *where* $a \in (-\infty, t_0)_{\mathbb{T}}$ *is a right-dense point of* $\mathbb{T}$*;*
3. *if* $t_0 \neq \inf \mathbb{T}$ *and* $t_0 \neq \sup \mathbb{T}$ *then* $I_{\mathbb{T}} = (a, +\infty)_{\mathbb{T}}$ *or* $I_{\mathbb{T}} = (-\infty, b)_{\mathbb{T}}$ *or* $I_{\mathbb{T}} = (a, b)_{\mathbb{T}}$ *where* $a \in (-\infty, t_0)_{\mathbb{T}}$ *is a right-dense point of* $\mathbb{T}$ *and* $b \in (t_0, +\infty)_{\mathbb{T}}$ *is a left-dense point of* $\mathbb{T}$*;*

and moreover, for every $K \in \mathcal{K}$ *there exists* $t \in I_{\mathbb{T}}$ *(close to* a *or* b *depending on the cases listed above) such that* $q(t) \in \Omega \backslash K$.

THEOREM 6. *If* $t_0 = \max \mathbb{T}$*,* $\Omega = \mathbb{R}^n$*, if* f *satisfies* (H_∞)*, that is* f *is locally bounded on* $\mathbb{R}^n \times \mathbb{T}\backslash\{\sup \mathbb{T}\}$*, and if* f *satisfies* (H^{glob}_{Lip})*, that is* f *is globally Lipschitz continuous in its first variable, then the shifted Δ-Cauchy problem* $(\Delta\text{-CP}^{\sigma})$ *has a unique maximal solution* $(q, I_{\mathbb{T}})$*, which is moreover global.*

Remark 4. As in Remark 3, in the linear case the maximal solution of any shifted Δ-Cauchy problem is automatically global.

5. Proofs of the results

5.1 Proof of Theorem 1

If f satisfies (H_∞), then for all $(a,b) \in \mathbb{T}^2$ such that $a < b$, there holds

$$f(q,t) \in L^\infty_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R}^n) \subset L^1_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R}^n), \tag{4}$$

for every $q \in \mathscr{C}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$. Then, from Section 2.3, we have the following Δ-integral characterization of the solutions of (Δ-CP).

LEMMA 1. *Let* $I_{\mathbb{T}} \in \mathbb{I}$ *and let* $q : I_{\mathbb{T}} \to \Omega$. *If* f *satisfies* ($H_\infty$)*, then the couple* $(q, I_{\mathbb{T}})$ *is a solution of* (Δ-CP) *if and only if for all* $a, b \in I_{\mathbb{T}}$ *satisfying* $a \trianglelefteq t_0 \trianglelefteq b$*, one has* $q \in \mathscr{C}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ *and*

$$q(t) = \begin{cases} q_0 + \int_{[t_0,t)_{\mathbb{T}}} f(q(\tau),\tau)\Delta\tau & \text{if } t \geq t_0, \\ q_0 - \int_{[t,t_0)_{\mathbb{T}}} f(q(\tau),\tau)\Delta\tau & \text{if } t \leq t_0. \end{cases}$$

for every $t \in [a,b]_{\mathbb{T}}$.

Note that, if $t < t_0$ is right-scattered, then $q(t)$ appears in the two sides of the above equation. Therefore this equation is implicit in $q(t)$ and a regressivity assumption is then required to ensure the existence of a solution.

The characterization of the solutions given by Lemma 1 allows one to prove the following result.

LEMMA 2. *If* f *satisfies* (H_∞)*, then every solution of* (Δ-CP) *can be extended to a maximal solution.*

Proof. Let $(q, I_{\mathbb{T}})$ be a solution of (Δ-CP). Let us define the non-empty set $\mathscr{F}$ of extensions of $(q, I_{\mathbb{T}})$. The set $\mathscr{F}$ is ordered by

$$\left(q_1, I^1_{\mathbb{T}}\right) \leq \left(q_2, I^2_{\mathbb{T}}\right) \quad \text{if and only if} \quad \left(q_2, I^2_{\mathbb{T}}\right) \text{ is an extension of } \left(q_1, I^1_{\mathbb{T}}\right).$$

Let us prove that $\mathscr{F}$ is inductive. Let $\mathscr{G} = \cup_{p\in\mathscr{P}}\{(q_p, I^p_{\mathbb{T}})\}$ be a non-empty totally ordered subset of $\mathscr{F}$. Let us prove that $\mathscr{G}$ admits an upper bound.

Let us define $\bar{I} = \cup_{p\in\mathscr{P}} I^p$. This is an interval of $\mathbb{R}$, since $t_0 \in \cap_{p\in\mathscr{P}} I^p$. Then $\bar{I}_{\mathbb{T}} = \cup_{p\in\mathscr{P}} I^p_{\mathbb{T}} \in \mathbb{I}$. For every $t \in \bar{I}_{\mathbb{T}}$, there exists $p \in \mathscr{P}$ such that $t \in I^p_{\mathbb{T}}$ and, since $\mathscr{G}$ is totally ordered, if $t \in I^{p_1}_{\mathbb{T}} \cap I^{p_2}_{\mathbb{T}}$ then $q_{p_1}(t) = q_{p_2}(t)$. Consequently, we can define $\bar{q}$ by

$$\forall t \in \bar{I}_{\mathbb{T}}, \bar{q}(t) = q_p(t) \in \Omega \quad \text{where } t \in I^p_{\mathbb{T}}. \tag{5}$$

Our aim is to prove that $(\bar{q}, \bar{I}_{\mathbb{T}})$ is a solution of (Δ-CP). Let $a, b \in \bar{I}_{\mathbb{T}}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$. Since $\mathscr{G}$ is totally ordered, there exists $p \in \mathscr{P}$ such that $[a,b]_{\mathbb{T}} \subset I^p_{\mathbb{T}}$ and $\bar{q} = q_p$ on $[a,b]_{\mathbb{T}}$. Since $(q_p, I^p_{\mathbb{T}})$ is a solution of (Δ-CP), we obtain that q_p satisfies the necessary and sufficient condition of Lemma 1 on $[a,b]_{\mathbb{T}}$. Consequently, this holds true as well for $\bar{q}$ on

$[a,b]_{\mathbb{T}}$. Finally, since this last sentence is true for all $a,b \in \bar{I}_{\mathbb{T}}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$, we infer from Lemma 1 that $(\bar{q}, \bar{I}_{\mathbb{T}})$ is a solution of (Δ-CP). Since $(\bar{q}, \bar{I}_{\mathbb{T}})$ is obviously an extension of any element of $\mathscr{G}$, we obtain that $\mathscr{G}$ admits an upper bound and then, $\mathscr{F}$ is inductive.

Finally, $\mathscr{F}$ is a non-empty ordered inductive set and consequently, from Zorn's lemma, admits a maximal element. The proof is complete. □

Proposition 3. (Existence of a local solution). *There exist $a,b \in \mathbb{T}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$ and $q : [a,b]_{\mathbb{T}} \to \Omega$ such that $(q,[a,b]_{\mathbb{T}})$ is a solution of* (Δ-CP).

Proof. We only prove this proposition in the third case of Theorem 1 (the two first cases are derived similarly) for which $t_0 \neq \inf \mathbb{T}$ and $t_0 \neq \sup \mathbb{T}$. We distinguish between four situations.

First case. t_0 is a left- and a right-scattered point of $\mathbb{T}$. In this case, it is sufficient to consider $a = \rho(t_0) \in (-\infty, t_0)_{\mathbb{T}}$, $b = \sigma(t_0) \in (t_0, +\infty)_{\mathbb{T}}$ and the function q defined on $[a,b]_{\mathbb{T}} = \{a, t_0, b\}$ with values in Ω by $q(a) = G^+(a)^{-1}(q_0)$, $q(t_0) = q_0$ and $q(b) = G^+(t_0)(q_0)$. We note that $q(a)$ is well defined in Ω from ($\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}}$) and $q(b) \in \Omega$ from ($\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}}$).

Second case. t_0 is a left- and a right-dense point of $\mathbb{T}$. Let R', δ' and L' associated with q_0 and t_0 in ($\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}}$) and let R'', δ'' and L'' associated with q_0 and t_0 in ($\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}}$). We define $R = \min(R', R'') > 0$ and $L = \max(L', L'') \geq 0$. Let M associated with $\bar{B}(q_0, R) \in \mathcal{K}$ and $[t_0 - \delta', t_0 + \delta'')_{\mathbb{T}}$ in (H_∞). Consider $0 < \delta_1 \leq \delta'$ and $0 < \delta_2 \leq \delta''$ such that $a = t_0 - \delta_1 \in (-\infty, t_0)_{\mathbb{T}}$, $b = t_0 + \delta_2 \in (t_0, +\infty)_{\mathbb{T}}$ and δ_1 and δ_2 are sufficiently small in order to have $\max(\delta_1, \delta_2)M \leq R$ and $\max(\delta_1, \delta_2)L < 1$. Then, we can construct the $\max(\delta_1, \delta_2)L$-contraction map with respect to the norm $\|\cdot\|_\infty$

$$F : \mathscr{C}([a,b]_{\mathbb{T}}, \bar{B}(q_0,R)) \to \mathscr{C}([a,b]_{\mathbb{T}}, \bar{B}(q_0,R)) \quad q \mapsto F(q),$$

with

$$F(q) : [a,b]_{\mathbb{T}} \to \bar{B}(q_0,R) \quad t \mapsto \begin{cases} q_0 + \int_{[t_0,t)_{\mathbb{T}}} f(q(\tau),\tau)\Delta\tau & \text{if } t \geq t_0, \\ q_0 - \int_{[t,t_0)_{\mathbb{T}}} f(q(\tau),\tau)\Delta\tau & \text{if } t \leq t_0. \end{cases}$$

It follows from the Banach fixed-point theorem that F has a unique fixed point denoted by q, and then $(q,[a,b]_{\mathbb{T}})$ is a solution of (Δ-CP).

Third case. t_0 is a left-scattered and a right-dense point of $\mathbb{T}$. Let R, δ and L associated with q_0 and t_0 in ($\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}}$). Let M associated with $\bar{B}(q_0,R) \in \mathcal{K}$ and $[t_0, t_0+\delta)_{\mathbb{T}}$ in (H_∞). Consider $0 < \delta_1 \leq \delta$ such that $b = t_0 + \delta_1 \in (t_0, +\infty)_{\mathbb{T}}$ and δ_1 is sufficiently small in order to have $\delta_1 M \leq R$ and $\delta_1 L < 1$. Then, we can construct the $\delta_1 L$-contraction map with respect to the norm $\|\cdot\|_\infty$

$$F : \mathscr{C}([t_0,b]_{\mathbb{T}}, \bar{B}(q_0,R)) \to \mathscr{C}([t_0,b]_{\mathbb{T}}, \bar{B}(q_0,R)) \quad q \mapsto F(q)$$

with

$$F(q) : [t_0,b]_{\mathbb{T}} \to \bar{B}(q_0,R) \quad t \mapsto q_0 + \int_{[t_0,t)_{\mathbb{T}}} f(q(\tau),\tau)\Delta\tau.$$

It follows from the Banach fixed-point theorem that F has a unique fixed point denoted by q defined on $[t_0,b]_{\mathbb{T}}$. Finally, since t_0 is a left-scattered point of $\mathbb{T}$ and from $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$, we define $a=\rho(t_0)\in(-\infty,t_0)_{\mathbb{T}}$ and $q(a)=G^+(a)^{-1}(q_0)\in\Omega$. We have thus obtained a solution $(q,[a,b]_{\mathbb{T}})$ of (Δ-CP).

Fourth case. t_0 is a left-dense and a right-scattered point of $\mathbb{T}$. Let R, δ and L associated with q_0 and t_0 in $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$. Let M associated with $\bar{B}(q_0,R)\in\mathcal{K}$ and $[t_0-\delta,t_0)_{\mathbb{T}}$ in (H_∞). Consider $0<\delta_1\leq\delta$ such that $a=t_0-\delta_1\in(-\infty,t_0)_{\mathbb{T}}$ and δ_1 is sufficiently small in order to have $\delta_1 M\leq R$ and $\delta_1 L<1$. Then, we can construct the $\delta_1 L$-contraction map with respect to the norm $\|\cdot\|_\infty$

$$F:\mathscr{C}([a,t_0]_{\mathbb{T}},\bar{B}(q_0,R))\longrightarrow\mathscr{C}([a,t_0]_{\mathbb{T}},\bar{B}(q_0,R))\quad q\longmapsto F(q)$$

with

$$F(q):[a,t_0]_{\mathbb{T}}\longrightarrow\bar{B}(q_0,R)\quad t\longmapsto q_0-\int_{[t,t_0)_{\mathbb{T}}}f(q(\tau),\tau)\Delta\tau.$$

It follows from the Banach fixed-point theorem that F admits a unique fixed point denoted by q defined on $[a,t_0]_{\mathbb{T}}$. Since t_0 is a right-scattered point of $\mathbb{T}$, and from $(\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}})$, we define $b=\sigma(t_0)\in(t_0,+\infty)_{\mathbb{T}}$ and $q(b)=G^+(t_0)(q_0)\in\Omega$. We have thus obtained a solution $(q,[a,b]_{\mathbb{T}})$ of (Δ-CP). □

From Lemma 2, we can extend the solution given in Proposition 3 and we obtain the existence of a maximal solution. The following result proves that it is unique.

Proposition 4. (Local uniqueness of a solution). *Let* $(q_1,I^1_{\mathbb{T}})$ *and* $(q_2,I^2_{\mathbb{T}})$ *be two solutions of* (Δ-CP). *Then,* $q_1=q_2$ *on* $I^1_{\mathbb{T}}\cap I^2_{\mathbb{T}}$.

Proof. As before, we only prove this proposition in the third case of Theorem 1. We denote by $I=I^1\cap I^2$ (interval of $\mathbb{R}$). One can easily prove that $I_{\mathbb{T}}=I^1_{\mathbb{T}}\cap I^2_{\mathbb{T}}\in\mathbb{I}$. It is sufficient to prove that $q_1=q_2$ on $[a,b]_{\mathbb{T}}$ for all $a,b\in I_{\mathbb{T}}$ satisfying $a\trianglelefteq t_0\trianglelefteq b$. Let $a,b\in I_{\mathbb{T}}$ satisfying $a\trianglelefteq t_0\trianglelefteq b$. Set

$$A=\{t\in[a,t_0]_{\mathbb{T}},q_1(t)\neq q_2(t)\},$$

and

$$B=\{t\in[t_0,b]_{\mathbb{T}},q_1(t)\neq q_2(t)\}.$$

Let us prove by contradiction that $A\cup B=\emptyset$. Assume that $A\neq\emptyset$ and let $\bar{t}=\sup A$. Note that $\bar{t}\in[a,t_0]_{\mathbb{T}}$ (since $\mathbb{T}$ is closed) and that $q_1=q_2$ on $]\bar{t},t_0]_{\mathbb{T}}$. In order to raise a contradiction, we first derive the four following facts.

1. *Fact* 1: $\bar{t}<t_0$. If t_0 is a left-scattered point of $\mathbb{T}$, this claim is obvious since $q_1(t_0)=q_2(t_0)=q_0$ and $q_1(\rho(t_0))=q_2(\rho(t_0))=G^+(\rho(t_0))^{-1}(q_0)$ from $(\mathrm{H}^{\mathrm{back}}_{\mathrm{regr}})$. If t_0 is a left-dense point of $\mathbb{T}$, let R, δ and L associated with q_0 and t_0 in $(\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}})$. Let M associated with $\bar{B}(q_0,R)\in\mathcal{K}$ and $[t_0-\delta,t_0)_{\mathbb{T}}$ in (H_∞). Consider $0<\delta_1\leq\delta$ such that $c=t_0-\delta_1\in[a,t_0)_{\mathbb{T}}$ and δ_1 is sufficiently small in order to have $\delta_1 M\leq R$, $\delta_1 L<1$ and q_1, $q_2\in\mathscr{C}([c,t_0]_{\mathbb{T}},\bar{B}(q_0,R))$. Since q_1 and q_2 are solutions of (Δ-CP)

on $[a,b]_{\mathbb{T}}$, they are in particular fixed points of the $\delta_1 L$-contraction map

$$F : \mathscr{C}([c,t_0]_{\mathbb{T}}, \bar{B}(q_0,R)) \longrightarrow \mathscr{C}([c,t_0]_{\mathbb{T}}, \bar{B}(q_0,R)) \quad q \longmapsto F(q)$$

with

$$F(q) : [c,t_0]_{\mathbb{T}} \longrightarrow \bar{B}(q_0,R) \quad t \longmapsto q_0 - \int_{[t,t_0)_{\mathbb{T}}} f(q(\tau),\tau)\Delta\tau.$$

Since F has a unique fixed point from the Banach fixed-point theorem, we conclude that $q_1 = q_2$ on $[c,t_0]_{\mathbb{T}}$. Hence $\bar{t} < t_0$.

2. *Fact* 2: $q_1(\bar{t}) = q_2(\bar{t})$. If $\bar{t}$ is a right-scattered point of $\mathbb{T}$, then $\sigma(\bar{t})$ is a left-scattered point of $\mathbb{T}$ and $q_1(\sigma(\bar{t})) = q_2(\sigma(\bar{t}))$. As a consequence, $q_1(\bar{t}) = q_2(\bar{t}) = G^+(\bar{t})^{-1}(q_1(\sigma(\bar{t})))$. If $\bar{t}$ is a right-dense point of $\mathbb{T}$, then $q_1(\bar{t}) = q_2(\bar{t})$ from the continuity of q_1 and q_2 and since $q_1 = q_2$ on $]\bar{t},t_0]_{\mathbb{T}}$.
3. *Fact* 3: $\bar{t} > a$. Indeed, if $\bar{t} = a$ then $A = \emptyset$ since $q_1(\bar{t}) = q_2(\bar{t})$;
4. *Fact* 4: *$\bar{t}$ is a left-dense point of* $\mathbb{T}$. Indeed, if $\bar{t}$ were to be a left-scattered point of $\mathbb{T}$, since $q_1(\bar{t}) = q_2(\bar{t})$, then $q_1(\rho(\bar{t})) = q_2(\rho(\bar{t})) = G^+(\rho(\bar{t}))^{-1}(q_1(\bar{t}))$ and then it would raise a contradiction with the definition of $\bar{t}$.

Let us denote by $\bar{x} = q_1(\bar{t}) = q_2(\bar{t})$. Let R, δ and L associated with $\bar{t}$ and $\bar{x}$ in ($\mathrm{H}^{\mathrm{rd}}_{\text{loc-Lip}}$). Let M associated with $\bar{B}(\bar{x},R) \in \mathcal{K}$ and $[\bar{t}-\delta,\bar{t})_{\mathbb{T}}$ in (H_∞). Consider $0 < \delta_1 \leq \delta$ such that $c_0 = \bar{t} - \delta_1 \in [a,\bar{t})_{\mathbb{T}}$ and δ_1 is sufficiently small in order to have $\delta_1 M \leq R$, $\delta_1 L < 1$ and $q_1, q_2 \in \mathscr{C}([c_0,\bar{t}]_{\mathbb{T}}, \bar{B}(\bar{x},R))$. Since q_1 and q_2 are solutions of (Δ-CP) on $[a,b]_{\mathbb{T}}$, they are in particular fixed points of the $\delta_1 L$-contraction map

$$F_0 : \mathscr{C}([c_0,\bar{t}]_{\mathbb{T}}, \bar{B}(\bar{x},R)) \longrightarrow \mathscr{C}([c_0,\bar{t}]_{\mathbb{T}}, \bar{B}(\bar{x},R)) \quad q \longmapsto F_0(q)$$

with

$$F_0(q) : [c_0,\bar{t}]_{\mathbb{T}} \longrightarrow \bar{B}(\bar{x},R) \quad t \longmapsto \bar{x} - \int_{[t,\bar{t})_{\mathbb{T}}} f(q(\tau),\tau)\Delta\tau.$$

Since F_0 has a unique fixed point from the Banach fixed-point theorem, we conclude that $q_1 = q_2$ on $[c_0,\bar{t}]_{\mathbb{T}}$, and this is a contradiction. Consequently $A = \emptyset$.

In the same way, we prove that $B = \emptyset$ and the proof is complete. □

Theorem 1 follows from Lemma 2, Propositions 3 and 4.

5.2 *Proof of Theorem 2*

PROPOSITION 5. *Under the assumptions of Theorem* 1, *let* $(q, I_{\mathbb{T}})$ *be the maximal solution of* (Δ-CP). *Then either* $I_{\mathbb{T}} = \mathbb{T}$, *that is the solution* $(q, I_{\mathbb{T}})$ *is global, or*

1. *if* $t_0 = \min \mathbb{T}$ *then* $I_{\mathbb{T}} = [t_0,b)_{\mathbb{T}}$ *where* $b \in (t_0,+\infty)_{\mathbb{T}}$ *is a left-dense point of* $\mathbb{T}$;
2. *if* $t_0 = \max \mathbb{T}$ *then* $I_{\mathbb{T}} = (a,t_0]_{\mathbb{T}}$ *where* $a \in (-\infty,t_0)_{\mathbb{T}}$ *is a right-dense point of* $\mathbb{T}$;
3. *if* $t_0 \neq \inf \mathbb{T}$ *and* $t_0 \neq \sup \mathbb{T}$ *then* $I_{\mathbb{T}} = (a,+\infty)_{\mathbb{T}}$ *or* $I_{\mathbb{T}} = (-\infty,b)_{\mathbb{T}}$ *or* $I_{\mathbb{T}} = (a,b)_{\mathbb{T}}$, *where* $a \in (-\infty,t_0)_{\mathbb{T}}$ *is a right-dense point of* $\mathbb{T}$ *and* $b \in (t_0,+\infty)_{\mathbb{T}}$ *is a left-dense point of* $\mathbb{T}$.

Proof. We only prove this proposition in the first case of Theorem 1 (the other ones are derived similarly).

Let us first prove that if $I_{\mathbb{T}} = [t_0, b]_{\mathbb{T}}$ then $b = \max \mathbb{T}$ (and thus $I_{\mathbb{T}} = \mathbb{T}$). By contradiction, assume that $I_{\mathbb{T}} = [t_0, b]_{\mathbb{T}}$ with $b < \sup \mathbb{T}$. Consider the Δ-Cauchy problem

$$z^{\Delta}(t) = f(z(t), t), \quad z(b) = q(b).$$

As in Proposition 3, we can prove that it has a solution $(z, [b, b_1]_{\mathbb{T}})$ with $b_1 \in]b, +\infty)_{\mathbb{T}}$. Then, we define q_1 by

$$q_1(t) = \begin{cases} q(t) & \text{if } t \in [t_0, b]_{\mathbb{T}}, \\ z(t) & \text{if } t \in [b, b_1]_{\mathbb{T}}, \end{cases} \tag{6}$$

for every $t \in [t_0, b_1]_{\mathbb{T}}$. Then $q_1 \in \mathscr{C}([t_0, b_1]_{\mathbb{T}})$ and one can easily prove that

$$q_1(t) = q_0 + \int_{[t_0,t)_{\mathbb{T}}} f(q_1(\tau), \tau)\Delta\tau.$$

for every $t \in [t_0, b_1]_{\mathbb{T}}$. It follows from Lemma 1 that $(q_1, [t_0, b_1]_{\mathbb{T}})$ is a solution of (Δ-CP) and is a strict extension of $(q, [t_0, b]_{\mathbb{T}})$. It is a contradiction with the maximality of $(q, [t_0, b]_{\mathbb{T}})$.

If $I_{\mathbb{T}} = [t_0, b)_{\mathbb{T}}$ with b a left-scattered point of $\mathbb{T}$, then $I_{\mathbb{T}} = [a, \rho(b)]_{\mathbb{T}}$ with $\rho(b) < \sup \mathbb{T}$ and we recover to the previous contradiction. □

Lemma 3. *Under the assumptions of Theorem 1, let $(q, I_{\mathbb{T}})$ be the maximal solution of* (Δ-CP). *If $(q, I_{\mathbb{T}})$ is not global, then* q *cannot be continuously extended with a value in Ω at $t = a$ or at $t = b$ (see Proposition 5 for* a *and* b).

Proof. We only prove this lemma in the first case of Theorem 1. By contradiction, let us assume that q can be continuously extended with a value in Ω at $t = b$, that is $\lim_{t \to b, t \in [t_0, b)_{\mathbb{T}}} q(t) = q_b \in \Omega$. Then, we define q_1 by

$$q_1(t) = \begin{cases} q(t) & \text{if } t \in [t_0, b)_{\mathbb{T}}, \\ q_b & \text{if } t = b, \end{cases}$$

for every $t \in [t_0, b]_{\mathbb{T}}$. In particular $q_1 : [t_0, b]_{\mathbb{T}} \to \Omega$ and $q_1 \in \mathscr{C}([t_0, b]_{\mathbb{T}}, \mathbb{R}^n)$. Our aim is to prove that $(q_1, [t_0, b]_{\mathbb{T}})$ is a solution of (Δ-CP).

Since $(q, [t_0, b)_{\mathbb{T}})$ is a solution of (Δ-CP), it follows from Lemma 1 that

$$q_1(t) = q(t) = q_0 + \int_{[t_0,t)_{\mathbb{T}}} f(q(\tau), \tau)\Delta\tau = q_0 + \int_{[t_0,t)_{\mathbb{T}}} f(q_1(\tau), \tau)\Delta\tau, \tag{7}$$

for every $b' \in (t_0, b)_{\mathbb{T}}$ and every $t \in [t_0, b']_{\mathbb{T}}$. Since $f(q_1, t) \in L^1_{\mathbb{T}}([t_0, b)_{\mathbb{T}}, \mathbb{R}^n)$ (see (4)), we infer from Lebesgue's dominated convergence theorem that

$$q_1(b) = q_b = q_0 + \int_{[t_0,b)_{\mathbb{T}}} f(q_1(\tau), \tau)\Delta\tau.$$

Therefore (7) also holds for $b' = b$. It follows from Lemma 1 that $(q_1, [t_0, b]_{\mathbb{T}})$ is a solution of (Δ-CP) and is a strict extension of $(q,[t_0,b)_{\mathbb{T}})$. It is a contradiction with the maximality of $(q,[t_0,b)_{\mathbb{T}})$. □

Lemma 4. *Under the assumptions of Theorem* 1, *let* $(q, I_{\mathbb{T}})$ *be the maximal solution of* (Δ-CP). *If* $(q, I_{\mathbb{T}})$ *is not global, then for every* $K \in \mathcal{K}$ *there exists* $t \in I_{\mathbb{T}}$ (*close to a or b depending on the cases listed in the theorem*) *such that* $q(t) \in \Omega \backslash K$.

Proof. We only prove this lemma in the first case of Theorem 1. By contradiction, assume that there exists $K \in \mathcal{K}$ such that q takes its values in K on $I_{\mathbb{T}} = [t_0, b)_{\mathbb{T}}$ with b a left-dense point of $\mathbb{T}$. Consider $M \geq 0$ associated with $K \in \mathcal{K}$ and $[t_0,b)_{\mathbb{T}}$ in (H_∞). For all $t_1 \leq t_2$ elements of $[t_0,b)_{\mathbb{T}}$, one has

$$\|q(t_2) - q(t_1)\| \leq \int_{[t_1,t_2)_{\mathbb{T}}} \|f(q(\tau),\tau)\|\Delta\tau \leq M(t_2 - t_1).$$

Therefore q is Lipschitz continuous and thus uniformly continuous on $[t_0,b)_{\mathbb{T}}$ with b a left-dense point of $\mathbb{T}$. Hence q can be continuously extended at $t = b$ with a value $q_b \in \mathbb{R}^n$. Moreover, since q takes its values in the compact $K \subset \Omega$, it follows that $q_b \in \Omega$. Using Lemma 3, this raises a contradiction. □

The proof of Theorem 2 follows from Proposition 5 and Lemma 4.

5.3 Proof of Theorem 3

Note that since $\Omega = \mathbb{R}^n$ and since f satisfies ($\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}}$), f automatically satisfies ($\mathrm{H}^{\mathrm{forw}}_{\mathrm{stab}}$) and ($\mathrm{H}^{\mathrm{rd}}_{\mathrm{loc\text{-}Lip}}$). Since $t_0 = \min \mathbb{T}$, (Δ-CP) admits a unique maximal solution from Theorem 1. Proving that this maximal solution is global requires the following result.

Lemma 5. *If* $t_0 = \min \mathbb{T}$ *then*

$$\int_{[t_0,t)_{\mathbb{T}}} (\tau - t_0)^k \Delta\tau \leq \frac{(t-t_0)^{k+1}}{k+1},$$

for every $k \in \mathbb{N}$ *and every* $t \in \mathbb{T}$.

Proof. One has

$$\int_{[t_0,t)_{\mathbb{T}}} (\tau - t_0)^k \Delta\tau = \int_{[t_0,t)_{\mathbb{T}}} (\tau - t_0)^k \,\mathrm{d}\tau + \sum_{r\in[t_0,t)_{\mathbb{T}}\cap\mathscr{R}} \mu(r)(r-t_0)^k,$$

for every $k \in \mathbb{N}$ and every $t \in \mathbb{T}$. Since

$$\sum_{r\in[t_0,t)_{\mathbb{T}}\cap\mathscr{R}} \mu(r)(r-t_0)^k = \sum_{r\in[t_0,t)_{\mathbb{T}}\cap\mathscr{R}} \int_{(r,\sigma(r))} (r-t_0)^k \,\mathrm{d}\tau \quad \leq \sum_{r\in[t_0,t)_{\mathbb{T}}\cap\mathscr{R}} \int_{(r,\sigma(r))} (\tau - t_0)^k \,\mathrm{d}\tau,$$

it follows that

$$\int_{[t_0,t)_{\mathbb{T}}} (\tau - t_0)^k \Delta\tau \leq \int_{[t_0,t)} (\tau - t_0)^k \, \mathrm{d}\tau = \frac{(t - t_0)^{k+1}}{k+1},$$

and the proof is complete. □

For every $b \in \mathbb{T}\backslash\{t_0\}$, we define the mapping

$$F_b : \mathscr{C}([t_0, b]_{\mathbb{T}}, \mathbb{R}^n) \longrightarrow \mathscr{C}([t_0, b]_{\mathbb{T}}, \mathbb{R}^n) \quad q \longmapsto F(q)$$

with

$$F_b(q) : [t_0, b]_{\mathbb{T}} \longrightarrow \mathbb{R}^n \quad t \longmapsto q_0 + \int_{[t_0,t)_{\mathbb{T}}} f(q(\tau), \tau)\Delta\tau.$$

From Lemma 5 and Assumption ($\mathrm{H}^{\mathrm{glob}}_{\mathrm{Lip}}$), one can easily prove by induction that

$$\left\| F_b^k(q_1)(t) - F_b^k(q_2)(t) \right\| \leq \frac{L^k}{k!} \|q_1 - q_2\|_\infty (t - t_0)^k,$$

for every $k \in \mathbb{N}^*$, all $q_1, q_2 \in \mathscr{C}([t_0, b]_{\mathbb{T}}, \mathbb{R}^n)$, and every $t \in [t_0, b]_{\mathbb{T}}$. Then,

$$\left\| F_b^k(q_1) - F_b^k(q_2) \right\|_\infty \leq \frac{(L(b - t_0))^k}{k!} \|q_1 - q_2\|_\infty,$$

for every $k \in \mathbb{N}^*$, all $q_1, q_2 \in \mathscr{C}([t_0, b]_{\mathbb{T}}, \mathbb{R}^n)$. Therefore F_b admits a contraction iterate and thus has a unique fixed point that is a solution on $[t_0, b]_{\mathbb{T}}$ of (Δ-CP). In the case of a bounded time scale $\mathbb{T}$, it suffices to take $b = \max \mathbb{T}$. In the case where $\mathbb{T}$ is not bounded, it suffices to make b tend to $+\infty$. This last comment concludes the proof of Theorem 3.

5.4 Further comments for the shifted case

An important remark in the *shifted* case is the following. Let $(a, b) \in \mathbb{T}^2$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$ and let $q : [a, b]_{\mathbb{T}} \longrightarrow \Omega$. Since $\sigma(t) \in [a, b]_{\mathbb{T}}$ for every $t \in [a, b)_{\mathbb{T}}$, q^σ is well defined on $[a, b)_{\mathbb{T}}$. This remark permits to derive all results of Section 3 in a similar way since Δ-integrals are considered on intervals of the form $[a, b)_{\mathbb{T}}$.

For example, if f satisfies (H_∞), then for all $(a, b) \in \mathbb{T}^2$ such that $a < b$,

$$f(q^\sigma, t) \in \mathrm{L}^\infty_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R}^n) \subset \mathrm{L}^1_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R}^n),$$

for every $q \in \mathscr{C}([a, b]_{\mathbb{T}}, \mathbb{R}^n)$. This remark permits to prove (from Section 2.3) the following Δ-integral characterization of the solutions of (Δ-CP$^\sigma$).

LEMMA 6. *Let $I_{\mathbb{T}} \in \mathbb{I}$ and $q : I_{\mathbb{T}} \longrightarrow \Omega$. If f satisfies* (H_∞)*, then the couple $(q, I_{\mathbb{T}})$ is a solution of* (Δ-CP$^\sigma$) *if and only if for all $a, b \in I_{\mathbb{T}}$ satisfying $a \trianglelefteq t_0 \trianglelefteq b$, one has $q \in \mathscr{C}([a, b]_{\mathbb{T}}, \mathbb{R}^n)$*

and

$$q(t) = \begin{cases} q_0 + \int_{[t_0,t)_{\mathbb{T}}} f(q^\sigma(\tau),\tau)\Delta\tau & \text{if } t \geq t_0, \\ q_0 - \int_{[t,t_0)_{\mathbb{T}}} f(q^\sigma(\tau),\tau)\Delta\tau & \text{if } t \leq t_0. \end{cases}$$

for every $t \in [a,b]_{\mathbb{T}}$.

All results permitting to prove Theorems 4 and 5 can be derived as in Section 5. Nevertheless, in order to derive Theorem 6, the following result is required.

LEMMA 7. *If* $t_0 = \max \mathbb{T}$ *then*

$$\int_{[t,t_0)_{\mathbb{T}}} (t_0 - \sigma(\tau))^k \Delta\tau \leq \frac{(t_0 - t)^{k+1}}{k+1},$$

for every $k \in \mathbb{N}$ *and every* $t \in \mathbb{T}$.

Proof. One has

$$\int_{[t,t_0)_{\mathbb{T}}} (t_0 - \sigma(\tau))^k \Delta\tau = \int_{[t,t_0)_{\mathbb{T}}} (t_0 - \tau)^k \, \mathrm{d}\tau + \sum_{r \in [t,t_0)_{\mathbb{T}} \cap \mathscr{R}} \mu(r)(t_0 - \sigma(r))^k,$$

for every $k \in \mathbb{N}$ and every $t \in \mathbb{T}$. Since

$$\begin{aligned} \sum_{r \in [t,t_0)_{\mathbb{T}} \cap \mathscr{R}} \mu(r)(t_0 - \sigma(r))^k &= \sum_{r \in [t,t_0)_{\mathbb{T}} \cap \mathscr{R}} \int_{(r,\sigma(r))} (t_0 - \sigma(r))^k \, \mathrm{d}\tau \\ &\leq \sum_{r \in [t,t_0)_{\mathbb{T}} \cap \mathscr{R}} \int_{(r,\sigma(r))} (t_0 - \tau)^k \, \mathrm{d}\tau, \end{aligned}$$

we infer that

$$\int_{[t,t_0)_{\mathbb{T}}} (t_0 - \sigma(\tau))^k \Delta\tau \leq \int_{[t,t_0)} (t_0 - \tau)^k \, \mathrm{d}\tau = \frac{(t_0 - t)^{k+1}}{k+1},$$

and the statement follows. □

Notes

1. Email: emmanuel.trelat@upmc.fr
2. Actually, this paper was motivated by the needs of completing the existing results on Cauchy–Lipschitz theory on time scales, in order to investigate nonlinear control systems with measurable controls, and finally to derive a strong version of the Pontryagin maximum principle in optimal control theory on time scales (see [8]).
3. Indeed, in the discrete case and in the case of an initial condition, such an assertion would imply that *an implicit discrete equation is equivalent to an explicit discrete equation*. But this is wrong: an implicit equation does not necessarily admit a solution while an explicit equation always does. For example, let us consider $\mathbb{T} = \mathbb{N}$ and $t_0 = 0$. In this case, the non-shifted Δ-Cauchy problem $q^\Delta(t) = f(q(t),t)$, $q(0) = 0$, has a unique global solution for any function *f*. At the opposite, the shifted Δ-Cauchy problem $q^\Delta(t) = f(q^\sigma(t),t)$, $q(0) = 0$, has no solution whenever $f(q,t) = q + 1$ for example. Hence, this shifted problem cannot be reduced into an equivalent

non-shifted problem. It can be noted that the reduction procedure mentioned in [19] is based in a crucial way on a regressivity assumption (denoted by ($\mathrm{A1}^{\sigma}$) in this paper) on *f*. We insist that we do not make such an assumption in our paper.

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