Addendum to Pontryagin’s maximum principle for dynamic systems on time scales

Loïc Bourdin\textsuperscript{a}, Oleksandr Stanzhytskyi\textsuperscript{b} and Emmanuel Trélat\textsuperscript{c}

\textsuperscript{a}XLIM Research Institute, University of Limoges, CNRS UMR 7252, Limoges, France; \textsuperscript{b}Taras Shevchenko National University of Kyiv, Kyiv, Ukraine; \textsuperscript{c}Laboratoire Jacques-Louis Lions, Sorbonne Universités, UPMC Univ Paris 06, CNRS UMR 7598, Paris, France

ABSTRACT
This note is an addendum to [L. Bourdin and E. Trélat, SIAM J. Cont. Optim., 2013] and [M. Bohner, K. Kenzhebaev, O. Lavrova and O. Stanzhytskyi, J. Differ. Equ. Appl., 2017], pointing out the differences between these papers and raising open questions.

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The main differences. In view of establishing a time scale version of the Pontryagin Maximum Principle (PMP), the authors of [1, Theorem 1] have developed in 2013 a strategy of proof based on the Ekeland variational principle. This strategy was originally considered for the classical continuous case by Ivar Ekeland in his seminal paper [3].

The authors of [2, Theorem 2.11] developed in 2017 a different approach, with packages of needle-like variations and necessary conditions for an extreme in a cone. Note that the authors of [2] prove moreover in [2, Theorem 2.13] that the necessary conditions derived in the PMP are also sufficient in the linear-convex case.

In the sequel of this paragraph, we focus on the major pros and cons of each approach:

(1) In [1]:
   (a) The set Ω of control constraints is assumed to be closed. This is in order to apply the Ekeland variational principle on a complete metric space.
   (b) There is no assumption on the time scale \( \mathbb{T} \).
(2) In [2]:
   (a) The set Ω of control constraints is assumed to be convex, but need not to be closed.

CONTACT Loïc Bourdin \texttt{loic.bourdin@unilim.fr}

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(b) The time scale $\mathbb{T}$ is assumed to satisfy density conditions (see [2, Definition 2.4]) of the kind
\[
\lim_{\beta \to 0^+} \frac{\mu(s + \beta)}{s + \beta} = 0,
\]
for every right-dense points $s$, in order to guarantee that
\[
\lim_{\beta \to 0^+} \frac{1}{\beta} \int_{[s,s+\beta)_{\mathbb{T}}} x(\tau) \Delta \tau = x(s),
\]
for $\Delta$-integrable function $x$ and for right-dense $\Delta$-Lebesgue points $s$, even for $\beta > 0$ such that $s + \beta \notin \mathbb{T}$. Note that a discussion about this issue was provided in [1, Section 3.1].

Hence, the method developed in [2] allows to remove the closedness assumption done on $\Omega$ in [1], but this is at the price of an additional assumption on the time scale $\mathbb{T}$.

In [1, Section 3.1], the authors explained why other approaches (other than the Ekeland variational principle), based for instance on implicit function arguments, or on Brouwer fixed point arguments, or on separation (Hahn-Banach) arguments, fail for general time scales.

As a conclusion, a time scale version of the PMP without closedness assumption on $\Omega$ and without any assumption on the time scale $\mathbb{T}$ still remains an open challenge.

**Additional comments on the terminal constraints.** In [1] the authors considered constraints on the initial/final state of the kind $g(x(t_0), x(t_1)) \in S$, where $S$ is a nonempty closed convex set and $g$ is a general smooth function.

In [2] the authors considered terminal state constraints of the kind $\Phi_i(x(t_0), x(t_1)) = 0$ for $i = 1, \ldots, k$, and $\Phi_i(x(t_0), x(t_1)) \leq 0$ for $i = k+1, \ldots, n$, where $\Phi_i$ are general smooth functions.

Contrarily to what is claimed in [2], the terminal constraints considered in [2] are only a particular case of the ones considered in [1]. Indeed, it suffices to take
\[
g = (\Phi_1, \ldots, \Phi_k, \Phi_{k+1}, \ldots, \Phi_n)
\]
and
\[
S = \{0\} \times \ldots \times \{0\} \times \mathbb{R}^- \times \ldots \times \mathbb{R}^-.
\]

Moreover, note that the necessary condition $-\Psi \in \mathcal{O}_S(g(x(t_0), x(t_1)))$ obtained in [1, Theorem 1] encompasses both the sign condition (1) and the complementary slackness (2) obtained in [2, Theorem 2.11]. For the sign condition, it is sufficient to recall that the orthogonal of $\mathbb{R}^-$ at a point $x \in \mathbb{R}^-$ is included in $\mathbb{R}^+$. For the complementary slackness, it is sufficient to recall that the orthogonal of $S$ at $g(x(t_0), x(t_1))$ is reduced to $\{0\}$ when $g(x(t_0), x(t_1))$ belongs to the interior of $S$.

**Additional comments on the convexity of $\Omega$.** The set $\Omega$ is assumed to be convex in [2], while it is not in [1]. As explained in [1, Section 3.1], in order to apply necessary conditions of an extreme in a cone, the authors of [2] require that the parameters of perturbations live in intervals. As a consequence, in order to remove the convexity assumption on $\Omega$, one would need (local-directional) convexity of the set $\Omega$ for perturbations at right-scattered...
points, which is a concept that differs from the stable $\Omega$-dense directions used in [1]. Hence, in spite of the claim done in [2], the convexity assumption on $\Omega$ does not seem to be easily removable.

**On the universal Lagrange multipliers.** This paragraph is devoted to providing more details on the existence of universal Lagrange multipliers claimed in [2, p.25]. In the sequel, we use the notations of [2], and we denote by $S$ the unit sphere of $\mathbb{R}^{n+1}$.

A package $P$ consists of:

- $N \in \mathbb{N}$ and $\nu \in \mathbb{N}$;
- $\mathcal{T} = (\tau_1, \ldots, \tau_N)$ where $\tau_i$ are right-dense points of $\mathbb{T}$;
- $\mathcal{V} = (v_1, \ldots, v_N)$ where $v_i \in U$;
- $\overline{\mathcal{T}} = (r_1, \ldots, r_N)$ where $r_i$ are right-scattered points of $\mathbb{T}$;
- $\mathcal{Z} = (z_1, \ldots, z_\nu)$ where $z_i \in U$.

Let $(P_i)_{i \in I}$ denotes the set of all possible packages.

Following the proof of [2, Theorem 2.11], for every $i \in I$, there exists a nonzero vector $\lambda = (\lambda_0, \ldots, \lambda_n)$ (that we renormalize in $S$) of Lagrange multipliers such that:

1. (1) and (2) in [2, Theorem 2.11] are satisfied;
2. the adjoint vector $\Psi$ solution of (2.9), with the final condition (3.65) which depends on $\lambda$, satisfies the initial condition $\Psi(t_0) = L_{\lambda_0}$;
3. (4a) and (4b) in [2, Theorem 2.11] are satisfied, but only at the points contained in $\mathcal{T}$ and $\mathcal{V}$ respectively.

For every $i \in I$, the above vector $\lambda$ is not necessarily unique. Then, for every $i \in I$, we denote by $K_i$ the set of all nonzero and renormalized Lagrange multiplier vectors associated with $P_i$ satisfying the above properties.

By continuity of the adjoint vector $\Psi$ with respect to the Lagrange multipliers (dependence from its final condition), we infer that $K_i$ is a nonempty closed subset contained in the compact $S$. This is true for every $i \in I$.

Now, let us prove that the family $(K_i)_{i \in I}$ satisfies the finite intersection property. Let $J \subset I$ be a finite subset and let us prove that $\cap_{i \in J} K_i \neq \emptyset$. Note that we can construct a package $P$ corresponding to the union of all packages $P_i$ with $i \in J$. It follows that $P \in (P_i)_{i \in I}$, and thus there exists a nonzero and renormalized Lagrange multiplier vector $\lambda$ associated with $P$ satisfying the above properties. Since $\lambda \in K_i$ for every $i \in J$, we conclude that $\cap_{i \in J} K_i \neq \emptyset$.

It follows from the lemma of a centered system in a compact set that $\cap_{i \in I} K_i \neq \emptyset$, and we deduce the existence of a universal Lagrange multiplier vector.

**On the density conditions and the Cantor set.** Contrarily to what is claimed in [2, Example 2.5], the classical Cantor set does not satisfy the density conditions. However, generalized versions of the Cantor set (see, e.g., [4]) that satisfy density conditions can be constructed as follows.

Let $(\alpha_k)_{k \in \mathbb{N}}$ be a real sequence such that $0 < \alpha_k < \frac{1}{2}$ for all $k \in \mathbb{N}$, and such that $\lim_{k \to +\infty} \alpha_k = \frac{1}{2}$. Let $(A_k)_{k \in \mathbb{N}}$ be the sequence of compact subsets defined by the induction

$$A_0 = [0, 1], \quad A_{k+1} = T_k(A_k) \quad \forall k \in \mathbb{N},$$
where $T_k$ denotes the operator removing the open $(\alpha_k, 1 - \alpha_k)$-central part of all intervals. Note that the classical Cantor set corresponds to the case where $\alpha_k = \frac{1}{3}$ for every $k \in \mathbb{N}$.

In our situation, we obtain

$$A_1 = [0, \alpha_0] \cup [1 - \alpha_0, 1],$$

$$A_2 = \left( [0, \alpha_1 \alpha_0] \cup [(1 - \alpha_1) \alpha_0, \alpha_0] \right) \cup \left( [1 - \alpha_0, 1 - (1 - \alpha_1) \alpha_0) \cup [1 - \alpha_1 \alpha_0, 1] \right),$$

etc. We define the generalized Cantor set $\mathbb{T} = \cap_{k \in \mathbb{N}} A_k$. In order to prove that the time scale $\mathbb{T}$ satisfies the density conditions, from the fractal properties of $\mathbb{T}$, it suffices to prove that the density condition is satisfied at the right-dense point $0 \in \mathbb{T}$. More precisely, it is sufficient to prove that

$$\lim_{\beta \to 0^+} \frac{\mu(\beta)}{\beta} = 0.$$

Since $\mu(\beta) = 0$ for every right-dense point $\beta$, we only have to consider the case where $\beta$ is a right-scattered point of $\mathbb{T}$. In that case, one can easily see that $\frac{\mu(\beta)}{\beta} \leq \frac{1 - 2\alpha_k}{\alpha_k}$ for some $k \in \mathbb{N}$ and that $k$ tends to $+\infty$ when $\beta$ tends to 0. The conclusion follows from the fact that $\lim_{k \to +\infty} \alpha_k = \frac{1}{2}$.

References


