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# THE TRANSCENDENCE NEEDED TO COMPUTE THE SPHERE AND THE WAVE FRONT IN MARTINET SR-GEOMETRY

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## 1. Preliminaries

Consider the local SR-geometry $(U, D, g)$, where $U$ is a neighborhood of $0 \in \mathbb{R}^3$, $D$ is a Martinet-type distribution, which can be taken in the normal form $D = \operatorname{Ker}\omega$, $\omega = dz - \frac{y^2}{2}dx$, and $g$ is a $C^\omega$ metric on $D$, which can be written (see [1]) in the normal form $a(q)dx^2 + c(q)dy^2$, $a = 1 + yF(q)$, $c = 1 + G(q)$, $G_{|x=y=0} = 0$; $a$ and $c$ can be expanded in Taylor series by using the following weights: $x$ and $y$ of weight 1 and $z$ of weight 3 given by the *privileged coordinates system* $q = (x, y, z)$ at $O$ (see [9]). Hence we obtain the *orthonormal basis*

$$F_1 = \frac{1}{\sqrt{a}}G_1, \quad G_1 = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}, \quad F_2 = \frac{1}{\sqrt{c}}G_2, \quad G_2 = \frac{\partial}{\partial y}.$$

Expanding $F_1$ and $F_2$ in Taylor series according to the previous weights and identifying at the order $p$ two elements whose Taylor series are the same at the order $p$, we obtain the following normal forms of order $-1$ and 0:

- Normal form of order $-1$:

$$g = dx^2 + dy^2 \qquad \text{(flat case);}$$

- Normal form of order 0:

$$g = (1 + \alpha y)^2 dx^2 + (1 + \beta x + \gamma y)^2 dy^2, \qquad \alpha, \beta, \gamma \in \mathbb{R}.$$

**1.1. Geodesics equations.** The *energy* minimization problem equivalent to the SR-problem is the following *optimal control problem*:

$$\begin{cases} \dfrac{dq^{(t)}}{dt} = \displaystyle\sum_{i=1}^{2} u_i(t)G_i(q(t)), \\ \displaystyle\min_{u(\cdot)} \int_0^T \big(a(q(t))u_1^2(t) + c(q(t))u_2^2(t)\big)\,dt; \end{cases}$$

from *Pontryagin's maximum principle* [9], minimizing solutions are solutions of the following equations:

$$\dot{q} = \frac{\partial H_\nu}{\partial p}, \quad \dot{p} = -\frac{\partial H_\nu}{\partial q}, \quad \frac{\partial H_\nu}{\partial u} = 0, \tag{1}$$

where $H_\nu$ is the pseudo-Hamiltonian

$$H_\nu = \sum_{i=1}^{2} u_i \langle p, G(q)\rangle - \nu(au_1^2 + cu_2^2),$$

where $\nu$ is a constant normalized to 0 or 1/2. A solution of the previous equations is called an *extremal*; when $\nu = 1/2$ (resp. $\nu = 0$), the solutions are called *normal* (resp. *abnormal*), and their projections onto the state space are called the *geodesics*.

They can be easily computed:

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- *Abnormal case.* If $\nu = 0$, $D = \text{Span}\{G_1, G_2\} = \text{Ker}\,\omega$ and they depend only on the distribution $D$. If $\omega = dz - \frac{y^2}{2}dx$, they are contained in the plane $y = 0$ called the *Martinet plane* and are the straight-lines $z = z_0$. In particular, the line passing through 0 is given by $t \longmapsto (\pm t, 0, 0)$ and is called the *abnormal direction*.
- *Normal case.* For $\nu = 1/2$ with $g = a(q)dx^2 + c(q)dy^2$, we obtain

$$H_{1/2} = \sum_{i=1}^{2} u_i G_i(q) - \frac{1}{2}(au_1^2 + cu_2^2).$$

Solving the equation $\frac{\partial H_{1/2}}{\partial u} = 0$, we obtain

$$u_1 = \frac{1}{a}(p_x + p_z \frac{y^2}{2}), \quad u_2 = \frac{p_y}{c};$$

plugging $(u_1, u_2)$ into $H_{1/2}$, we obtain the Hamilton function

$$H_n(q, p) = \frac{1}{2}\left[\frac{(p_x + p_z\frac{y^2}{2})^2}{a} + \frac{p_y^2}{c}\right],$$

where $p = (p_x, p_y, p_z)$, and (1) takes the form

$$\dot{q} = \frac{\partial H_n}{\partial p}, \quad \dot{p} = -\frac{\partial H_n}{\partial q}.$$

Another representation is obtained by using the frame $F_1$, $F_2$, and $F_3 = \frac{\partial}{\partial z}$, and defining $P = (P_1, P_2, P_3)$ with $P_i = \langle p, F_i(q)\rangle$, i.e., $P_1 = \frac{p_x + p_z\frac{y^2}{2}}{\sqrt{a}}$, $P_2 = \frac{p_y}{\sqrt{c}}$, $P_3 = p_z$. The Hamiltonian has the form $H_n = \frac{1}{2}(P_1^2 + P_2^2)$. Assuming $g$ to be independent of $z$ (*isoperimetric situation*), the normal extremals are solutions of the following equations:

$$\begin{aligned}
\dot{x} &= \frac{1}{a}\left(p_x + p_z\frac{y^2}{2}\right), \\
\dot{y} &= \frac{p_y}{c}, \\
\dot{z} &= \frac{y^2}{2a}\left(p_x + p_z\frac{y^2}{2}\right), \\
\dot{p_x} &= \frac{p_y^2 c_x}{2c^2} + \frac{\left(p_x + p_z\frac{y^2}{2}\right)^2}{2a^2}a_x, \\
\dot{p_y} &= \frac{p_y^2 c_y}{2c^2} + \frac{\left(p_x + p_z\frac{y^2}{2}\right)^2}{2a^2} - \frac{\left(p_x + p_z\frac{y^2}{2}\right)}{a}p_z y, \\
\dot{p_z} &= 0,
\end{aligned} \tag{2}$$

which take the following form in $(q, P)$ coordinates:

$$\begin{aligned}
\dot{x} &= \frac{P_1}{\sqrt{a}}, \\
\dot{y} &= \frac{P_2}{\sqrt{c}}, \\
\dot{z} &= \frac{y^2}{2}\frac{P_1}{\sqrt{a}}, \\
\dot{P}_1 &= \frac{P_2}{\sqrt{a}\sqrt{c}}\left(yP_3 - \frac{a_y}{2\sqrt{a}}P_1 + \frac{c_x}{2\sqrt{c}}P_2\right), \\
\dot{P}_2 &= -\frac{P_1}{\sqrt{a}\sqrt{c}}\left(yP_3 - \frac{a_y}{2\sqrt{a}}P_1 + \frac{c_x}{2\sqrt{c}}P_2\right), \\
\dot{P}_3 &= 0.
\end{aligned} \tag{3}$$

**1.2. Sphere and the wave front.** Let $r > 0$. The *wave front* $W(0, r)$ at 0 is the endpoints of geodesics of SR-length $r$ that start from 0; the *sphere* $S(O, r)$ is the endpoints of *minimizing geodesics* of length $r$ and that start from 0. We are interested in the local problem near 0; hence, we choose $r$ *small enough*; in this case, using the Filippov existence theorem on minimizers, we have $S(O, r) \subset W(0, r)$.

The *exponential mapping* $\exp_0$ is defined as follows. Consider a solution $(q, p)$ with $q(0) = 0$ corresponding to the Hamiltonian $H_n$ and *parametrized by arc-length* : $H_n = 1/2$. We set $\exp_0 : (p(0), t) \longmapsto q(t)$.

**Integrability problem.** Two basic questions to compute the sphere and the wave front are the following:

- *Question* 1. Are the geodesics equations (2) *integrable in quadratures*?
- *Question* 2. If the geodesics equations are integrable in quadratures, what kind of functions do we need to make the computations: *elementary functions* (exp, log, cos, sin, ...), *elliptic functions* (cn, sn, dn, E, K, etc.), or others?

In particular, if we can parametrize the solutions with no more transcendence than elliptic functions, the sphere and the wave front can be represented by using minimal computations with the Mathematica or Maple packages.

*In this paper, we make a complete analysis concerning these two problems with the graded normal form of order* 0: $g = (1 + \alpha y)^2 dx^2 + (1 + \beta x + \gamma y)^2 dy^2$.

**1.3. Singularity problem and the exp-log category.** The general theory (see, e.g., [2]) tells us that the abnormal geodesic $t \longmapsto (\pm t, 0, 0)$ is a global minimizer if its length is small enough; hence its endpoints of length $r$, where $r$ is small, given by $(\pm r, 0, 0)$ belong to the sphere $S(0, r)$. Near these endpoints the sphere has *singularities that do not belong in general to the analytic category*. In particular, this will cause numerical problems to compute the sphere near these points even in the *"integrable" case*. The goal of this article is to indicate how to deal with this problem in the "integrable" case; we compute *converging asymptotic expansions in the exp-log category*, which is the extension of subanalytic functions by the exp-log functions (see [7]). We give the scale of the asymptotic expansions.

**1.4. General research program.** More generally, the results presented in this article fit in a general research program to explain the role of abnormal minimizers in SR-geometry on the transcendence of the sphere. The main lines of this program are the following:

(1) prove that the SR-sphere is not subanalytic if there exist abnormal minimizers;
(2) prove that the SR-sphere is in the *exp-log category* if the geodesics equations are integrable by quadratures;
(3) investigate if the SR-sphere is pfaffian in the general case.

This article gives the main lines of the proof of the two first propositions in the SR-Martinet integrable case. The third, more difficult problem is briefly discussed in Sec. 4.

This research program is parallel to a research program of Agrachev–Sarychev to prove that the SR-sphere is subanalytic if there exist no abnormal minimizers (see [3]).

## 2. Integrability Problem

**2.1. Isoperimetric situation.** Since the metric does not depend on $z$, the $z$-coordinate is a *cyclic coordinate* for the Hamilton function $H_n = \frac{1}{2}(P_1^2 + P_2^2)$; hence, $p_z$ is a *first integral*, and the integrability of Eqs. (2) can be reduced to the integrability of the vector field

$$\begin{aligned} \dot{x} &= \frac{1}{a}\left(p_x + p_z\frac{y^2}{2}\right), \\ \dot{y} &= \frac{p_y}{c}, \\ \dot{p_x} &= \frac{p_y^2 c_x}{2c^2} + \frac{\left(p_x + p_z\frac{y^2}{2}\right)^2}{2a^2}a_x, \\ \dot{p_y} &= \frac{p_y^2 c_y}{2c^2} + \frac{\left(p_x + p_z\frac{y^2}{2}\right)^2}{2a^2} - \frac{\left(p_x + p_z\frac{y^2}{2}\right)}{a}p_z y, \end{aligned} \tag{4}$$

with $p_z = \lambda$ constant. The geodesics corresponding to $\lambda = 0$ are called *exceptional.* They have a geometric interpretation. If we denote by $g_R$ the Riemannian metric $a(x,y)dx^2 + c(x,y)dy^2$ induced by $g$ on the plane $(x, y)$ identified with the quotient space $\mathbb{R}^3/_{\frac{\partial}{\partial z}}$, then the trajectories of (4) with $\lambda = 0$ in the plane $(x, y)$ are the *geodesics of the Riemannian metric.*

**2.2. Metrics of the form $g = a(y)dx^2 + c(y)dy^2$.** In this case, $H_n$ does not depend on $x$ and $x$ is a *cyclic coordinate*; therefore, $p_x$ is a first integral; one more first integral is the Hamiltonian $H_n$. Hence the system has three first integrals: $p_x$, $p_z$, and $H_n$ with commuting Poisson brackets. Therefore, the system is integrable in quadratures.

We proceed as follows. If we parametrize the geodesics by arclength, we obtain $H_n = 1/2$, and the equation

$$P_1^2 + P_2^2 = 1 \tag{5}$$

with $P_1 = \dfrac{p_x + p_z\frac{y^2}{2}}{\sqrt{a}}$ and $P_2 = \dfrac{p_y}{\sqrt{c}}$, where $p_x$ and $p_z$ are constant, is called the *characteristic equation*; it can be written in the form

$$(\sqrt{c}\,\dot{y})^2 + \left(\frac{p_x + p_z\frac{y^2}{2}}{\sqrt{a}}\right)^2 = 1. \tag{6}$$

Using the time $d\eta = \dfrac{dt}{\sqrt{c}}$, we can rewrite it as follows:

$$\left(\frac{dy}{d\eta}\right)^2 + \left(\frac{p_x + p_z\frac{y^2}{2}}{\sqrt{a}}\right)^2 = 1.$$

It corresponds to the *evolution of a particle* of $\mathbb{R}$ of *mass* 2 *whose energy is* 1 in a *potential field*: $V(y) = P_1^2(y)$.

**2.3. The general gradated case of order 0: $g = (1 + \alpha y)^2 dx^2 + (1 + \beta x + \gamma y)^2 dy^2$.** If we parametrize the geodesics by arclength, we can set $P_1 = \cos\theta$, $P_2 = \sin\theta$. Moreover, if $P_3 = p_z = \lambda$ and if $\theta \neq k\pi$, we obtain the geodesics equations in *cylindrical coordinates*:

$$
\begin{aligned}
\dot{x} &= \frac{\cos\theta}{\sqrt{a}},\\
\dot{y} &= \frac{\sin\theta}{\sqrt{c}},\\
\dot{z} &= \frac{y^2}{2}\frac{\cos\theta}{\sqrt{a}},\\
\dot{\theta} &= -\frac{1}{\sqrt{a}\sqrt{c}}\left[y\lambda - \frac{a_y}{2\sqrt{a}}\cos\theta + \frac{c_x}{2\sqrt{c}}\sin\theta\right],
\end{aligned} \tag{7}
$$

and the last equation can be written as follows:

$$
\dot{\theta} = -\frac{1}{\sqrt{a}\sqrt{c}}(y\lambda - \alpha\cos\theta + \beta\sin\theta).
$$

Making the change of parametrization $\sqrt{a}\sqrt{c}\dfrac{d}{dt} = \dfrac{d}{d\tau}$ and denoting by $'$ the derivative with respect to $\tau$, we obtain

$$
\begin{aligned}
x' &= \cos\theta(1 + \beta x + \gamma y),\\
y' &= \sin\theta(1 + \alpha y),\\
z' &= \frac{y^2}{2}\cos\theta(1 + \beta x + \gamma y),\\
\theta' &= -(y\lambda - \alpha\cos\theta + \beta\sin\theta).
\end{aligned} \tag{8}
$$

The vector field can be projected onto the symplectic space $(y, \theta)$ endowed with the local symplectic form $\cos\theta dy \wedge d\theta$.

**Asymptotic integrability.** The parameters $\alpha, \beta$, and $\gamma$ are given by the metric. The exponential mapping is defined on the cylinder $(\theta, \lambda)$; the relevant behavior is when $|\lambda| \to +\infty$. Hence we make the following assumption.

**Assumption.** $|\lambda| \gg \alpha, \beta, \gamma$.

Moreover, we examine the case $\lambda > 0$; the case $\lambda < 0$ is similar.

Consider the projection of the equations on the plane $(y, \theta)$:

$$
\begin{aligned}
y' &= \sin\theta(1 + \alpha y),\\
\theta' &= -(y\lambda - \alpha\cos\theta + \beta\sin\theta).
\end{aligned} \tag{9}
$$

The singular points localized near 0 are given by $\theta = 0$, $y = \dfrac{\alpha}{\lambda}$ and $\theta = \pi$, $y = -\dfrac{\alpha}{\lambda}$, where $y \to 0$ as $\lambda \to +\infty$.

Differentiating the second equation and simplifying, we obtain

$$
\theta'' + \lambda\sin\theta + \alpha^2\sin\theta\cos\theta - \alpha\beta\sin^2\theta + \beta\cos\theta\,\theta' = 0. \tag{10}
$$

Setting $ds = \sqrt{\lambda}\,d\tau$, we obtain the equation

$$
\frac{d^2\theta}{ds^2} + \sin\theta + \varepsilon\beta\cos\theta\frac{d\theta}{ds} + \varepsilon^2\alpha\sin\theta(\alpha\cos\theta - \beta\sin\theta) = 0, \tag{11}
$$

where $\varepsilon = \dfrac{1}{\sqrt{\lambda}}$ is a *small parameter*; the remaining equations are

$$
\begin{aligned}
\frac{dx}{ds} &= \varepsilon\cos\theta(1 + \beta x + \gamma y),\\
\frac{dz}{ds} &= \varepsilon\frac{y^2}{2}\varepsilon\cos\theta(1 + \beta x + \gamma y),
\end{aligned}
$$

where $y$ is given by the second equation of (9).

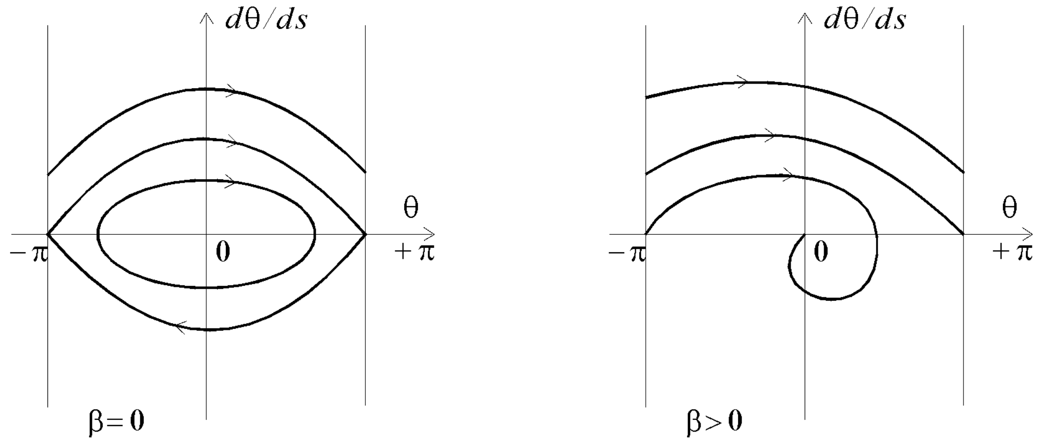


Fig. 1. The conservative and the dissipative case.

For $\varepsilon = \dfrac{1}{\sqrt{\lambda}}$, Eq. (11) defines a *one-dimensional foliation* $(\mathcal{F})$ on the *cylinder* $(e^{i\theta}, \dfrac{d\theta}{ds})$.

**Local analysis.** The foliation $(\mathcal{F})$ has two fixed singular points corresponding to $M_1 : (\theta = 0, \theta' = 0)$ and $M_2 : (\theta = \pi, \theta' = 0)$. The behavior near these two points *can be studied by linearization* of the set of equations

$$\dot\theta = v,$$
$$\dot v = -(\sin\theta + \varepsilon\beta\cos\theta\, v + \varepsilon^2\alpha\sin\theta(\alpha\cos\theta - \beta\sin\theta)).$$

We obtain the following.

- *Near* $M_1$. The linearized system is

$$\dot\theta = v,$$
$$\dot v = -(\theta(1+\varepsilon^2\alpha^2) + \varepsilon\beta v),$$

  and the eigenvalues are the complex numbers

$$\sigma_\pm = \frac{-\varepsilon\beta \pm 2i\sqrt{1+\varepsilon^2(\alpha^2 - \frac{\beta^2}{4})}}{2}.$$

  In particular, for $\beta \neq 0$, the point $M_1$ is a focus.
- *Near* $M_2$. We set $\psi = \theta - \pi$; the linearized system is

$$\dot\psi = v,$$
$$\dot v = -(-\psi - \varepsilon\beta v + \varepsilon^2\alpha^2\psi),$$

  and the eigenvalues are the two real numbers

$$\eta_\pm = \frac{\varepsilon\beta \pm 2\sqrt{1+\varepsilon^2(\frac{\beta^2}{4} - \alpha^2)}}{2};$$

  the point $M_2$ is a saddle.

**Integrability properties of $\mathcal{F}$.**

- *Near* $M_1$. We must distinguish between two cases:
  - *Case* $\beta \neq 0$. $M_1$ is a focus. The equation can be *linearized in the real analytic category*; hence, the system can be integrated locally by using the following elementary functions: exp, sin, and cos. This does not mean that the system can be (even locally) integrated in quadratures. In particular, *a focus does not admit any continuous first integral*.
  - *Case* $\beta = 0$. In this case, $M_1$ is a *center*. The global solution will be given later.

- *Near $M_2$.* The integrability problem is much more complex, since $M_2$ is a *saddle.* The *formal linearization* depends on the *resonant situation* $\eta_+/\eta_- \in \mathbb{Q}$ and the *nonresonant situation* $\eta_+/\eta_- \notin \mathbb{Q}$; but, *in both cases, there exists a formal first integral* (see [6] and the discussion of Sec. 4).

The analytic integrability is difficult to decide, but we make the following conjecture.

**Conjecture.** For $\beta \neq 0$, there exists $\varepsilon$ such that the saddle is not integrable in the real analytic category.

**Global discussion.** For $\beta = 0$, the foliation $(\mathcal{F})$ is described by

$$\theta'' + \sin\theta + \varepsilon^2\alpha^2 \sin\theta\cos\theta = 0, \tag{12}$$

where $\theta'$ is the derivative with respect to $s$.

This equation is integrable and is indeed a standard equation from elasticity theory (see [14, 4]). It can be integrated as follows. Multiplying by $\theta'$, we obtain

$$\theta''\theta' + \sin\theta\,\theta' + \varepsilon^2\alpha^2\sin\theta\cos\theta\,\theta' = 0.$$

Integrating, we obtain

$$\theta'^2(s) - \theta'^2(0) = \cos\theta(s) - \cos\theta(0) + \frac{\varepsilon^2\alpha^2}{2}(\cos^2\theta(s) - \cos^2\theta(0)). \tag{13}$$

**Remark.** In our problem, $\dfrac{d\theta}{ds}_{|s=0}$ is computed by using the relation $q(0) = (x(0), y(0), z(0)) = 0$.

We observe that $\theta$ can be integrated with *only one quadrature* by using Eq. (13). We easily deduce the following proposition.

**Proposition 2.1.** *The foliation $(\mathcal{F})$ is integrable in the $C^0$-category if and only if $\beta = 0$. In this case, it is integrable in the $C^\omega$-category. The condition $\beta = 0$ is equivalent to the fact that $\theta$ can be integrated by using one quadrature.*

**Integration in the case $\beta = 0$.** This case is called the *conservative case* and the Hamiltonian $H_n$ has two cyclic coordinates: $x$ and $z$; the geodesics equations have the three first integrals: $p_x$, $p_z$, and $H_n = \frac{1}{2}(P_1^2+P_2^2)$ whose Poisson brackets are zero. The angle $\theta$ can be computed by using one quadrature, and the same is true for $y$ by using the relation between $y$ and $\theta$ that comes from the equation $p_x =$ constant.

*Using the analogy with the pendulum, where the derivative of the angle can be represented by the Jacobi functions cn and dn* (see [10]), we can compute $y$ by using the same Jacobi functions. This comes from the following analysis. The *characteristic equation* (6) can be written by using the parametrization $d\tau = \dfrac{dt}{\sqrt{a}\sqrt{c}}$:

$$\left(\frac{dy}{d\tau}\right)^2 + \left(p_x + p_z\frac{y^2}{2}\right)^2 = a, \tag{14}$$

where $a = (1+\alpha y)^2$.

Hence, setting $F(y) = (1+\alpha y)^2 - \left(p_x + p_z\dfrac{y^2}{2}\right)^2$, we see that $F$ is a quartic, which can be written as $F_1F_2$ with

$$F_1 = (1+\alpha y) - \left(p_x + p_z\frac{y^2}{2}\right) \quad \text{and} \quad F_2 = (1+\alpha y) + \left(p_x + p_z\frac{y^2}{2}\right);$$

now we can write

$$F(y) = \left(2m^2 - \frac{\lambda}{2}\left(y - \frac{\alpha}{\lambda}\right)^2\right)\left(2m'' + \frac{\lambda}{2}\left(y + \frac{\alpha}{\lambda}\right)^2\right),$$

where $2m^2 = 1 - p_x + \dfrac{\alpha^2}{2\lambda}$, $2m'' = 1 + p_x - \dfrac{\alpha^2}{2\lambda}$, and $m^2 + m'' = 1$.

We have $p_x = \cos\theta(0)$; hence, $|p_x| \leqslant 1$. Then $m^2 > 0$ if $\alpha \neq 0$; $m^2 > 0$ if $\alpha = 0$ and $\theta(0) \neq n\pi$.

If we set

$$\eta = \frac{\sqrt{\lambda}\,y}{2m} - \frac{\alpha}{2m\sqrt{\lambda}} \quad \text{and} \quad \bar{\eta} = \frac{\sqrt{\lambda}\,y}{2m} + \frac{\alpha}{2m\sqrt{\lambda}},$$

we can write

$$F(y) = 4m^2(1-\eta^2)(m'' + m^2\bar{\eta}^2); \tag{15}$$

$F$ is a quartic whose roots on $\mathbb{C}$ are $\eta = \pm 1$ and $\bar{\eta} = \pm\frac{\sqrt{m''}}{m}$. The case $m'' = 0$ is called *critical*; it corresponds to a double root for $F$.

**Lemma 2.2.** *If $\alpha \neq 0$ in the graded normal form of order 0, then there exist geodesics starting from 0 that are critical.*

**Geometric interpretation.** If $\alpha = 0$, the geodesics can be integrated as in the flat case studied in [2]: $m'' = k'^2 = \sqrt{1-k^2}$, where $k$ is the modulus of the elliptic functions and $k'$ is the complement of the modulus. When $p_x \to -1$ and $k' \to 0$, $y$ behaves as sech. In the $(\theta, \dot{\theta})$ projection, the system has a *saddle connection* and the projections of the geodesics tend to the *separatrix*.

When $\alpha \neq 0$, the separatrix is the projection of a geodesic starting from 0.

*The role of the parameter $\alpha$ is to make the separatrix and hence some rotating trajectories of the pendulum as the projection of geodesics starting from* 0.

**Normal form.** The characteristic equation can be normalized by using a classical method (see [10, p. 55]). We proceed as follows: $F$ is factorized as $F_1F_2$, and we consider the *pencil* $F_1 + \nu F_2$ of two quadratic forms.

If $\alpha \neq 0$, there exist two distinct real numbers $\nu_1$ and $\nu_2$ such that $F_1 + \nu F_2$ is a *perfect square* $K_1(y-p)^2$, $K_2(y-q)^2$.

Using the homographic transformation

$$u = \frac{y-p}{y-q}, \tag{16}$$

we can write the characteristic equation in the normal form

$$\frac{dy}{\sqrt{F(y)}} = \frac{(p-q)^{-1}du}{\sqrt{(A_1u^2+B_1)(A_2u^2+B^2)}}. \tag{17}$$

Except for the critical case $m'' = 0$, the solution in the $u$-coordinate can be computed as follows:

- if the quartic $F$ admits two real roots, $u$ can be given by using the cn Jacobi function;
- if the quartic $F$ admits four real roots, $u$ can be given by using the dn Jacobi function.

If $\alpha = 0$, the analysis is simpler; indeed $F(y)$ can be written as

$$F(y) = 4k^2(1-\eta^2)(k'^2 + k^2\eta^2),$$

where $\eta = \frac{\sqrt{\lambda}\,y}{2k}$, and $\eta$ can be computed by using only the cn function.

Hence we have proved the following assertion.

**Proposition 2.3.** *We have the following two cases*:

(i) *If $\alpha = 0$, then $y = \frac{2k}{\sqrt{\lambda}}\eta$, where $\eta$ is the cn Jacobi function.*

(ii) *If $\alpha \neq 0$, then $y$ is the image by a homography of the cn or dn Jacobi function.*

**Geometric interpretation.** If $\alpha = 0$, the motion of $y$ is the cn function whose amplitude is $\frac{2k}{\sqrt{\lambda}}$; in particular, the motion is symmetric with respect to $y = 0$ and the amplitude tends to 0 as $\lambda$ tends to the infinity.

If $\alpha \neq 0$, we can expand the homography $y = \dfrac{uq - p}{u - 1}$ near $u = 0$. The motion of $y$ is no longer symmetric with respect to $y = 0$; there is a constant term in the expansion. Hence $y$ can be approximated for $u$ small enough by a shift plus a cn or dn motion.

**Integrating $x$ or $z$.** Both $x(\tau)$ and $z(\tau)$ can be computed by using only one integral. The integrand is a polynomial function of $y$. Moreover, $y$ can be expanded into a power series in $u$. Hence the transcendence needed to compute $x$ or $z$ is given by primitives of the form

$$J_m = \int \operatorname{cn} mu\, du \quad \text{and} \quad K_m = \int \operatorname{dn}^m u\, du.$$

These primitives are computed by recurrence in [10, p. 87]. It involves a new transcendence: the *Jacobi epsilon function* $E(u, k)$ defined by

$$E(u, k) = \int_0^u \operatorname{dn}^2(v, k)\, dv.$$

*This function was already needed in the flat case* (see [2]).

**Arclength parameter.** To recover the length parameter, we use the formula $dt = (1 + \alpha y)(1 + \gamma y) d\tau$. As previously, $y$ can be computed as a power series in $u$; hence, it can be evaluated by using the same primitives $J_m$ and $K_m$.

**2.4. Application: computation of conjugate points.** One interesting and nontrivial application of the previous parametrizations is the computation of the conjugate points; they are solutions of the equation

$$\frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \theta_0} - \frac{\partial y}{\partial \lambda}\frac{\partial x}{\partial \theta_0} = 0,$$

where $\theta_0 = \theta(0)$ and $x$ and $y$ are the two first components of a normal geodesic. This equation was used in the flat case in order to evaluate the conjugate points.

## 3. Transcendence of the Sphere and the Wave Front near the Abnormal Line. The exp-log Category

**3.1. The geometric framework.** In order to study the structure of the sphere or the wave front near the abnormal direction, it is convenient to consider the following *traces*:

$$\widetilde{S}(0, r) = S(0, r) \cap \{y = 0\} \quad \text{and} \quad \widetilde{W}(0, r) = W(0, r) \cap \{y = 0\}.$$

This leads to the concept of the return mapping.

**Definition 3.1.** Let $e : t \in [0, T] \longmapsto (x(t), y(t), z(t))$ be a normal geodesic parametrized by arclength. If $y(t) \not\equiv 0$, we can define $0 < t_1 < \cdots < t_N \leqslant T$, the instants of time that correspond to $y(t_i) = 0$. The first *return mapping* is

$$R_1 : (\lambda, \theta(0)) \longmapsto (x(t_1), z(t_1));$$

more generally, the $n$th mapping is the map

$$R_n : (\lambda, \theta(0)) \longmapsto (x(t_n), z(t_n)).$$

If the length is fixed at $r$, we observe that $\widetilde{W}$ is the union of the image of the return mapping with $x = \pm r$, $z = 0$.

The following proposition is straightforward (see [4]):

**Proposition 3.2.** *For each $n \geqslant 1$, the return mapping $R_n$ is not proper.*

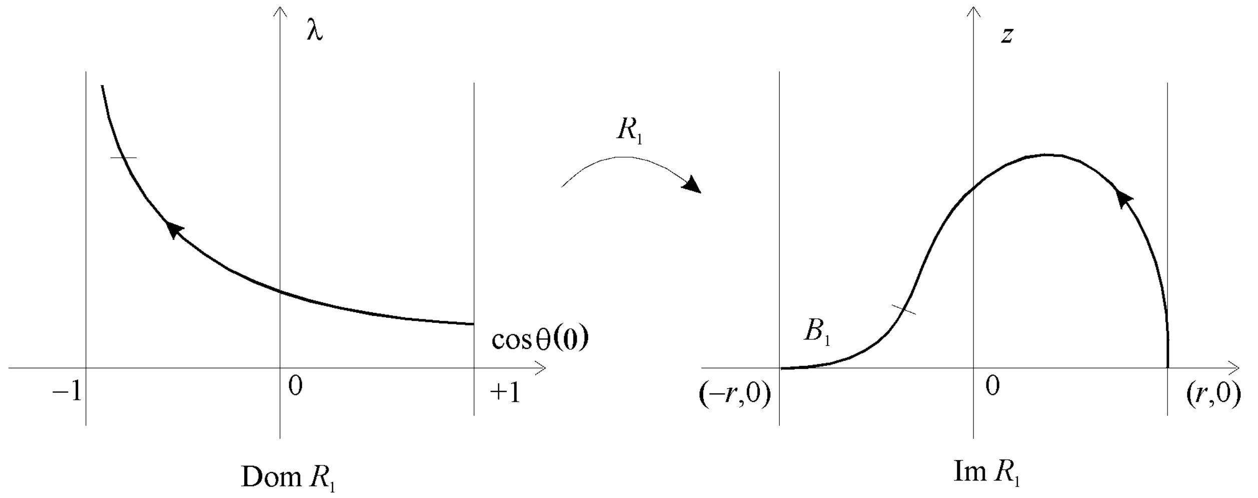


Fig. 2. The first return mapping in the flat case.

**3.2. Formulas in the conservative case.** If the metric $g$ does not depend on $x$, it is convenient to use the following formulas (see [8]); we introduce

$$\sigma = \begin{cases} \operatorname{sign} \dot{y}(0) & \text{if} \quad \dot{y}(0) \neq 0, \\ \operatorname{sign} \ddot{y}(0) & \text{if} \quad \dot{y}(0) = 0. \end{cases}$$

If the motion of $y$ is periodic with period $\mathcal{P}$, we set

$$y_+ = \max_{t \in [0,\mathcal{P}]} y(t) \quad \text{and} \quad y_- = \min_{t \in [0,\mathcal{P}]} y(t).$$

Parametrizing the geodesics by $y$, we should solve the equations

$$\frac{dx}{dy} = \frac{\sqrt{c}}{\sqrt{a}} \frac{P_1}{P_2}, \quad \frac{dz}{dy} = \frac{y^2}{2} \frac{\sqrt{c}}{\sqrt{a}} \frac{P_1}{P_2}, \quad dt = \frac{\sqrt{c}}{P_2} dy,$$

where $P_2(y) = \sigma\sqrt{1 - P_1^2(y)}$ for $t \in [0, t_1]$.

If $y(T) = 0$ for $T = t_N$, we obtain the following formulas.

- *N odd*

$$\begin{aligned} x(T) &= 2 \int_0^{y_\sigma} \sigma \frac{\sqrt{c}}{\sqrt{a}} \frac{P_1(y)}{\sqrt{1 - P_1^2(y)}} dy + (N-1) \int_{y_-}^{y_+} \frac{\sqrt{c}}{\sqrt{a}} \frac{P_1(y)}{\sqrt{1 - P_1^2(y)}} dy, \\ z(T) &= \int_0^{y_\sigma} \sigma \frac{\sqrt{c}}{\sqrt{a}} \frac{y^2 P_1(y)}{\sqrt{1 - P_1^2(y)}} dy + (N-1) \int_{y_-}^{y_+} \frac{\sqrt{c}}{2\sqrt{a}} \frac{y^2 P_1(y)}{\sqrt{1 - P_1^2(y)}} dy. \end{aligned} \tag{18}$$

- *N even*

$$\begin{aligned} x(T) &= N \int_{y_-}^{y_+} \frac{\sqrt{c}}{\sqrt{a}} \frac{P_1(y)}{\sqrt{1 - P_1^2(y)}} dy, \\ z(T) &= N \int_{y_-}^{y_+} \frac{\sqrt{c}}{2\sqrt{a}} \frac{y^2 P_1(y)}{\sqrt{1 - P_1^2(y)}} dy, \end{aligned} \tag{19}$$

and the period is given by

$$\mathcal{P} = 2 \int_{y_-}^{y_+} \frac{\sqrt{c}}{\sqrt{1 - P_1^2(y)}} dy. \tag{20}$$

**3.3. Computations in the flat case.** The basis of the general algorithm to compute the image of the return mapping is coming from the flat case: $g = dx^2 + dy^2$. The algorithm is as follows. Both sets $\widetilde{S}(0,r)$ and $\widetilde{W}(0,r)$ are symmetric with respect to 0, and we can assume that $z > 0$. From [2], the image of $R_1$ in $z > 0$ is parametrized by

$$\begin{aligned} x(k,\lambda) &= -t + \frac{4E}{\sqrt{\lambda}}, \\ z(k,\lambda) &= \frac{2}{3\lambda^{3/2}}[2(2k^2-1)E + k'^2 K], \end{aligned} \tag{21}$$

where $K$ and $E$ are complete elliptic integrals with modulus $k = \sqrt{\frac{1-p_x}{2}}$, $p_x = \cos\theta(0)$, $k' = \sqrt{1-k^2}$, and $\theta(0) \in [-\pi, 0]$:

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad E = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\, d\theta$$

and the period $\mathcal{P} = \frac{4}{\sqrt{\lambda}} K(k)$.

Both parameters $\lambda$ and $k$ are related when we fix the length to $r$:

$$t = r = \frac{2K}{\sqrt{\lambda}}. \tag{22}$$

Hence the image of $R_1$ in $z > 0$ is given by

$$\begin{aligned} x &= -r + 2r\frac{E}{K}, \\ z &= \frac{r^3}{6K^3}\left[(2k^2-1)E + k'^2 K\right]. \end{aligned} \tag{23}$$

It is a parametric curve and the parameter is $k$. It is semi-analytic except for the case where $\theta(0) \to -\pi$ and $k \to 1^-$. This can be seen from the following expansions for $E$ and $K$ when $k' \to 0$:

$$\begin{aligned} E &= u_1(k'^2)\ln\frac{4}{k'} + u_2(k'^2), \\ K &= u_3(k'^2)\ln\frac{4}{k'} + u_4(k'^2), \end{aligned}$$

where $u_i$ are analytic functions, and, moreover,

$$\begin{aligned} u_1(k'^2) &= \frac{k'^2}{2} + o(k'^3), & u_2(k'^2) &= 1 - \frac{k'^2}{4} + o(k'^3), \\ [4pt] u_3(k'^2) &= 1 + \frac{k'^2}{4} + o(k'^3), & u_4(k'^2) &= -\frac{k'^2}{4} + o(k'^3). \end{aligned}$$

In particular, both $E$ and $K$ have a *logarithmic singularity* as $k' \to 0$; hence, using [11], the branch of (23) near $x = -r$, denoted by $B_1$, can be computed in the *exp-log category*. More precisely, we proceed as follows.

Let $X = \dfrac{x+r}{2r}$ and $Z = \dfrac{z}{r^3}$. We obtain

$$X = \frac{E}{K} = \frac{u_1(k'^2)\ln\frac{4}{k'} + u_2(k'^2)}{u_3(k'^2)\ln\frac{4}{k'} + u_4(k'^2)}, \tag{24}$$

$$Z = \frac{1}{6K^3}\left[(2k^2-1)E + k'^2 K\right]. \tag{25}$$

Then

*Step* 1. *"Compactification."* If we introduce

$$X_1 = k' \quad \text{and} \quad X_2 = \frac{1}{\ln\frac{4}{k'}},$$

then we have $X_1, X_2 \to 0$ as $k' \to 0^+$ and both $X$ and $Z$ are *analytic functions* of $X_1$ and $X_2$.

*Step* 2. *"Finding equivalents."* An easy computation with the use of (24) shows the following:

$$X_1 \sim 4e^{-\frac{1}{X}}, \quad X_2 \sim X \qquad \text{as} \quad X \to 0^+;$$

we can write

$$X_1 = 4e^{-\frac{1}{X}}(1 + Y_1(X)), \quad X_2 = X(1 + Y_2(X)),$$

where $Y_1, Y_2 \to 0$ as $X \to 0^+$.

Both $Y_1$ and $Y_2$ *can be compared*, and a computation gives us

$$Y_2 = X\, A_1(X, Y_1), \quad Y_1 \sim \frac{Y_2}{X} \quad \text{as} \quad X \to 0^+,$$

where $A_1$ is the germ of an analytic function at 0.

*Step* 3. *"Solving Eq.* (24) *in the analytic category."* Equation (24) can be solved in the variables $Y_1, X_1$, and $X_2$ by using the *implicit function theorem in the analytic category*; the computations show the following:

$$Y_1 = A_2\left(X, \frac{e^{-\frac{1}{X}}}{X}\right),$$

where $A_2$ is the germ at 0 of an analytic function.

Using this relation we arrive at

$$Z = F\left(X, \frac{e^{-\frac{1}{X}}}{X}\right),$$

where $F$ is the germ at 0 of an analytic function.

**N.B.** If we use only the fact that the functions $u_i$ are analytic with respect to $k'$, we obtain the scale $\dfrac{e^{-\frac{1}{X}}}{X^2}$.

**3.4. Computations in the general conservative case.** The algorithm is similar to the flat case using the *integral formulas* of Sec. 3.2, but the computations are much more complex. The additional complexity is coming from two phenomena called respectively the *double log* and the *period halfing*.

**3.4.1. Double log.** In the flat case, relation (22), which expresses the fact that the length is fixed to $r$, is trivial. In general, this is no longer true and we must solve an equation of the type

$$x = y \ln y, \qquad y \to +\infty.$$

We set

$$y = \frac{x}{\ln x}(1 + Y_1(x)) \quad \text{with} \quad Y_1 = o(1);$$

substituting $y$ into the equation and using the implicit function theorem in the analytic category, we obtain the relation

$$Y_1 = A(X_1, X_2),$$

where $A$ is the germ at 0 of an analytic function; $X_1$ and $X_2$ represent the scale factors, that is,

$$X_1 = \frac{1}{\ln x} \quad \text{and} \quad X_2 = \frac{\ln \ln x}{\ln x}.$$

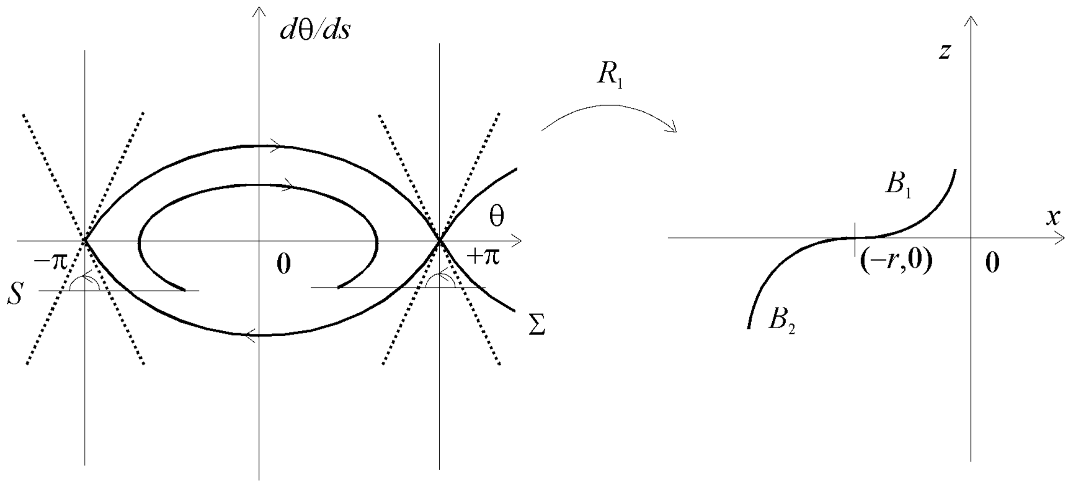


Fig. 3. The separatrix $\Sigma$.

**3.4.2. Period halfing.** In the flat case, the image of $R_1$ contains only one branch $B_1$ in the domain $z > 0$ that is not subanalytic. It corresponds to the limit behavior of the *oscillating* trajectories of the pendulum as $\theta(0) \to -\pi$, which tend to the separatrix. When $\alpha \neq 0$, we see from our analysis that we must consider, on one hand, *oscillating trajectories*, where $y$ is parametrized by the cn Jacobi elliptic function, and on the other hand, *rotating trajectories*, where $y$ is parametrized by the dn Jacobi elliptic function. This can be interpreted as a *period halfing phenomenon* by using the relation $\mathrm{dn}^2 s = k'^2 + k^2 \mathrm{cn}^2 s$ for fixed $k$ and taking $k \to 1$. In this case, the image of $R_1$ in the domain $z > 0$ contains *two branches* $B_1$ and $B_2$, which end respectively at $x = \pm r, z = 0$ and one *not subanalytic*. The branch $B_1$ corresponds to a cn-behavior and the branch $B_2$ to a dn-behavior. The branch $B_2$ shrinks to 0 as $\alpha \to 0$.

Figure 3 illustrates the role of the parameter $\alpha$. Indeed, imposing $y(0) = 0$ and $y(r) = 0$, this defines a section $S$ given in the space $\left(\theta, \dfrac{d\theta}{ds}\right)$ by the equation

$$\frac{d\theta}{ds} = \varepsilon(\alpha \cos\theta - \beta \sin\theta).$$

The role of the parameter $\alpha$ is to push the separatrix $\Sigma$ as an admissible trajectory; hence we get two nonsubanalytic branches $B_1$ and $B_2$. This phenomenon is illustrated in Fig. 3.

**3.5. Algorithm for computing $B_1$ and the complexity of $B_1$.** The aim is to give a precise transcendence of the branch $B_1$ in the general conservative case. From now on, $\mathrm{An}(\cdot)$ and $\mathrm{An}_0(\cdot)$ denote the germ of an analytic function at 0, and, moreover, $\mathrm{An}_0(0) = 0$.

Recall the general formulas that give a parametrization of this branch:

$$x(r) = -2\int_0^{y_{-1}} \frac{\sqrt{c}}{\sqrt{a}} \frac{P_1}{\sqrt{1-P_1^2}}\, dy, \tag{26}$$

$$z(r) = -\int_0^{y_{-1}} \frac{\sqrt{c}}{\sqrt{a}} \frac{y^2 P_1}{\sqrt{1-P_1^2}}\, dy, \tag{27}$$

$$r = -2\int_0^{y_{-1}} \sqrt{c}\, \frac{1}{\sqrt{1-P_1^2}}\, dy, \tag{28}$$

where

$$a(y) = \mathrm{An}(y) = 1 + \alpha y + \alpha' y^2 + \dots,$$
$$c(y) = \mathrm{An}(y) = 1 + \gamma y + \dots$$

and

$$P_1(y) = \frac{p_x + \frac{\lambda}{2} y^2}{\sqrt{a(y)}} = p_x - p_x \frac{\alpha}{2} y + \left(p_x(\frac{3}{8}\alpha^2 - \alpha') + \frac{\lambda}{2}\right) y^2 + \dots,$$

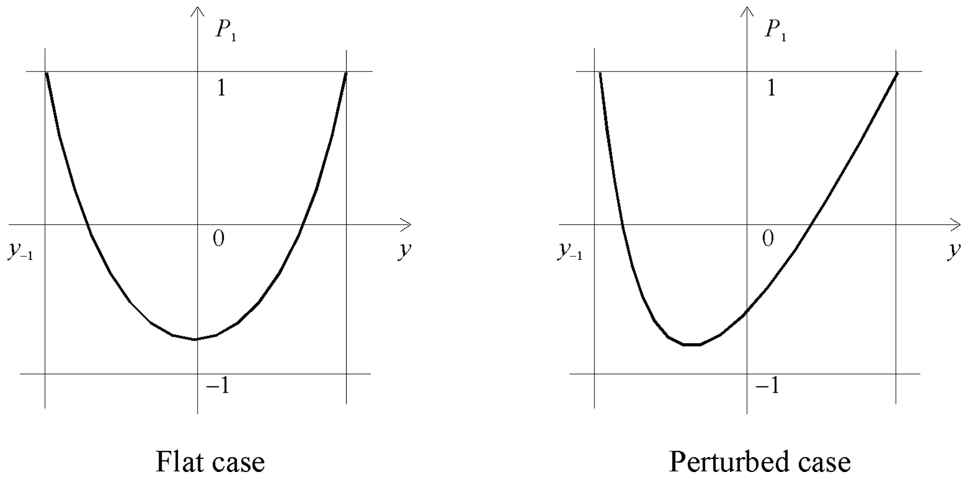


Fig. 4. The graph of $P_1$.

$y_{-1}$ is the negative root of $1 - P_1(y)$ (we will justify it later).

Our aim is to express $x$ and $z$ as a parametric curve in the exp-log category and compute the graph in the same category by elimination of the parameter.

**3.5.1. Precision on parameters.** We study the system near the abnormal direction, so we have

$$\lambda \to +\infty, \quad p_x \to -1.$$

**Precision on $y_{-1}$.** In the flat case, $P_1$'s graph is a parabola, represented in Fig. 4.

The general integrable case can be considered as a perturbation of the flat case where the parabola is deformed into a nonsymmetric graph (see Fig. 4).

Hence the existence of $y_{-1}$, as a negative simple root of $P_1 = 1$, is straightforward.

Using the implicit function theorem, we obtain

$$y_{-1} = \frac{1}{\sqrt{\lambda}} \operatorname{An}\left(p_x, \frac{1}{\sqrt{\lambda}}\right).$$

Moreover, by continuity with the flat case, $y_{-1} \underset{\sqrt{\lambda}\to+\infty}{\sim} -\sqrt{\frac{2}{\lambda}(1-p_x)}$.

**Precision on $P_1'(y)$ roots.**

**Proposition 3.3.** *In the foregoing domain there is a unique root $S$ of $P_1'$. Moreover,*

$$S = \frac{\alpha}{\lambda} \operatorname{An}(p_x, \frac{1}{\lambda}) \underset{\lambda\to+\infty}{\sim} \frac{p_x \alpha}{2\lambda} < 0.$$

**Remark 3.4.** $S = 0 \Leftrightarrow \alpha = 0$.

**Proof.** If $A$ is large enough, then $P_1''(y) > 0 \ \forall y$. Moreover, $P_1'\left(-\frac{2}{\sqrt{\lambda}}\right) P - 1'\left(\frac{2}{\sqrt{\lambda}}\right) < 0$. We can deduce that $P_1'$ admits a unique root in this domain. The remaining goes as before.

**Remark 3.5.** In the flat case, we have $S = 0$ and $P_1(S) = p_x$.

This remark leads us to define the following objects.

**Definition 3.6.** $k' = \sqrt{\dfrac{1+P_1(S)}{2}}$ and $k = \sqrt{\dfrac{1-P_1(S)}{2}}$.

As in the flat case, we have $k^2 + k'^2 = 1$; then $k = 1 + \mathrm{An}(k'^2)$. Note that $p_x \to -1$, so $k' \to 0$.

Our new parameters are $k'$ and $\lambda$. They are the initial conditions for the geodesics. Our aim is the following: from (26) and (28) we express $x$ and $z$ in terms of $k'$ and $\lambda$; from (27) we obtain an implicit relation between $k'$ and $\lambda$. Solving this implicit equation, we will get $\lambda$ in terms of $k'$, so that we get $x$ and $z$ in terms of $k'$. Then the problem is to eliminate the parameter $k'$, to get finally the graph $z(x)$.

As we will see, the previous expansions are in the exp-log category (see [11]), e.g., these are analytic expansions in $k'$ and some functions composed of exp and log. Hence the aim is to express the graph $z(x)$ in this category, with a precise scale.

We encounter two technical problems:

(1) justifying the *convergence of the expansions*,
(2) solving algorithmically this problem of elimination of the parameter in the *exp-log category*.

The problem of analyticity of the expansions is based on the following proposition.

**Proposition 3.7.** *Let $f_n(x) = \sum a_{n,p}x^p$, $n \in \mathbb{N}$, be a family of entire series that converge for $|x| < 1$. Assume that $\exists A\ \forall p \sum\limits_n |a_{n,p} \leqslant A$. Then $f(x) = \sum\limits_n f_n(x) = \sum\limits_p \left(\sum\limits_n a_{n,p}\right) x^p$ is analytic and converges for $|x| < 1$.*

In what follows, we will not detail all these calculations, which would be too long. Note that, to do this, formal computations using the Maple package was very helpful.

We will now express all our parameters in terms of $k'$ and $\frac{1}{\sqrt{\lambda}}$.

**Expression of $p_x$.** By definition, $k'^2 = \dfrac{1 + P_1(S)}{2}$ with $S = \dfrac{\alpha}{\lambda}\,\mathrm{An}\left(p_x, \dfrac{1}{\lambda}\right)$.

Thus, $2k'^2 - 1 = P_1(S) = p_x + \dfrac{1}{\lambda}\,\mathrm{An}\left(p_x, \dfrac{1}{\lambda}\right)$.

Using the implicit function theorem, we conclude that

$$p_x = \mathrm{An}\left(k'^2, \frac{1}{\lambda}\right) \sim -1. \tag{29}$$

**Expression of $S$.** We easily obtain

$$S = \frac{1}{\lambda}\,\mathrm{An}\left(k'^2, \frac{1}{\lambda}\right) \sim -\frac{\alpha}{2\lambda}. \tag{30}$$

**Expression of $y_{-1}$.** We obtain

$$y_{-1} = \frac{1}{\sqrt{\lambda}}\,\mathrm{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}\right) \sim -\frac{2}{\sqrt{\lambda}}. \tag{31}$$

**3.5.2. Preliminaries before calculating integrals.** The aim is to expand analytically all the integrands, so that very simple reference integrals appear, which will give the precise transcendence of the branch.

**Expansion of $P_1$ with the new parameters.** $P_1$ appears in all formulas, so it is natural to deal with its expression.

Recall that $P_1(y) = \dfrac{p_x + \frac{\lambda}{2}y^2}{\sqrt{a(y)}} = \mathrm{An}\left(p_x, \dfrac{1}{\sqrt{\lambda}}, \sqrt{\lambda}\,y\right)$.

Let us make the *change of variable*

$$y = \frac{1}{\sqrt{\lambda}}\left(2k\eta + S\sqrt{\lambda}\right). \tag{32}$$

Recalling that $P_1'(S) = 0$, we actually obtain

$$P_1(y) = P_1(S) + 2k^2\eta^2\left(1 + \frac{1}{\lambda}F\left(p_x, \frac{1}{\sqrt{\lambda}}\right)\right)\frac{\eta^3}{\sqrt{\lambda}}\,\mathrm{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, \frac{\eta}{\sqrt{\lambda}}\right), \tag{33}$$

where $F$ is analytic.

**Expansion of $\dfrac{1}{\sqrt{1-P_1(y)}}$.** From (33), we obtain

$$1 - P_1(y) = 2k^2 - 2k^2\eta^2 \left(1 + \frac{1}{\lambda}\operatorname{An}\left(p_x, \frac{1}{\sqrt{\lambda}}\right)\right) - \frac{\eta^3}{\sqrt{\lambda}}\operatorname{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, \frac{\eta}{\sqrt{\lambda}}\right).$$

Hence, noting that $k = \operatorname{An}(k'^2)$, we have

$$\frac{1}{\sqrt{1-P_1(y)}} = \frac{1}{\sqrt{2}}\left(1 + \operatorname{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, \eta^2, \frac{\eta}{\sqrt{\lambda}}\right)\right) \tag{34}$$

(this is valid if $|P_1(y)| < 1$; this will be the case in our integrals).

**Expansion of $\dfrac{1}{\sqrt{1+P_1(y)}}$.** From (33), we obtain

$$1 + P_1(y) = 2k'^2 + 2k^2\eta^2 \left(1 + \frac{1}{\lambda}F\left(p_x, \frac{1}{\sqrt{\lambda}}\right)\right) + \frac{\eta^3}{\sqrt{\lambda}}\operatorname{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, \frac{\eta}{\sqrt{\lambda}}\right).$$

If we make the change of variable $\eta = \dfrac{k'}{k}\dfrac{1}{1+\frac{1}{\lambda}F\left(p_x, \frac{1}{\sqrt{\lambda}}\right)}u$, then

$$\begin{aligned} 1 + P_1(y) &= 2k'^2\left(1 + u^2 + \frac{k'u^3}{\sqrt{\lambda}}\operatorname{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}}\right)\right) \\ &= 2k'^2(1+u^2)\left(1 + \frac{k'u^3}{\sqrt{\lambda}(1+u^2)}\operatorname{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}}\right)\right). \end{aligned}$$

Hence

$$\frac{1}{\sqrt{1+P_1(y)}} = \frac{1}{k'\sqrt{2}}\frac{1}{\sqrt{1+u^2}}\left(1 + \operatorname{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}}, \frac{k'u^3}{\sqrt{\lambda}(1+u^2)}\right)\right) \tag{35}$$

(the same remark on the validity of the expansion holds).

### 3.5.3. Expansions of the integrals. Reference integrals.

As we will see later, the following integrals are useful. They will be in our expansions of $x$, $z$, and $r$ and in our reference integrals.

**Proposition 3.8.** *Let $p, i \in \mathbb{N}$. Then*

$$\begin{aligned} \int_0^x \frac{u^{2i}}{\left(\sqrt{1+u^2}\right)^{2p+1}}\,du = \frac{1}{\left(\sqrt{1+x^2}\right)^{2p-1}}&\Bigg(\mu_0\left(\sqrt{1+x^2}\right)^{2p-1}\ln(x+\sqrt{1+x^2}) \\ &+\mu_1 x + \mu_3 x^3 + \cdots + \mu_{2i-1}x^{2i-1}\Bigg), \end{aligned}$$

$$\int_0^x \frac{u^{2i+1}}{\left(\sqrt{1+u^2}\right)^{2p+1}}\,du = \mu_1 + \frac{1}{\left(\sqrt{1+x^2}\right)^{2p-1}}\left(\mu_0 + \mu_2 x^2 + \cdots + \mu_{2i}x^{2i}\right).$$

**Proof.** The proof is elementary. Let $I_{i,p}$ be one of the two integrals being studied. We have immediately

$$I_{i+1,p} = I_{i,p-1} - I_{i,p}.$$

Thus, it is enough to calculate $(I_{i,0})_{i\in\mathbb{N}}$ and $(I_{0,p})_{p\in\mathbb{N}}$ and to get all $I_{i,p}$; this is easy.

**Proposition 3.9.** *Let $i \in \mathbb{N}$. Then*

$$\text{If } i \geqslant 1, \text{then} \int_0^x t^{2i}\sqrt{1+t^2}\,dt = \lambda_0 \ln(x+\sqrt{1+x^2}) + \lambda_1 x\sqrt{1+x^2}.$$

$$[5pt] \qquad + (1+x^2)^{3/2}(\mu_1 x + \mu_3 x^3 + \cdots + \mu_{2i-1}x^{2i-1})$$

$$[7pt]\text{If } i = 1, \text{then} \int_0^x t^2\sqrt{1+t^2}\,dt = \lambda_0 \ln(x+\sqrt{1+x^2}) + \lambda_1 x\sqrt{1+x^2}.$$

$$[7pt]\text{If } i = 0, \text{then} \int_0^x \sqrt{1+t^2}\,dt = \lambda_0 \ln(x+\sqrt{1+x^2}).$$

**Proof.** The change of variable $t = \sinh(u)$ reduces this to easy calculations.

**Expansion of the length.** Recall the formula

$$r = \int_0^{y_{-1}} \sqrt{c(y)}\,\frac{1}{\sqrt{1-P_1^2(y)}}\,dy.$$

It is an improper integral, since $P_1(y_{-1}) = 1$. Since $y_{-1}$ is a simple root, the integral exists. We see easily that the formal expansions done previously are relevant; then we can interchange $\int$ and $\sum$, which is not obvious *a priori*.

From our previous calculations and both changes of variables, we obtain

$$r = 2\int_A^B C\,du,$$

where

$$A = \frac{1}{k'}\left(-1 + \mathrm{An}(k'^2, \frac{1}{\sqrt{\lambda}})\right),$$

$$B = \frac{\alpha}{4\sqrt{\lambda}k'}\left(1 + \mathrm{An}(k'^2, \frac{1}{\sqrt{\lambda}})\right),$$

$$C = \sqrt{c} \times 1/\sqrt{1-P_1} \times 1/\sqrt{1+P_1} \times dy$$

$$= \left(1 + \mathrm{An}(k'^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}})\right) \times \frac{1}{\sqrt{2}}\left(1 + \mathrm{An}(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2u^2, \frac{k'u}{\sqrt{\lambda}})\right)$$

$$\frac{1}{k'\sqrt{2}}\frac{1}{\sqrt{1+u^2}}\left(1 + \mathrm{An}(k'^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}}, \frac{k'u^3}{\sqrt{\lambda}(1+u^2)})\right) \times \frac{2k}{\sqrt{\lambda}}k'\left(1 + \mathrm{An}(k'^2, \frac{1}{\sqrt{\lambda}})\right).$$

Hence

$$\frac{r\sqrt{\lambda}}{2} = \int_A^B \frac{du}{\sqrt{1+u^2}} + \int_A^B \frac{1}{\sqrt{1+u^2}}\,\mathrm{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, \frac{k'u}{\sqrt{\lambda}}, \frac{k'u^3}{\sqrt{\lambda}(1+u^2)}, k'^2u^2\right)du = I_1 + I_2.$$

**Calculation of $I_2$.** $I_2$ is sum of the following integrals:

$$J_{p,m,n} = \int_A^B \frac{k'^{p+m+2n}u^{3p+m+2n}}{\left(\sqrt{\lambda}\right)^{p+m}\left(\sqrt{1+u^2}\right)^{2p+1}}\,du, \qquad p,m,n \in \mathbb{N}.$$

Hence, from Propositions 3.8 and 3.9, we obtain

$$J_{p,m,n} = \begin{cases} \mathrm{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\ln\frac{1}{k'}, k'^2\ln\sqrt{\lambda}\right) & \text{if} \quad \alpha \neq 0, \\ \left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\ln\frac{1}{k'}\right) & \text{if} \quad \alpha = 0. \end{cases}$$

Notice that $k'$ always appears *squared*. This implies that the expression of $z$ as a function of $x+r$ will not contain any $\sqrt{x+r}$.

Moreover, a detailed analysis yields

$$I_2 = \begin{cases} \ln 2 + \mathrm{An}_0\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2 \ln \frac{1}{k'}, k'^2 \ln \sqrt{\lambda}\right) & \text{if} \quad \alpha \neq 0, \\ \ln 2 + \mathrm{An}_0\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2 \ln \frac{1}{k'}\right) & \text{if} \quad \alpha = 0. \end{cases}$$

**Calculation of $I_1$.**

$$I_1 = \mathrm{Argsinh}(B) - \mathrm{Argsinh}(A).$$

But, if $X \to +\infty$, we have

$$\mathrm{Argsinh}(X) = \ln(X + \sqrt{1 + X^2}) = \ln X + \ln 2 + \mathrm{An}_0\left(\frac{1}{X^2}\right).$$

Hence

$$I_1 = \begin{cases} 2\ln\dfrac{1}{k'} - \ln\sqrt{\lambda} + \ln(\alpha) + \mathrm{An}_0\left(k'^2, \dfrac{1}{\sqrt{\lambda}}, k'^2\lambda\right) & \text{if} \quad \alpha \neq 0, \\ \ln\dfrac{1}{k'} + \ln 2 + \mathrm{An}_0\left(k'^2, \dfrac{1}{\sqrt{\lambda}}, k'^2\lambda\right) & \text{if} \quad \alpha = 0. \end{cases}$$

**Remark 3.10.** Here we can state that there is a "period doubling" if and only if $\alpha \neq 0$.

We obtain actually the following implicit equation in $\sqrt{\lambda}$:

$$\frac{r\sqrt{\lambda}}{2} = \begin{cases} 2\ln\frac{1}{k'} - \ln\sqrt{\lambda} + \ln 2\alpha + \mathrm{An}_0\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\ln\frac{1}{k'}, k'^2\lambda, k'^2\ln\sqrt{\lambda}\right) & \text{if} \quad \alpha \neq 0, \\ \ln\frac{1}{k'} + 2\ln 2 + \mathrm{An}_0\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\lambda\right) & \text{if} \quad \alpha = 0. \end{cases}$$

**Resolution of the implicit equation.** From now on, we set $t = k'^2$.

We must distinguish the following two cases:

- *Case $\alpha \neq 0$.* We have

$$\frac{r\sqrt{\lambda}}{2} = \ln\frac{1}{t} - \ln\left(\frac{r\sqrt{\lambda}}{2}\right) + \ln(r\alpha) + \mathrm{An}_0\left(t, \frac{1}{\sqrt{\lambda}}, t\ln\frac{1}{t}, t\lambda, t\ln\sqrt{\lambda}\right).$$

It is easy to see that $\dfrac{r\sqrt{\lambda}}{2} \sim \ln\dfrac{1}{t}$. We set $\dfrac{r\sqrt{\lambda}}{2} = \ln\dfrac{1}{t} + u$. Hence

$$u = -\ln\ln\frac{1}{t} + \ln(r\alpha) + \mathrm{An}_0\left(t, \frac{1}{\ln\frac{1}{t}}, \frac{u}{\ln\frac{1}{t}}, t\ln\frac{1}{t}, t\ln^2\frac{1}{t}, t\ln\ln\frac{1}{t}\right).$$

Then we set $u = -\ln\ln\dfrac{1}{t} + \ln(r\alpha) + v$. We obtain

$$v = \mathrm{An}_0\left(t, \frac{1}{\ln\frac{1}{t}}, \frac{\ln\ln\frac{1}{t}}{\ln\frac{1}{t}}, t\ln\frac{1}{t}, t\ln^2\frac{1}{t}, t\ln\ln\frac{1}{t}, v\right).$$

The *implicit-function theorem* allows us to conclude that

$$\begin{aligned} v &= \mathrm{An}_0\left(t, \frac{1}{\ln\frac{1}{t}}, \frac{\ln\ln\frac{1}{t}}{\ln\frac{1}{t}}, t\ln\frac{1}{t}, t\ln^2\frac{1}{t}, t\ln\ln\frac{1}{t}\right) \\ &= \mathrm{An}_0\left(\frac{1}{\ln\frac{1}{t}}, \frac{\ln\ln\frac{1}{t}}{\ln\frac{1}{t}}, t\ln^2\frac{1}{t}, t\ln\ln\frac{1}{t}, v\right), \end{aligned}$$

<u>ccl</u>: $$\frac{r\sqrt{\lambda}}{2} = \ln\frac{1}{t} - \ln\ln\frac{1}{t} + \ln(r\alpha) + \mathrm{An}_0\left(\frac{1}{\ln\frac{1}{t}}, \frac{\ln\ln\frac{1}{t}}{\ln\frac{1}{t}}, t\ln^2\frac{1}{t}, t\ln\ln\frac{1}{t}, v\right).$$

- *Case* $\alpha = 0$. We set $\dfrac{r\sqrt{\lambda}}{2} = \dfrac{1}{2}\ln\dfrac{1}{t} + 2\ln 2 + u$. Hence

$$u = \mathrm{An}_0\left(t, \frac{1}{\ln\frac{1}{t}}, t\ln\frac{1}{t}, t\ln^2\frac{1}{t}, u\right).$$

In the same way, due to the *implicit-function theorem*, we obtain

<u>ccl</u>: $\dfrac{r\sqrt{\lambda}}{2} = \dfrac{1}{2}\ln\dfrac{1}{t} + 2\ln 2 + \mathrm{An}_0\left(\dfrac{1}{\ln\dfrac{1}{t}}, t\ln^2\dfrac{1}{t}\right)$.

**Expansion of $x$.** Recall that

$$x(r) = -2\int_0^{y_{\text{-}1}} \frac{\sqrt{c}}{\sqrt{a}}\frac{P_1}{\sqrt{1-P_1^2}}\,dy.$$

We write

$$\frac{P_1}{\sqrt{1-P_1^2}} = \frac{1+P_1}{\sqrt{1-P_1^2}} - \frac{1}{\sqrt{1-P_1^2}} = \frac{\sqrt{1+P_1}}{\sqrt{1-P_1^2}} - \frac{1}{\sqrt{1-P_1^2}}.$$

Using

$$r = -2\int_0^{y_{\text{-}1}} \sqrt{c}\,\frac{1}{\sqrt{1-P_1^2}}\,dy,$$

we obtain

$$\frac{x+r}{2} = \int_{y_{\text{-}1}}^0 \frac{\sqrt{c}}{\sqrt{a}}\frac{\sqrt{1+P_1}}{\sqrt{1-P_1}}\,dy - \int_{y_{\text{-}1}}^0 \left(\frac{\sqrt{c}}{\sqrt{a}} - 1\right)\frac{1}{\sqrt{1-P_1^2}}\,dy.$$

Now, using the previous notation, we have

- $\dfrac{\sqrt{c}}{\sqrt{a}} = 1 + \mathrm{An}\left(k'^2, \dfrac{1}{\sqrt{\lambda}}, \dfrac{k'u}{\sqrt{\lambda}}\right)$.
- $\sqrt{1+P_1} = k'\sqrt{2}\sqrt{1+u^2}\left(1 + \mathrm{An}\left(k'^2, \dfrac{1}{\sqrt{\lambda}}, \dfrac{k'u}{\sqrt{\lambda}}, \dfrac{k'u^3}{\sqrt{\lambda}(1+u^2)})\right)\right)$.

Thus, we obtain $X = \dfrac{x+r}{2}$ as an analytic sum of integrals of following type:

$$J_{p,n,m} = \int_A^B \frac{k'^{p+m+2n+2}u^{3p+m+2n}}{(\sqrt{\lambda})^{p+m+1}(\sqrt{1+u^2})^{2p-1}}\,du.$$

From Proposition 3.8, we obtain

$$J_{p,n,m} = \begin{cases} \frac{1}{\sqrt{\lambda}}\,\mathrm{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\ln\frac{1}{k'}, k'^2\ln\sqrt{\lambda}\right) & \text{if } \alpha \neq 0,\\ \frac{1}{\sqrt{\lambda}}\,\mathrm{An}\left(k'^2, \frac{1}{\sqrt{\lambda}}, k'^2\ln\frac{1}{k'}\right) & \text{if } \alpha = 0. \end{cases}$$

Knowing $\dfrac{1}{\sqrt{\lambda}}$, we obtain

$$\begin{aligned} &\text{if } \alpha \neq 0, \text{then} \qquad X = \frac{x+r}{2} = \frac{1}{\ln\frac{1}{t}}\,\mathrm{An}\left(\frac{1}{\ln\frac{1}{t}}, \frac{\ln\ln\frac{1}{t}}{\ln\frac{1}{t}}, t\ln^2\frac{1}{t}, t\ln\ln\frac{1}{t}\right)\\ &= \frac{2}{\sqrt{\lambda}} + \frac{C}{\lambda} + O(\frac{1}{\lambda^{\frac{3}{2}}}); \qquad (36)\\ &\text{if } \alpha = 0, \text{then} \qquad X = \frac{x+r}{2} = \frac{1}{\ln\frac{1}{t}}\,\mathrm{An}\left(\frac{1}{\ln\frac{1}{t}}, t\ln^2\frac{1}{t}\right). \end{aligned}$$

**Expansion of $z$.** Recall that

$$Z = \frac{z(r)}{r} = \int_{y_{-1}}^{0} \frac{\sqrt{c}}{\sqrt{a}} \frac{y^2 P_1}{\sqrt{1-P_1^2}}\, dy.$$

In the same way, we prove the following assertion:

$$\begin{aligned} &\text{if } \alpha \neq 0, \text{ then} \quad Z = \frac{1}{\ln^3 \frac{1}{t}} \operatorname{An}\left(\frac{1}{\ln \frac{1}{t}}, \frac{\ln\ln \frac{1}{t}}{\ln \frac{1}{t}}, t\ln^2 \frac{1}{t}, t \ln\ln \frac{1}{t}\right) = \frac{4}{\lambda^{\frac{3}{2}}} + o\left(\frac{1}{\lambda^{\frac{3}{2}}}\right), \\ &\text{if } \alpha = 0, \text{ then} \quad Z = \frac{1}{\ln^3 \frac{1}{t}} \operatorname{An}\left(\frac{1}{\ln \frac{1}{t}}, t\ln^2 \frac{1}{t}\right). \end{aligned} \tag{37}$$

Now we have a parametrization $(X(t), Z(t))$; the problem is *to eliminate the parameter* $t$.

**3.5.4. Inversion of the parameter $t$ as a function of $X$.** Set $X = \dfrac{x+r}{2r}$. The method of expressing $t$ as a function of $X$ is general in the category of functions that we are dealing with. In our example, $X$ is an analytic function of $t$ and some functions composed of ln and $t$ that tend to 0 as $t$ tends to 0. Thus, we are dealing with a *subclass* of the exp-log category (see [11]), denoted by LE. The general theory from [11] tells us that $t = F(X)$ with $F \in$ LE. But this general theorem, whose proof is based on the Weierstrass preparation theorem, is not algorithmic. Our problem is more specific: we are dealing with a *specific scale.* In this case, we can develop an algorithm for computing precisely $F$, and thus, find the subclass of LE that is needed to express $t$ as a function of $X$.

First we give the algorithm in the particular case of our example; then we give a general result.

**Algorithm.** Let $u = \dfrac{1}{\ln \frac{1}{t}}$. Then

$$X = \operatorname{An}\left(u, u\ln\frac{1}{u}, \frac{e^{-\frac{1}{u}}}{u^2}\right) = u + u^2 \ln \frac{1}{u} + Cu^2 + o(u^2), \tag{38}$$

where $C$ is a constant, which can be precisely computed.

We easily obtain

$$u = X - X^2 \ln \frac{1}{X} - cX^2 + o(X^2),$$

which leads us to set

$$u = X - X^2 \ln \frac{1}{X} - cX^2 + x^2 v.$$

Then

$$\begin{aligned} \frac{1}{u} &= \frac{1}{X} + \ln \frac{1}{X} + C - v + \operatorname{An}\left(X, X\ln^2 \frac{1}{X}, X \ln \frac{1}{X}, v\right), \\ e^{-\frac{1}{u}} &= e^{-C} X e^{-\frac{1}{X}} (1 + v \operatorname{An}(v)), \\ \ln \frac{1}{u} &= \ln \frac{1}{X} + \operatorname{An}\left(X, X \ln \frac{1}{X}, Xv\right). \end{aligned}$$

Substituting into (38), we obtain

$$0 = X^2 v + \operatorname{An}\left(X, X \ln \frac{1}{X}, \frac{e^{-\frac{1}{X}}}{X}, v\right),$$

the analytic function being an $o(X^2)$, which allows us to divide this equation by $X^2$. Then

$$0 = v + \operatorname{An}\left(X, X\ln \frac{1}{X}, X\ln^2 \frac{1}{X}, X \ln^3 \frac{1}{X}, \frac{e^{-\frac{1}{X}}}{X^3}, v\right).$$

Applying the *implicit function theorem*, we have

$$v = \mathrm{An}\left(X, X\ln\frac{1}{X}, X\ln^2\frac{1}{X}, X\ln^3\frac{1}{X}, \frac{e^{-\frac{1}{X}}}{X^3}\right);$$

the same goes for $u$.

Substituting this relation in the expansion of $Z$, we conclude that

$$Z = \mathrm{An}\left(X, X\ln\frac{1}{X}, X\ln^2\frac{1}{X}, X\ln^3\frac{1}{X}, \frac{e^{-\frac{1}{X}}}{X^3}\right). \tag{39}$$

**Remark 3.11.** We can be more specific on the first terms of the expansion, that is,

$$Z = \frac{1}{6}X^3 + X^4\,\mathrm{An}(X) + O(X^4 e^{-\frac{1}{X}}).$$

Our computations have proved the following theorem.

**Theorem 3.12.** *The sub-Riemannian sphere in the general Martinet conservative case is in the log-exp category.*

**3.5.5. Generalization of the algorithm.** The previous algorithm can be generalized in the following manner. Our aim is to build a subclass of the general *log-exp category* (see [11]) with the following functions:

$$h_1(t) = t, \quad h_2(t) = \ln\frac{1}{t}, \quad h_3(t) = e^{-\frac{1}{t}}.$$

**Notation.** $h^p$ means $h \times h \times \ldots \times h$ ($p$ times). $h^{[p]}$ means $h \circ h \circ \cdots \circ h$ ($p$ times).

**Definition 3.13.** We set

$$\mathcal{E}_1 = \left\{ h_1^p \prod_{0<i_1<i_2<\cdots<i_m} \left(h_2^{[i_k]}\right)^{i'_k} \;/\; i'_k \in \mathbb{Z}^*,\ m \in \mathbb{N},\ (p=1 \ \text{ and } \ i'_1 < 0), \ \text{ or } (p \geqslant 2) \right\};$$

$$\mathcal{I} = \left\{ h_1^p \prod_{0<i_1<i_2<\cdots<i_m} \left(h_2^{[i_k]}\right)^{i'_k} \;/\; (p \geqslant 1)\text{or} \ \ (p = 0 \ \text{ and } \ i_1 = 1,\ i'_1 \leqslant -2) \right\};$$

$$\mathcal{E}_2 = \left\{ h_1^q \prod_{0<i_1<i_2<\cdots<i_m} \left(h_2^{[i_k]}\right)^{i'_k} e^{-\frac{1}{f}} \;/\; p \in \mathbb{Z},\ i'_k \in \mathbb{Z},\ f \in \mathcal{I} \right\}.$$

**Proposition 3.14.** *Let $F(X_0, X_1, \ldots, X_n)$ be an analytic function near 0, so that $F \underset{0}{\sim} X_0$. Let $X(t) = F(t, f_1(t), \ldots, f_n(t))$, where $f_i \in \mathcal{E}_1 \cup \mathcal{E}_2$. Let $r$ be the greatest degree of meromorphy in the expressions of the $f_i$'s, i.e., the greatest power to which appears $\frac{1}{t}$ in the expressions of the $f_i$'s. Let $g_1(t)$ be such that $X(t) = g_1(t) + o(t^{r+2})$ (in fact, $g_1 \in \mathrm{Vect}(h_1, \mathcal{E}_1)$). Then*

1. $\exists g_2 \in \mathrm{Vect}(h_1, \mathcal{E}_1) \;/\; t = g_2(X) + o(X^{r+2})$.
2. *If we set $t = g_2(X) + X^{r+2}u$, then $u$ can be analytically computed using the implicit function theorem.*

**Remark 3.15.** This algorithm implies not only that $t$ can be expressed as a function of $X$ in the *log-exp category*; it gives a *precise scale.*

## 4. The General Gradated Case of Order 0

To investigate the general case, one method is *to consider the general case as a perturbation of the integrable case.* This point of view is similar to the one used to solve the 16th Hilbert problem about limit cycles.

We proceed as follows: if $\beta \neq 0$ in the gradated normal form of order 0, the basic second-order equation (11) describes a nonconservative pendulum. The *asymptotic expansions* of the SR-distance near the abnormal direction can be evaluated by estimating the solutions near the saddle.

Since this saddle is not a priori integrable in the analytic category for any value of the parameter $\varepsilon = \frac{1}{\sqrt{\lambda}}$, we use the procedure of [13, p. 91] to compute the Poincaré transition map near a hyperbolic saddle point depending on a parameter.

It is based on the *existence of a formal first integral* and uses the following normal form near a saddle:

$$X_\varepsilon \sim x\frac{\partial}{\partial x} + y\left(-r(\varepsilon) + \sum_{i=1}^{N} \alpha_{i+1}(\varepsilon)(xy)^i\right)\frac{\partial}{\partial y},$$

where $r(\varepsilon)$ is defined using the linearized system

$$X_\varepsilon(0) = x\frac{\partial}{\partial x} - r(\varepsilon)y\frac{\partial}{\partial y}, \qquad r(\varepsilon) = \left|\frac{\lambda_2(\varepsilon)}{\lambda_1(\varepsilon)}\right|,$$

where $\lambda_1, \lambda_2$ are the two eigenvalues of the saddle, and $r(0) = 1$ (flat case) in our situation.

The previous vector field can be integrated by making the following (toric) *blowing-up*: $u = xy$, $v = x$. This procedure allows us to compute asymptotic expansions for the solutions near the saddle. In essence, this method will not provide *converging expansions*.

This procedure is based on the use of our normal form. Moreover, for computing $x$ and $z$ we require one more integration. Hence we can imagine that the final expansions are converging. Another method that could be used to compute converging expansions involves the use of the *Briot–Bouquet theory*. This method is as follows.

Similarly to the general conservative case, the objective is to express $X$ and $Z$ in terms of $k'$. To understand precisely the role of the parameter $\beta$, one may study the system with the following particular metric:

$$a = 1, \quad c = (1 + \beta x)^2.$$

In this particular case, the general differential system (3) is simpler. Indeed, dividing by $\dot{y}$ we obtain

$$\frac{dx}{dy} = \sigma(1+\beta x)\frac{P_1}{\sqrt{1-P_1^2}}, \tag{40}$$

$$\frac{dz}{dy} = \sigma(1+\beta x)\frac{y^2}{2}\frac{P_1}{\sqrt{1-P_1^2}}, \tag{41}$$

$$\frac{P_1}{dy} = \lambda y + \varepsilon\sigma\sqrt{1-P_1^2}, \tag{42}$$

where $\sigma = \mathrm{sign}(\dot{y})$.

Moreover, we fix the length to $r$, hence

$$r = 2\int_{y_{-1}}^{y_1} (1+\beta x)\frac{P_1}{\sqrt{1-P_1^2}}dy. \tag{43}$$

Contrary to the conservative case where $P_1$ was given explicitly, $P_1$ is solution of a differential equation.

It seems reasonable to think that one could express $P_1$ analytically in some class of functions. Indeed plugging in (40), one would express $x(y)$, then the relation (43) would give $\lambda(k')$, and finally it would go as before. In this analysis the key-equation is Eq. (42).

If we set $P_1 = -1 + 2f^2$, $f(0) = k'$, and $\eta = \sqrt{\lambda}y$, we get

$$2f\frac{df}{d\eta} = \frac{1}{2}\eta + \varepsilon f\sqrt{1-f^2} = \frac{1}{2}\eta + \varepsilon f + \varepsilon\sum_{n=1}^{\infty} a_n f^{2n+1}.$$

This is a *Briot–Bouquet equation*, studied by Boutroux (see [5]). We can expect to get *sectorially converging expansions* of $P_1$, which could help us to compute expansions of $X$ and $Z$. More precisely we conjecture that

for $D_1 = \{0 \leqslant \eta < k'\}$, $P_1$ has a convergent Taylor series (which is computable thanks to (42)).

for $D_2 = \{k' < \eta < 1\}$, $P_1$ can be analytically expanded in some scale.

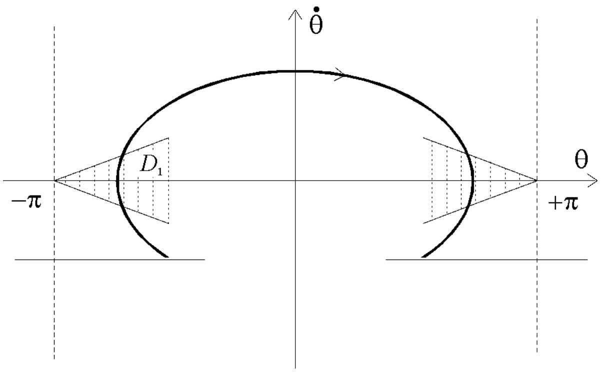


Fig. 5.

## REFERENCES


1. A. Agrachev, B. Bonnard, M. Chyba, and I. Kupka, "Sub-Riemannian sphere in the Martinet flat case," *ESAIM/COCV*, **2**, 377-448 (1998).
2. A. Agrachev and A. V. Sarychev, "Strong minimality of abnormal geodesics for 2-distributions," *J. Dyn. Contr. Syst.*, **1**, No. 2, 139-176 (1995).
3. A. Agrachev and A. V. Sarychev, "Sub-Riemannian metrics: Minimality of abnormal geodesics versus sub-analyticity," to appear.
4. B. Bonnard and M. Chyba, "Méthodes géométriques et analytiques pour étudier l'application exponentielle, la sphère et le front d'onde en géométrie SR dans le cas Martinet," (in press).
5. P. Boutroux, *Leçons sur les Fonctions Définies par les Équations Différentielles du Premier Ordre*, Gauthier-Villard, Paris (1908).
6. A. D. Brjuno, *Local Methods in Nonlinear Differential Equations*, Springer-Verlag, New York (1989).
7. L. V. D. Dries, A. Macintyre, and D. Maker, "The elementary theory of restricted analytic fields with exponentiation," *Ann. Math.*, **140**, 183-205 (1994).
8. I. Kupka, "Abnormal extremals," *Preprint* (1992).
9. I. Kupka, "Géométrie sous-Riemannienne," *Sém. Bourbaki* (1996).
10. D. F. Lawden, *Elliptic Functions and Applications*, Springer-Verlag, New York (1989).
11. J. M. Lion and J. P. Rolin, "Théorème de préparation pour les fonctions logarithmico-exponentielles," *Ann. Inst. Fourier*, **47**, No. 3, 859-884 (1997).
12. L. S. Pontryagin, V. G. Boltyanski, R. V. Gamkrelidze and E. F. Mishchenko, *Mathematics of Optimal Processes* [in Russian], Nauka, Moscow (1976).
13. R. Roussarie, *Bifurcations of Planar Vector Fields and Hilbert's* 16*th Problem*, Birkhâuser, Berlin (1998).
14. J. Stoker, *Nonlinear Elasticity*, Gordon and Breach, London (1968).