Journal of Dynamical and Control Systems, Vol. 9, No. 2, April 2003, 155–199 (©2003)

OPTIMAL CONTROL WITH STATE CONSTRAINTS AND THE SPACE SHUTTLE RE-ENTRY PROBLEM

B. BONNARD, L. FAUBOURG, G. LAUNAY, and E. TRÉLAT

ABSTRACT. In this article, we initialize the analysis under generic assumptions of the *small time optimal synthesis* for single input systems with *state constraints*. We use geometric methods to evaluate the *small time reachable set* and necessary optimality conditions. Our work is motivated by the *optimal control of the atmospheric arc for the re-entry of a space shuttle*, where the vehicle is subject to constraints on the thermal flux and on the normal acceleration. A *multiple shooting technique* is finally applied to the computation of the optimal longitudinal arc.

1. INTRODUCTION

The objective of this article is to initialize the classification of the local closed loop time optimal control for the single input affine systems: $\dot{q} = X(q)+uY(q)$, where $q \in \mathbb{R}^2$ or $\mathbb{R}^3$, $|u| \le 1$, with *state constraints*: $c(q) \le 0$. This analysis is motivated by the optimal control of the *atmospheric arc for the re-entry of the space shuttle*, where the cost is the total amount of thermal flux and where the constraints are on the thermal flux and on normal acceleration, the control being the angle of bank. The system is modeled by an equation of the form:

$$m\ddot{q} = \overrightarrow{K}(q) + \overrightarrow{D}(q) + \overrightarrow{L}(q), \tag{1.1}$$

where m is the vehicle mass, q is the position in an inertial frame, $\overrightarrow{K}$ is the standard *Keplerian force*, $\overrightarrow{D}$ is the *drag force* and $\overrightarrow{L}$ is the *lift force* controlled by the bank angle. The target T is a point in an Earth fixed frame. For reasons explained later, we will express the dynamics in the Earth fixed frame which is rotating with respect to the inertial frame and consequently we have additional *Coriolis* and *centripetal forces* and the system is more complicated.

Pioneering necessary optimality conditions concerning the *optimality status of a boundary arc* and *junction* or *reflection* with the boundary are a consequence of Weierstrass theory (see [3]) applied to Riemannian theory

2000 *Mathematics Subject Classification.* 49K15, 70M20, 49M15.

Key words and phrases. Optimal control with state constraints, minimum principle, control of the atmospheric arc, multiple shooting techniques.

1079-2724/03/0400-0155/0

with obstacles. The author compares the length of a reference arc with adjacent arcs to deduce geometric necessary conditions. This approach was generalized by Pontryagin and his co-authors [23] to obtain a *minimum principle* under regularity assumptions on the constraints. A general minimum principle based on Kuhn–Tucker theorem and nonsmooth analysis is presented in [15]. In these principles, the adjoint vector p dual to the state vector q can suffer discontinuities at contact points with the boundary of the domain or in the boundary. Following [16] and [19], these discontinuities can be specified if we assume that the system is a single input and if the *order of the constraint* is constant; by definition, the order is the first integer such that the control u appears explicitly in the time derivative of the constraint $t \mapsto c(q(t))$ evaluated along a boundary arc of the system. For the space shuttle re-entry problem, the constraints are of order 2 and although the system is in dimension 6, by [4] we can mainly restrict our analysis to a subsystem in dimension 3, called the *longitudinal subsystem*, controlling the altitude, the modules of the relative speed and the flight path angle.

The evaluation of the small time reachable set and its boundary which can be parametrized by the minimum principle with application to the optimal synthesis was a research program initialized by Sussmann [27] for planar system and pursued in dimension 3 by [18], see also [6] for problems with a target of codimension one. The objective of this article is mainly to outline such an analysis in the case of optimal control with state constraints. Here the geometry is different and we must classify up to changes of coordinate triplets (X, Y, c) using the order of the constraints. We make direct evaluation of the reachable set for the constrained system, using normal forms. One of the main problems is the characterization the *optimality status of a boundary arc*. Under a suitable generic assumption, we obtain necessary and sufficient conditions which are compared with the necessary conditions of the minimum principle.

Our geometric work, completed by the preliminary study [7], is finally applied to the re-entry problem. A quasi-optimal trajectory consisting of concatenation of bang and boundary arcs is given and the exact trajectory corresponding to the boundary conditions imposed by [12] is computed using a *multiple shooting algorithm* and numerical simulations.

2. Generalities

2.1. **Definitions.** We consider a smooth (C^∞ or C^ω) single input affine system

$$\dot{q} = X(q) + uY(q), \tag{2.1}$$

with $|u| \leq 1$, $q \in U \subset \mathbb{R}^n$ with the state constraint $c(q) \leq 0$, where $c : \mathbb{R}^n \to \mathbb{R}$ and the time optimal problem with fixed boundary conditions:

$q(0) = q_0$, $q(T) = q_1$. We denote by $(\mathcal{P}_0)$ the minimization problem which can be embedded in the one parameter family of problems $(\mathcal{P}_\alpha)$, where the constraint is $c(q) \leq \alpha$ for sufficiently small α; α is a *homotopy parameter.*

If f is a function and Z is a vector field, it acts on f by the *Lie derivative* $Zf = \frac{\partial f}{\partial q}(q)Z(q)$ and if Z_1 and Z_2 are two vector fields, the *Lie bracket* is computed with the convention: $[X, Y] = X \circ Y - Y \circ X$, that is in local coordinates:

$$[X, Y](q) = \frac{\partial Y}{\partial q}(q)X(q) - \frac{\partial X}{\partial q}(q)Y(q).$$

The *generic order* of the constraint is the integer m such that

$$Yc = YXc = \ldots = YX^{m-2}c = 0 \quad \text{and} \quad YX^{m-1}c \neq 0.$$

A *boundary arc* $t \mapsto \gamma_b(t)$ is an arc (not reduced to a point) of the system contained in $c = 0$. If the order is m, a boundary arc and the associated feedback control can be generically computed by differentiating m times the mapping $t \mapsto c(q(t))$ and solving with respect to u the linear equation:

$$c^{(m)} = X^m c + uYX^{m-1}c = 0.$$

A boundary arc is contained in

$$c = \dot{c} = \ldots = c^{(m-1)} = 0,$$

and the constraint $c = 0$ is called *primary* and constraints $\dot{c} = \ldots = c^{(m-1)} = 0$ are called *secondary*. We denote by

$$u_b = -\frac{X^m c}{YX^{m-1}c}$$

the boundary feedback control.

2.2. **Assumptions** C. Let $t \mapsto \gamma_b(t)$, $t \in [0, T]$, be a boundary arc associated with u_b. We need to introduce the following assumptions:

C_1. $YX^{m-1}c_{|\gamma_b} \neq 0$, where m is the order of the constraint.
C_2. $|u_b| < 1$ for $t \in]0, T[$, i.e., the boundary control is admissible.
C_3. $|u_b| < 1$ for $t \in [0, T]$, i.e., the boundary control is not saturating.

2.3. **A minimum principle with state constraints.** We recall the necessary conditions due to [16] and [19] that we will use in our study. Consider the single input affine system (2.1), $\dot{q} = X(q) + uY(q)$, $q \in U \subset \mathbb{R}^n$, $|u| \leq 1$, and a cost to be minimized of the form

$$J(u) = G(q(T)),$$

where the transfer time is fixed and q satisfies the constraint

$$c(q) \leq 0,$$

and the boundary conditions are

$$q(0) = 0, \quad \chi(q(T)) = 0$$

with $\chi = (\chi_1, \dots, \chi_k)$ and $k \leq n$.

2.3.1. *Statement of the necessary optimality conditions.* Assume that $t \mapsto q(t)$, $t \in [0,T]$, is a piecewise smooth optimal solution which hits the boundary $c(q) = 0$ at times t_{2i-1}, $i = 1, 2, \dots, M$, and leaves the boundary at times t_{2i}, $i = 1, 2, \dots, M$, and, moreover, assume that along each boundary arc, Assumptions C_1 and C_2 are satisfied at contact or junction times. Define the Hamiltonian by

$$H(q, p, u, \eta) = \langle p, X + uY \rangle + \eta c,$$

where $\langle\ ,\ \rangle$ is the standard scalar product, p is the adjoint vector, and η is the Lagrange multiplier of the constraint. The necessary optimality conditions are as follows.

1. There exists $t \mapsto \eta(t) \geq 0$, a real number η_0, and $\tau \in \mathbb{R}^k$ such that the adjoint vector satisfies:

$$\dot{p} = -p\left(\frac{\partial X}{\partial q} + u\frac{\partial Y}{\partial q}\right) - \eta\frac{\partial c}{\partial q} \quad \text{a.e.,} \tag{2.2}$$

$$p(T) = \eta_0\frac{\partial G}{\partial q}(q(T)) + \tau\frac{\partial \chi}{\partial q}(q(T)). \tag{2.3}$$

2. The mapping $t \mapsto \eta(t)$ is continuous along the boundary arc and satisfies:

$$\eta(t)c(q(t)) = 0 \quad \forall t \in [0,T].$$

3. At a contact or junction time t_i with the boundary, we have

$$H(t_i+) = H(t_i-), \tag{2.4}$$

$$p(t_i+) = p(t_i-) - \nu_i\frac{\partial c}{\partial q}(q(t_i)), \quad \nu_i \geq 0. \tag{2.5}$$

4. The optimal control minimizes almost everywhere the Hamiltonian

$$H(q(t), p(t), u(t), \eta(t)) = \min_{|v|\leq 1} H(q(t), p(t), v, \eta(t)). \tag{2.6}$$

2.3.2. *Remarks.*

1. In this minimum principle only the primary constraint $c = 0$ is penalized. Other choices are possible using the secondary constraint (see [10] and [23]).
2. A minimum principle without assumptions on the order is stated in [15] and the adjoint vector is given as follows:

$$p(t) = -\int p(s)\left(\frac{\partial X}{\partial q}(q(s)) + u(s)\frac{\partial Y}{\partial q}(q(s))\right)ds - \sum\int\frac{\partial c}{\partial q}(q(s))d\mu_i,$$

where μ_i are nonnegative regular measures supported by $c = 0$. With the assumption of constant order, the measures on the boundary arcs take the form: $d\mu_i = \eta_i dt$, where η_i is C^0. If Assumption C_1 is not satisfied, we can have at a nongeneric point $YX^{m-1}c = 0$ and η_i can explode.

2.3.3. *Application to the time optimal control problem.* In the time minimizing problem the transfer time T is not fixed. We reparametrize the trajectories on $[0, 1]$ by setting $s = t/T$, $z = T$. The problem is to minimize $t(1)$ for the extended system:

$$\frac{dq}{ds} = (X + uY)z, \quad \frac{dt}{ds} = z, \quad \frac{dz}{ds} = 0.$$

The transversality conditions imply

$$p_t \geq 0 \text{ at } s = 1 \text{ and } p_z = 0 \text{ for } s = 0, 1.$$

The adjoint system decomposes into

$$\begin{aligned} \frac{dp}{ds} &= -p\left(\frac{\partial X}{\partial q} + u\frac{\partial Y}{\partial q}\right) - \eta\frac{\partial c}{\partial q}, \\ \frac{dp_t}{ds} &= 0, \\ \frac{dp_z}{ds} &= -p(X + uY) - p_t \end{aligned}$$

and, moreover,

$$M = \min_{|v|\leq 1} H = 0. \tag{2.7}$$

If we reparametrize by t and replace η by η/z and M by M/z, we obtain the following proposition.

Proposition 2.1. *The necessary optimal conditions for the time minimal control problem are*

$$\dot{q} = X + uY \qquad a.e., \tag{2.8}$$

$$\dot{p} = -p\left(\frac{\partial X}{\partial q} + u\frac{\partial Y}{\partial q}\right) - \eta\frac{\partial c}{\partial q} \qquad a.e., \tag{2.9}$$

$$u\langle p, Y\rangle = \min_{|v|\leq 1} v\langle p, Y\rangle \qquad a.e., \tag{2.10}$$

$$M = \min_{|v|\leq 1} \langle p, X + uY\rangle + p_t \equiv 0 \quad \text{with } p \neq 0. \tag{2.11}$$

At a contact or a junction with the boundary, we have

$$p(t_i+) = p(t_i-) - \nu_i\frac{\partial c}{\partial q}, \quad \nu_i \geq 0, \tag{2.12}$$

$$p_t \geq 0,\ \eta \geq 0 \quad \textit{with } \eta = 0 \textit{ for } c < 0$$
$$\textit{and } \eta \textit{ is } C^0 \textit{on the boundary } c = 0. \tag{2.13}$$

2.4. **Definitions.** An extremal is a solution (q,p) of the above equations. It is said to be *exceptional* if $p_t = 0$. In the nonexceptional case, we use the normalization $p_t = 1/2$. An extremal arc is said to be *bang-bang* if it corresponds to a piecewise constant control $u(t) = -\operatorname{sign}(\langle p(t), Y(q(t))\rangle)$; an extremal arc of the unconstrained problem is said to be *singular* if $\langle p(t), Y(q(t))\rangle = 0$. We denote by $\Phi = \langle p, Y(q)\rangle$ the *switching function* and by Σ_s the *switching set* formed by points q, where the optimal control is discontinuous.

2.5. **Computation of singular controls.**

Lemma 2.2. *Let $\Phi(t) = \langle p(t), Y(q(t))\rangle$ be the switching function evaluated along a smooth extremal $z(t) = (p(t), q(t))$ of the unconstrained problem. Then*:

$$\dot{\Phi}(t) = \langle p(t), [X,Y](q(t))\rangle,$$
$$\ddot{\Phi}(t) = \langle p(t), [X,[X,Y]](q(t))\rangle + u(t)\langle p(t), [Y,[X,Y]](q(t))\rangle.$$

Corollary 2.3. *A singular extremal $(p(t), q(t))$ satisfies the following equations*:

$$\langle p(t), Y(q(t))\rangle = \langle p(t), [X,Y](q(t))\rangle = 0 \quad a.e.,$$
$$\langle p(t), [X,[X,Y]](q(t))\rangle + u(t)\langle p(t), [Y,[X,Y]](q(t))\rangle = 0 \quad a.e.$$

2.6. **Geometric computations of the multipliers (η, ν_i) and the junction conditions.** One of the main contributions of [19] is the determination of the multipliers (η, ν_i) together with the analysis of the junction conditions. This is based on the concept of order and is related to the classification of extremals. We establish now these relations for orders $m = 1$ and $m = 2$. Also we make the computation geometric, i.e., related to iterated Lie brackets of (X, Y) acting on the constraint mapping c.

2.6.1. *The case $m = 1$.* For the first-order constraint, we have the following lemma.

Lemma 2.4. *Let $m = 1$. Then*:

1. *along the boundary,* $\eta = \dfrac{\langle p, [X,Y](q)\rangle}{(Yc)(q)}$;
2. *if the control is discontinuous at the contact or entrance-exit of a bang arc with the boundary, then* $\nu_i = 0$.

Proof. Along the boundary, $\Phi = \langle p, Y\rangle = 0$; by the differentiation, we obtain

$$0 = \dot{\Phi} = \langle p, [X,Y](q)\rangle - \eta Yc(q)$$

and $Yc \neq 0$ since the boundary arc is of order 1. Hence we have proved item 1.

Let us prove 2. We set $a = Xc$ and $b = Yc$. Hence $\dot{c} = a + ub$. Let Q be a contact point of a bang-bang extremal $t \mapsto q(t)$ with the boundary at time t_i. Let $\epsilon > 0$ be sufficiently small. We have:

$$c(q(t_i - \epsilon)) < 0, \quad c(q(t_i + \epsilon)) < 0.$$

Taking the limit as ϵ tends to 0, we obtain:

$$(a + bu)_{t_i-} \geq 0, \quad (a + bu)_{t_i+} \leq 0.$$

Hence making the difference, we obtain

$$b(q(t_i))\,(u(t_i-) - u(t_i+)) \geq 0. \tag{2.14}$$

For example, assume that $b(q(t_i)) > 0$. Hence $u(t_i-) - u(t_i+) > 0$ since $u(t_i-) \neq u(t_i+)$. By the minimum principle, we must have

$$\Phi(t_i-) \leq 0, \quad \Phi(t_i+) \geq 0.$$

By (2.12), we have

$$\Phi(t_i+) = \Phi(t_i-) - \nu_i b(q(t_i)) \tag{2.15}$$

and deduce $\nu_i b(q(t_i)) \leq 0$. The minimum principle implies $\nu_i \geq 0$. Consequently, if $\nu_i \neq 0$, then we have $b(q(t_i)) \leq 0$; this contradicts our assumption. The case $b(q(t_i)) < 0$ is similar. The discussion is similar at a junction point with a boundary arc. □

2.6.2. *The case $m = 2$.* For the second-order constraint, we have the following lemma.

Lemma 2.5. *Let $m = 2$. Then:*

1. *along a boundary arc, we have*

$$\eta = \frac{\langle p, [X,[X,Y]](q)\rangle + u_b \langle p, [Y,[X,Y]](q)\rangle}{([X,Y]c)(q)};$$

2. *at a contact or entrance-exit point, we have*

$$\Phi(t_i+) = \Phi(t_i-);$$

3. *we have*

$$\nu_i = \frac{\dot{\Phi}(t_i-)}{([X,Y]c)(q(t_i))}$$

at an entry point and

$$\nu_i = -\frac{\dot{\Phi}(t_i+)}{([X,Y]c)(q(t_i))}$$

at an exit point.

The proof is similar to the proof of Lemma 2.4.

Remark. Hence at an entry (respectively, exit) point, ν_i is determined by the extremal before reaching (respectively, after leaving) the constraint. The multiplier η is determined by (q, p) along the constraint.

2.7. **Small time reachable set, normality, and conjugate points along bang-bang extremals.**

2.7.1. *Definitions.* Consider a system of the form

$$\dot{q} = X + uY, \quad |u| \leq 1.$$

Let $q(0) = q_0$ be fixed; we denote by $q(t, q_0, u)$ the solution corresponding to $u(\cdot)$ and starting at time $t = 0$ from q_0. We denote by

$$R(q_0) = \bigcup_{u,\ t \text{ is sufficiently small}} q(t, q_0, u)$$

the *small time reachable set.* The time extended system is the system $\dot{q} = X + uY$, $\dot{q}^0 = 1$. We denote $\tilde{q} = (q, q^0)$; let $\tilde{R}(\tilde{q}_0)$ be the small time reachable set for the extended system, with $\tilde{q}_0 = (q_0, 0)$. We denote by $\tilde{B}(\tilde{q}_0)$ the boundary of $\tilde{R}(\tilde{q}_0)$ which contains both time minimal and time maximal trajectories of the system, parametrized by the minimum principle. Let $T > 0$; we denote by $B(q_0, T)$ the extremities of time minimal trajectories with time T (see Fig. 1). The *cut-locus* $C(q_0)$ is the set of points q_1, where there exist two distinct minimizing curves starting at q_0.

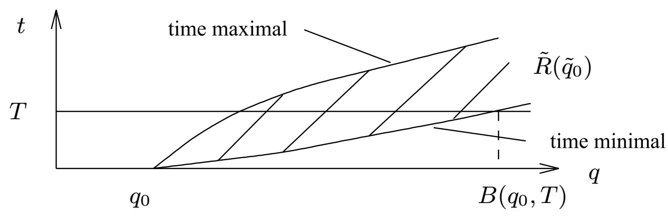


Fig. 1. Small time reachable set

We define similarly the small time reachable sets, their boundaries, and the cut locus for the constrained problem $c(q) \leq 0$ and mark them by a subscript b. *The purpose of this article is, in particular, to compute $C_b(q_0)$ and to stratify $B_b(q_0, T)$ for systems in dimensions 2 and 3 under generic assumptions.*

2.7.2. *Normality and conjugate points along bang-bang extremals for the unconstrained system.* We briefly recall the concept of normality and conjugate points introduced in [28]. Consider the family of vector fields

$$D = \{X + u_0 Y,\ |u_0| \leq 1\}.$$

If $Z \in D$, denote by $\exp tZ$ the one-parametric group of Z. Fix q_0 and $T > 0$. If $Z_1, \dots, Z_m \in D$, we denote by φ the mapping

$$\varphi(t_1, \dots, t_m) = \exp t_m Z_m \circ \cdots \circ \exp t_1 Z_1(q_0), \quad \sum_{i=1}^{m} t_i = T.$$

The point q_1 is said to be *normally accessible* (respectively, *quasi-normally accessible*) from q_0 in time T if there exists a mapping satisfying the condition $\varphi(\bar{t}) = q_1$ for some $\bar{t} = (\bar{t}_1, \dots, \bar{t}_m)$ with $\bar{t}_i > 0$ and such that φ is a submersion at $\bar{t}$ (respectively, open mapping). In both cases the image covers a neighborhood of q_1 and hence the corresponding trajectory is neither time minimal nor time maximal. Let $(z(\cdot), q(\cdot))$ be a bang-bang extremal defined on $[0, T]$. If n is the dimension of the state, assume that z has n switchings $0 \leq t_1 < \cdots < t_n = T$ on $[0, T]$. The corresponding points $q(t_1), q(t_n)$ of the trajectory are called *conjugate points*. Hence $\langle p(t_i), Y(q(t_i)) \rangle = 0$ for $i = 1, \dots, n$. Let Y_i^* be the vector $Y(q(t_i))$ transported from $q(t_i)$ to $q(t_n)$ by the flow of Z. Then we have $\langle p(t_n), Y_i^* \rangle = 0$ for $i = 1, \dots, n$ and since $p(t_n)$ is nonzero, the vectors Y_i^* are linearly dependent. The resulting relation between the switching times is called a *conjugate point relation*.

2.8. **Conclusion of this section.** The remnant of this article is devoted to the construction of the closed loop time optimal trajectories for a single input affine control system in dimensions 2 or 3 with application to the space shuttle re-entry problem. To guide the analysis, a standard result (see [18]) is as follows. Consider a system $\dot{q} = X + uY$ in $\mathbb{R}^3$ and take q_0 such that the vector fields X, Y, and $[X, Y]$ are linearly independent at q_0. Then the small time reachable set $R(q_0)$ has a nice structure. It is homeomorphic to a *convex cone*, whose boundary is formed by two surfaces S_1 and S_2 corresponding to bang-bang trajectories with at most one switching, see Fig. 2. The arcs γ_+ and γ_- are the trajectories corresponding to $u = 1$ and $u = -1$, respectively.

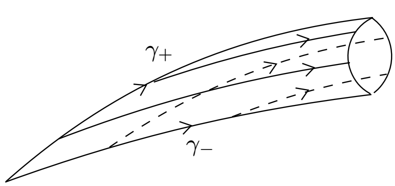


Fig. 2. Small time reachable set

3. Small time minimal syntheses for planar systems with state constraints

3.1. **Generalities.** We consider a system $\dot{q} = X + uY$, $|u| \leq 1$, $q = (x, y) \in \mathbb{R}^2$ with state constraints $c(q) \leq 0$. We denote by $\omega = p\,dq$ the *clock form* defined on the set where X and Y are independent by $\omega(X) = 1$ and $\omega(Y) = 0$. The singular trajectories are located on the set $S = \{q \in \mathbb{R}^2,\ \det(Y(q), [X,Y](q)) = 0\}$ and the singular control u_s is a solution of $\langle p, [X,[X,Y]](q)\rangle + u_s\langle p, [Y,[X,Y]](q)\rangle = 0$. The 2-form $d\omega$ is zero on the set S.

We take $q_0 \in \{q \in \mathbb{R}^2,\ c(q) = 0\}$ identified with 0. The problem is to determine local optimality status of a boundary arc $t \mapsto \gamma_b(t)$ corresponding to a control u_b and to describe time minimal syntheses near q_0. The first step is to construct a normal form, assuming the constraint of order one.

Lemma 3.1. *Assume that*

1. $X(q_0)$ *and* $Y(q_0)$ *are linearly independent*;
2. *the constraint is of order* 1, *i.e.*, $Yc(q_0) \neq 0$.

Then, replacing, if necessary, u *by* $-u$*, we can find a local diffeomorphism preserving* $q_0 = 0$ *such that the constrained system is*

$$\dot{x} = 1 + ya(q), \quad \dot{y} = b(q) + u, \qquad y \leq 0.$$

Proof. Using a local change of coordinates preserving 0, we can identify Y with $\partial/\partial y$ and the boundary arc with $\gamma_b : t \mapsto (t, 0)$. The admissible space is $y \leq 0$ or $y \geq 0$. Replacing, if necessary, u by $-u$, we can be identify it with $y \leq 0$. □

3.2. **The generic case A_1.** In this case, we impose the additional assumptions:

1. $Y(0), [X,Y](0)$ are linearly independent;
2. the boundary arc is admissible and not saturating at 0.

3.2.1. *Local model.* Under these assumptions, we have in the previous normal form: $a(0) \neq 0$ and $|b(0)| < 1$. To analyze the optimal synthesis near 0, we set $a = a(0)$ and $b = b(0)$ and the local model is

$$\dot{x} = 1 + ya, \quad \dot{y} = b + u, \qquad y \leq 0.$$

The clock form is

$$\omega = \frac{dx}{1 + ay}, \qquad d\omega = \frac{a}{(1 + ay)^2} dx \wedge dy.$$

3.2.2. *Local syntheses.* First, we consider the unconstrained case. The small time reachable set for the time extended system is represented in Fig. 2. Its boundary formed by the arc $\gamma_+\gamma_-$ or $\gamma_-\gamma_+$ (where $\gamma_+\gamma_-$ denotes an arc γ_+ followed by an arc γ_-) represents time minimal and time maximal trajectories. They are given by the minimum principle. Considering the model, we have two cases. If $a > 0$, then $d\omega > 0$ and each optimal trajectory is of the form $\gamma_+\gamma_-$, where $\gamma_-\gamma_+$ is time maximal. If $a < 0$, then $d\omega < 0$ and each optimal trajectory is of the form $\gamma_-\gamma_+$, where $\gamma_+\gamma_-$ is time maximal).

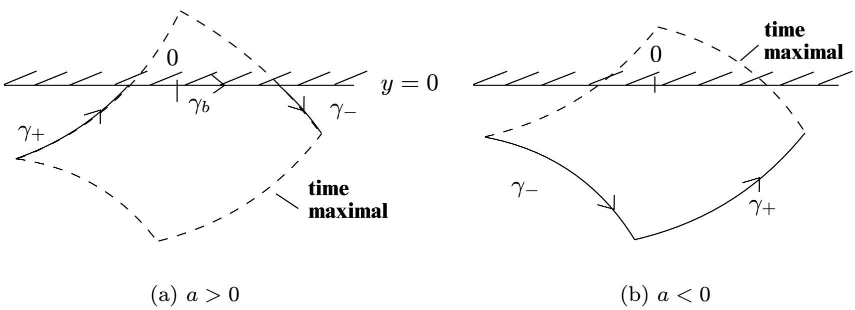


Fig. 3

For the constrained case, the same reasoning on the clock form shows that the boundary arc is optimal if and only if $a > 0$. The local optimal synthesis is represented in Fig. 3. We have proved the following lemma.

Lemma 3.2. *In the case A_1, we have*:

1. *for the unconstrained problem, if $a > 0$, then the arc $\gamma_+\gamma_-$ is time minimal and the arc $\gamma_-\gamma_+$ is time maximal and vice versa for $a < 0$;*
2. *for the constrained problem, a boundary arc is optimal if and only if $a > 0$; in this case, each optimal trajectory has the form $\gamma_+\gamma_b\gamma_-$. If $a < 0$, then each optimal arc has the form $\gamma_-\gamma_+$.*

3.2.3. *Connection with the minimum principle.* According to Lemma 2.4, along the boundary $\eta = \dfrac{\langle p, [X,Y](q)\rangle}{(Yc)(q)}$ with $\langle p, Y\rangle = 0$. Hence denoting $p = (p_x, p_y)$, we obtain $\eta = -ap_x$ and p_x is oriented by $\langle p, X+uY\rangle + p_t = 0$, $p_t \geq 0$. Hence $p_x < 0$. Therefore, $\mathrm{sign}(\eta) = \mathrm{sign}(a)$ and the *necessary optimality condition is violated if $a < 0$.*

3.3. The singular case B_1.

3.3.1. *Generalities and local model.* If Y and $[X, Y]$ are linearly dependent at 0, then $a(0) = 0$; we assume that the set

$$S = \{q \in \mathbb{R}^2,\ \det(Y(q), [X, Y](q)) = 0\}$$

is a simple curve. In Lemma 3.1, we have normalized Y to $\partial/\partial y$ and the boundary arc γ_b to $t \mapsto (t, 0)$. Hence the slope of S at 0 is an invariant. In the small time model, we approximate S by a straight line and the equations of the system are as follows:

$$\dot{x} = 1 + y(ay + bx), \quad \dot{y} = c + u, \qquad y \leq 0,$$

where the set S is identified with $\{(x, y) \in \mathbb{R}^2, 2ay + bx = 0\}$ and we assume that $a \neq 0$. Consider the system without state constraint and assume that $u \in \mathbb{R}$. According to [5], there exists a singular arc along S which can be time minimal or time maximal. The test to distinguish between the two cases is the Legendre–Clebsch condition and we have two cases:

- $a < 0$: the singular arc is time minimal;
- $a > 0$: the singular arc is time maximal.

The singular control u_s which makes S invariant is a solution of the equation

$$b(1 + y(ay + bx)) + 2a(c + us) = 0$$

and its value at 0 is

$$u_s = -c - b/2a.$$

Taking the constraint into account, we see that the condition of admissibility is $|c + b/2a| \leq 1$. In the previous normalizations, the clock form is

$$\omega = \frac{dx}{ay^2 + bxy}, \qquad d\omega = \frac{2ay + bx}{(ay^2 + bxy)^2} dx \wedge dy,$$

and we have $\operatorname{sign}(d\omega) = \operatorname{sign}(2ay + bx)$. We assume that the boundary arc is admissible and nonsaturating, $|c| < 1$. We have three generic cases to analyze. These cases are distinguished by the behavior of the bang-bang extremals for the unconstrained system near the switching surface. Differentiating twice $\Phi = \langle p, Y(q) \rangle$, we have:

$$\dot{\Phi} = \langle p, [X, Y](q) \rangle,$$
$$\ddot{\Phi}_u = \langle p, [X, [X, Y]](q) + u[Y, [X, Y]](q) \rangle,$$

where $u(t) = -\operatorname{sign}\langle p(t), Y(q(t)) \rangle$ and p is oriented by $\langle p, X + uY \rangle \leq 0$. The three generic cases are represented in Fig. 4; they are, respectively,

(a) hyperbolic case: $\ddot{\Phi}_+ < 0$, $\ddot{\Phi}_- > 0$, and $\langle p, Y \rangle = \langle p, [X, Y] \rangle = 0$;
(b) elliptic case: $\ddot{\Phi}_- < 0$, $\ddot{\Phi}_+ > 0$, and $\langle p, Y \rangle = \langle p, [X, Y] \rangle = 0$;
(c) parabolic case: $\ddot{\Phi}_+$ and $\ddot{\Phi}_-$ have the same sign at $\langle p, Y \rangle = \langle p, [X, Y] \rangle = 0$.

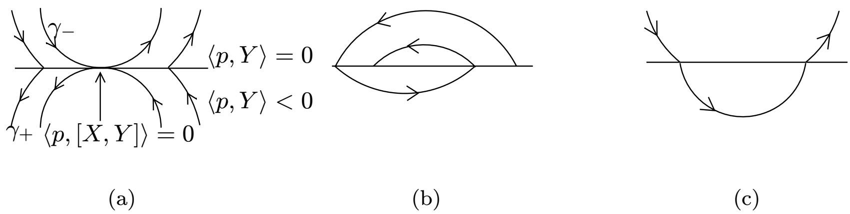


Fig. 4

In the hyperbolic case, the singular arc is admissible, nonsaturating, and time minimal. In the elliptic case, it is admissible, nonsaturating, and time maximal. In the parabolic case, $|u_s| > 1$ and the singular arc is not admissible. In the sequel, we analyze these three cases.

3.3.2. *Hyperbolic case: $a < 0$, $|c + b/2a| < 1$, $|c| < 1$, and $b \neq 0$.* We have two cases according to the slope at 0 of the singular arc (see Fig. 5).

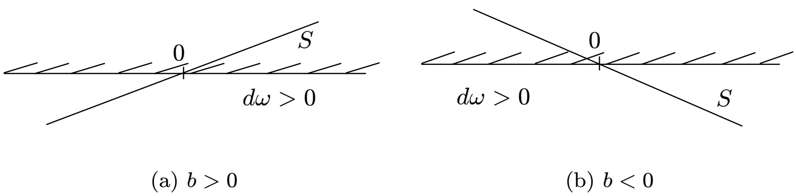


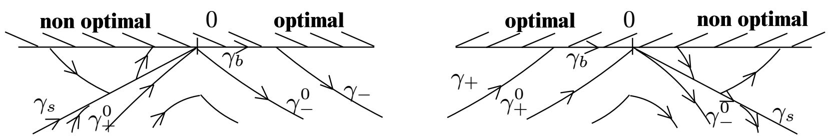
Fig. 5

The analysis is similar in both cases and we study only the case where $b > 0$. For the unconstrained problem, the singular arc is optimal and each optimal trajectory has at most two switchings and the local optimal synthesis is of the form $\gamma_\pm \gamma_s \gamma_\pm$. For the constrained problem, the using of the clock form and the Stokes theorem leads to the optimality of the boundary arc for $x \geq 0$ and nonoptimality for $x < 0$. The optimal synthesis joining two points of the boundary is then of the form $\gamma_- \gamma_s \gamma_b$. Each optimal curve in a neighborhood of 0 has at most 3 switchings and the local optimal synthesis is of the form $\gamma_\pm \gamma_s \gamma_b \gamma_\pm$. The situations are represented in Fig. 6 and are summarized in the following lemma.

Lemma 3.3. *Under our assumption, in the hyperbolic case each small time optimal trajectory has at most three switchings. Moreover,*

1. *for $b > 0$, a boundary arc is optimal if and only if $x \geq 0$ and each optimal arc has the form $\gamma_\pm \gamma_s \gamma_b \gamma_\pm$;*

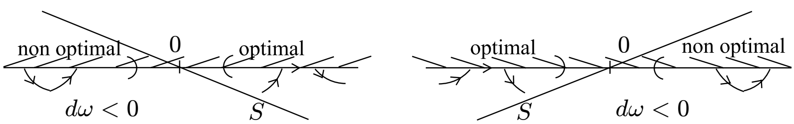
2. *for $b < 0$, a boundary arc is optimal if and only if $x \leq 0$ and each optimal arc has the form $\gamma_\pm \gamma_b \gamma_s \gamma_\pm$.*



(a) $b > 0$ (b) $b < 0$

Fig. 6

3.3.3. *Elliptic case: $a > 0$, $|c + b/2a| < 1$, $|c| < 1$, and $b \neq 0$.* Again, we have two cases according to the sign of the slope of the singular arc. Both cases are distinguished by the optimality status of the boundary arc, 0 being excluded (see Fig. 7). We study only the case where $b > 0$.



(a) $b > 0$ (b) $b < 0$

Fig. 7

For the optimal problem without state constraint, the situation is as follows. Each optimal control is bang-bang with at most one switching. This is proved by the concept of conjugate points (see [27]). Indeed, take near 0 a reference extremal with two switchings. It can be embedded into a one-parametric family of extremals with the same initial point represented in Fig. 8, which reflects on the switching locus.

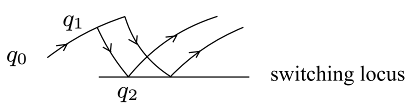


Fig. 8

Optimality status is lost after the second switching point which is conjugated. In particular, there exists a cut locus $C(0)$ for optimal trajectory starting from 0 (see Fig. 9(a)).

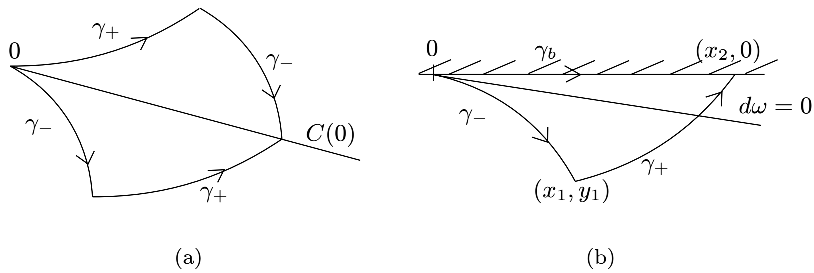
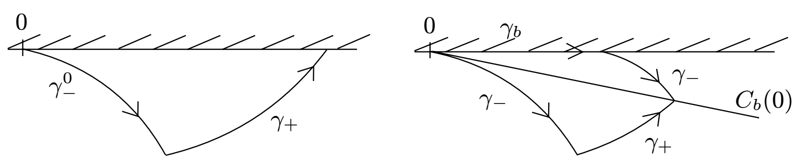


Fig. 9

This has the following consequence for the problem with state constraints. Let γ_-^0 be the arc associated with $u = -1$ starting at 0. Consider an arc $\gamma_-^0\gamma_+$ joining 0 with the boundary in this admissible domain (see Fig. 9(b)). Then it intersects $d\omega = 0$ and can be optimal or not. Using the model, we compare the time along the boundary arc γ_b and the time along $\gamma_-^0\gamma_+$ to decide about optimality. A straightforward computation yields the following lemma.

Lemma 3.4. *Consider the elliptic case where $b > 0$. If $\dfrac{b}{2a} > \dfrac{1-c^2}{c+3}$, then the boundary arc starting at 0 is optimal. If $\dfrac{b}{2a} < \dfrac{1-c^2}{c+3}$, then the optimal policy joining 0 with a nearby point of the boundary is $\gamma_-^0\gamma_+$. The small time reachable set from 0 is represented in Fig. 10.*



(a) $\frac{b}{2a} < \frac{1-c^2}{c+3}$ (b) $\frac{b}{2a} > \frac{1-c^2}{c+3}$

Fig. 10

If the boundary arc is admissible, there exists a cut-locus $C_b(0)$.

3.3.4. *Connection with the minimum principle.* In the hyperbolic and elliptic cases, the condition $\eta \geq 0$ along a boundary arc of the minimum principle detects the optimality outside 0 since $\eta = -bxp_x$. To decide at 0, we must use second-order condition taking the clock form. In the elliptic case, the bifurcation between the two cases can be obtained by the following standard argument. According to Lemma 2.4, for an extremal with a nontrivial boundary arc, the adjoint vector is continuous at the junction and hence $\langle p, Y\rangle = 0$, which determines p (using homogeneity) prior to the junction. This gives us a switching curve K for extremal curves with a boundary arc passing through 0. This curve can be computed and we obtain

$$y = 2(c+1)\frac{b/2a}{b/2a-(c+1)}x.$$

Depending on the slope, an extremal can reflect or cross this locus (see Fig. 11). The critical value is when the slope of γ_-^0 is equal to the slope of K, i.e., $\frac{b}{2a} = \frac{1-c^2}{c+3}$. *Hence optimality of the boundary arc corresponds to the crossing case.* This is coherent with [1] (see also Fig. 8).

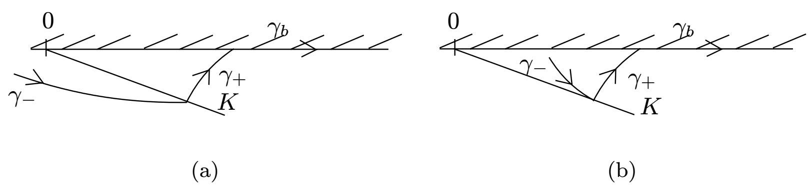


Fig. 11

3.3.5. *The parabolic case.* This case arises when the singular control is not admissible. The unconstrained case is easy to analyze and the optimal synthesis follows from the classification of extremals. Indeed, in the parabolic case, differentiating twice the switching function Φ, we have

$$\ddot{\Phi} = \langle p, [X,[X,Y]] + u[Y,[X,Y]]\rangle$$

and $\ddot{\Phi}$ has the same sign for $u=+1$ and $u=-1$. We have two situations represented in Fig. 12. Using the normal form, we obtain $\langle p, Y\rangle = 0$, i.e., $p_y = 0$:

$$\ddot{\Phi} = -p_x(b+2a(c+u)).$$

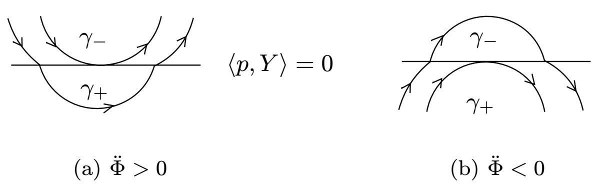


Fig. 12

Hence for the unconstrained problem, if $b+2a(c\pm1) > 0$, then the optimal solution is of the form $\gamma_-\gamma_+\gamma_-$ and if $b + 2a(c \pm 1) < 0$, then the optimal solutions are of the form $\gamma_+\gamma_-\gamma_+$ and the length of the intermediate arc is determined by the first switching point.

We analyze the optimal synthesis for the problem with state constraints. We have four cases, all similar. Consider, e.g., the case where $a > 0$ and $b > 0$. Since $|c| < 1$, we have $c+1 > 0$ and the optimal law for the problem without state constraint is of the form $\gamma_-\gamma_+\gamma_-$. The optimal status of the boundary arc, 0 excluded, is represented in Fig. 13 and the boundary arc is nonoptimal if $x < 0$ and optimal if $x > 0$. Moreover, the boundary arc γ_b^0 starting at 0 is optimal because the sector $d\omega > 0$ is positively invariant. The optimal synthesis follows from the minimum principle and Lemma 2.4.

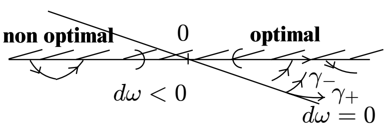


Fig. 13

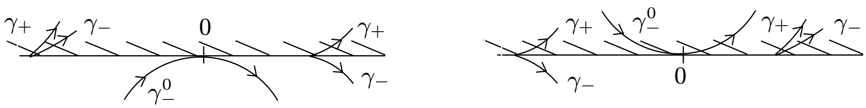
Lemma 3.5. *Consider the parabolic case and assume, e.g., that $a > 0$ and $b > 0$. Then:*

- *in the unconstrained case, each optimal policy is of the form $\gamma_-\gamma_+\gamma_-$;*
- *in the constrained case, each optimal policy is of the form $\gamma_-\gamma_+\gamma_b\gamma_-$.*

3.4. **The saturated case B_2.** Now we analyze the generic saturated case. Hence we assume that the boundary control u_b is saturated at 0, i.e., $u_b = \pm1$. We suppose that there exists no singular arc that passes through 0, i.e., a does not vanish at 0 in our model of Lemma 3.1. Consequently, if we denote $a(0) = a$ and $b(0) = b$, the local model is

$$\dot{x} = 1 + ya, \quad \dot{y} = b + u + cx + dy, \qquad y \leq 0,$$

where a, b, and c are constants and $b = \pm1$. Moreover, we assume that $c \neq 0$. The various cases are easy to analyze and we discuss in detail the case where $a > 0$ and $b = +1$. For the unconstrained problem, an optimal arc is of the form $\gamma_+\gamma_-$ and for the constrained problem, the saturating control is $u_b = -1$. We have two cases distinguished by the sign of c (see Fig. 14).

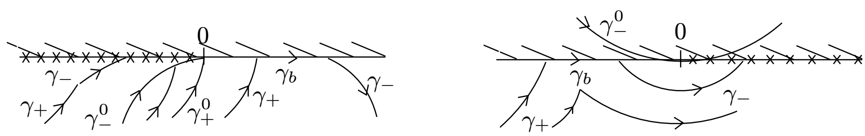


(a) $c > 0$ (b) $c < 0$

Fig. 14

If $c < 0$ (respectively, $c > 0$), the boundary arc is admissible only for $x \geq 0$ (respectively, $x \leq 0$). The local time minimal synthesis easily follows and is represented in Fig. 15.

Lemma 3.6. *Consider the saturated case where $a > 0$ and $b = 1$. Then:*
1. *if $c < 0$, then each local optimal arc has at most three switchings and is of the form $\gamma_+\gamma_-\gamma_b\gamma_-$;*
2. *if $c > 0$, then each local optimal arc has at most two switchings and is of the form $\gamma_+\gamma_b\gamma_-$.*



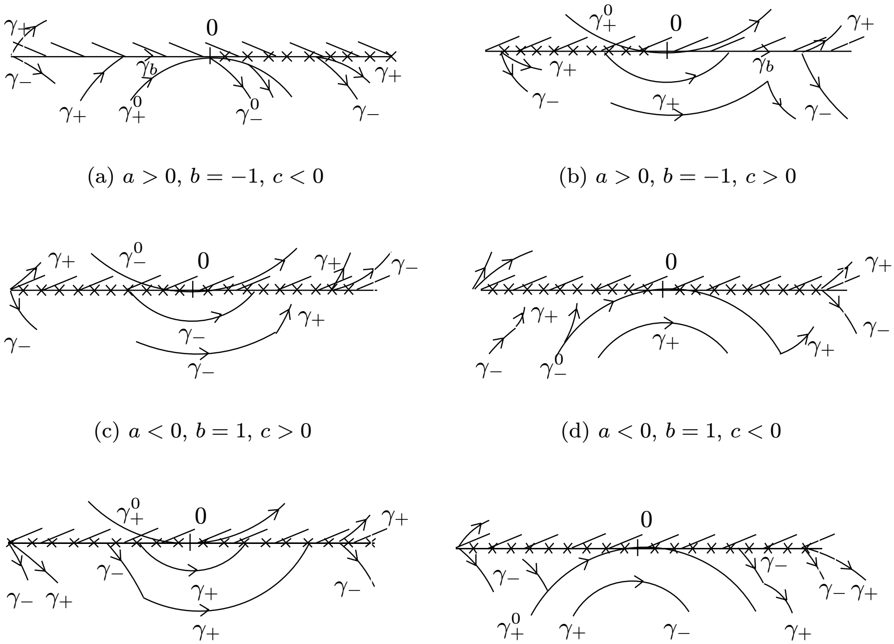
(a) $c > 0$ (b) $c < 0$

Fig. 15

The analysis is similar in the other cases and the syntheses are represented in Fig. 16.

4. Small time minimal synthesis for system in dimension 3 with state constraints

4.1. **Preliminaries.** We consider a system of the form $\dot{q} = X + uY$, $c(q) \leq 0$, $|u| \leq 1$, with $q = (x, y, z) \in \mathbb{R}^3$. The objective of this section is to initialize the classification of the optimal syntheses near a point q_0 identified with 0 on the boundary of the domain. First, consider the unconstrained case and assume that X, Y, and $[X, Y]$ are linearly independent at q_0. The small time reachable set $R(q_0)$ is represented in Fig. 2; its boundary is formed by the two surfaces S_1 and S_2 that are composed of extremal arcs of the form $\gamma_-\gamma_+$ and $\gamma_+\gamma_-$. To construct optimal trajectories, we



(a) $a > 0$, $b = -1$, $c < 0$

(b) $a > 0$, $b = -1$, $c > 0$

(c) $a < 0$, $b = 1$, $c > 0$

(d) $a < 0$, $b = 1$, $c < 0$

(e) $a < 0$, $b = -1$, $c > 0$

(f) $a < 0$, $b = -1$, $c < 0$

Fig. 16

must analyze the boundary of the small time reachable set for the time extended system. Its structure is described in [18] under generic assumptions. We proceed as follows. Differentiating twice the switching function $\Phi = \langle p(t), Y(q(t))\rangle$, we obtain

$$\dot{\Phi}(t) = \langle p(t), [X, Y](q(t))\rangle,$$
$$\ddot{\Phi}(t) = \langle p(t), [X + uY, [X, Y]](q(t))\rangle.$$

If $\langle p(t), [Y, [X, Y]](q(t))\rangle$ does not vanish, we can solve $\ddot{\Phi}(t) = 0$ to compute the singular control:

$$u_s = -\frac{\langle p, [X, [X, Y]](q)\rangle}{\langle p, [Y, [X, Y]](q)\rangle}.$$

If Y and $[X, Y]$ are independent, p can be eliminated by the homogeneity and u_s is computed as a feedback control. Introducing $D = \det(Y, [X, Y], [Y, [X, Y]])$ and $D' = \det(Y, [X, Y], [X, [X, Y]])$, we obtain $D'(q) + u_s D(q) = 0$. Hence in dimension 3, there exists a singular direction passing through each generic point. Moreover, as in the planar case, the Legendre–Clebsch condition allows one to distinguish between slow and fast directions in the nonexceptional case where X, Y, and $[X, Y]$ are noncollinear. We have two cases (see [5]).

Case 1: if X and $[[X, Y], Y]$ are on the opposite sides with respect to the plane generated by Y and $[X, Y]$, then the singular arc is locally time optimal if $u \in \mathbb{R}$;

Case 2: on the contrary, if X and $[[X, Y], Y]$ are on the same side, the singular arc is locally time maximal.

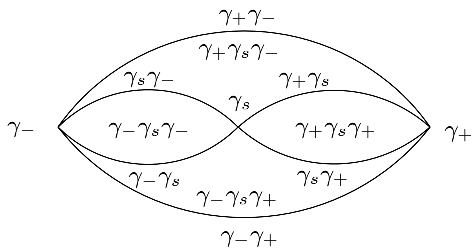
In these two cases, the constraint $|u_s| \leq 1$ is not taken into account and the singular control can be strictly admissible if $|u_s| < 1$, saturating if $|u_s| = 1$ at q_0, or nonadmissible if $|u_s| > 1$. We have three generic cases. Assume that X, Y, and $[X, Y]$ are noncollinear and p is oriented with the convention of the minimum principle: $\langle p(t), X + uY \rangle \leq 0$. Let t be a switching time of a bang-bang extremal: $\Phi = \langle p(t), Y(q(t)) \rangle = 0$. It is said to be of order one if $\dot{\Phi}(t) = \langle p(t), [X, Y](q(t)) \rangle \neq 0$ and of order two if $\dot{\Phi}(t) = 0$ but $\ddot{\Phi}(t) = \langle p(t), [X + uY, [X, Y]](q(t)) \rangle \neq 0$ for $u = \pm 1$. The classification of extremals near a point of order two is similar to the planar case (see Fig. 4). We have three cases:

- parabolic case: $\ddot{\Phi}_{\pm}$ have the same sign;
- elliptic case: $\ddot{\Phi}_{+} > 0$ and $\ddot{\Phi}_{-} < 0$;
- hyperbolic case: $\ddot{\Phi}_{+} < 0$ and $\ddot{\Phi}_{-} > 0$.

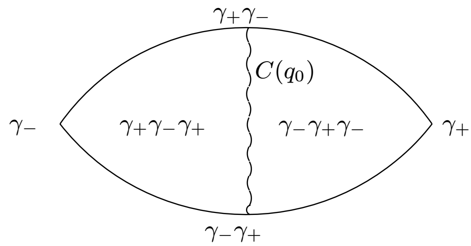
In both hyperbolic and parabolic cases, the local time optimal syntheses are obtained by using only the first-order conditions from the minimum principle and hence from extremality, together with Legendre–Clebsch condition in the hyperbolic case. More precisely, we have the following lemma.

Lemma 4.1. *In the hyperbolic or parabolic case, each extremal policy is locally time optimal. In the hyperbolic case each optimal policy is of the form* $\gamma_{\pm}\gamma_s\gamma_{\pm}$. *In the parabolic case, each optimal policy is bang-bang with at most two switchings.*

The set $B(q_0, T)$ describing the time minimal policy at fixed time is homeomorphic to a closed disk, whose boundary is formed by extremities of arcs $\gamma_-\gamma_+$ and $\gamma_+\gamma_-$ of length T and the stratification in the hyperbolic case is represented in Fig. 17(a).



(a) hyperbolic



(b) elliptic

Fig. 17

In the elliptic case, the situation is more complicated because there exists a cut locus. The analysis is related to the following crucial result based on the concept of conjugate points defined in Sec. 2.7.2 owing to [28].

Lemma 4.2. *Consider the system* $\dot{q} = X + uY$, $|u| \leq 1$, $q \in \mathbb{R}^3$. *Let* q_0 *be a point such that each of the two triplets* Y, $[X, Y]$, $[X-Y, [X, Y]]$ *and* Y, $[X, Y]$, $[X+Y, [X, Y]]$ *is linearly independent at* q_0. *Then each bang-bang locally time optimal trajectory has at most two switchings near* q_0.

The local time optimal policy at fixed time is represented in Fig. 17(b). There exists a *cut locus* $C(q_0)$ *whose extremities are conjugate points on the boundary of the reachable set.*

Now we analyze the constrained case. If the order of the constraint is one, the situation is similar to the planar case analyzed in Sec. 3. *Hence we assume that the constraint is of order* 2. *The case corresponding to the space shuttle problem is the parabolic case which is considered first.*

4.2. **Geometric normal form in the constrained parabolic case and local synthesis.** For the unconstrained problem, the situation in the parabolic case is clear. Indeed, X, Y, and $[X,Y]$ form a frame near q_0 and

$$[X \pm Y, [X,Y]] = aX + bY + c[X,Y];$$

the synthesis depends only on the sign of a at q_0. The small time reachable set is bounded by the surfaces formed by arcs $\gamma_-\gamma_+$ and $\gamma_+\gamma_-$. Each interior point can be reached by an arc $\gamma_-\gamma_+\gamma_-$ and an arc $\gamma_+\gamma_-\gamma_+$. According to Fig. 12, if $a<0$, then the time minimal policy is $\gamma_-\gamma_+\gamma_-$ and the time maximal policy is $\gamma_+\gamma_-\gamma_+$ and the vice versa if $a>0$. To construct the optimal synthesis, one can use a *nilpotent model*, where all Lie brackets of length greater than four are zero. In particular, the existence of a singular direction is irrelevant in the analysis and a model where $[Y,[X,Y]]$ is zero can be taken. This situation is called the *geometric model*. A similar model is constructed next taking into account the constraints, which are assumed to be of order 2. Moreover, we first suppose that conditions C_1 and C_3 hold along a boundary arc γ_b, i.e., $YXc \neq 0$ along γ_b and the boundary control is admissible and nonsaturating. We have the following lemma.

Lemma 4.3. *Under our assumptions, a local geometric model in the parabolic case is*

$$\begin{aligned}
\dot{x} &= a_1x + a_3z,\\
\dot{y} &= 1 + b_1x + b_3z,\\
\dot{z} &= (c+u) + c_1x + c_2y + c_3z, \qquad |u| \leq 1,
\end{aligned}$$

with $a_3>0$, where the constraint is $x \leq 0$ and the boundary arc is identified with $\gamma_b : t \mapsto (0,t,0)$. Moreover, we have

$$[X,Y] = -a_3\frac{\partial}{\partial x} - b_3\frac{\partial}{\partial y}, \quad [Y,[X,Y]] = 0,$$

$$[X,[X,Y]] = (a_1a_3 + a_3c_3)\frac{\partial}{\partial x} + (a_3b_1 + b_3c_3)\frac{\partial}{\partial y} + (a_3c_1 + b_3c_2 + c_3^2)\frac{\partial}{\partial z},$$

$$[X,[X,Y]] = aX \mod \{Y,[X,Y]\},$$

where $a = a_3b_1 - a_1b_3 \neq 0$. If the boundary arc is admissible and nonsaturating, we have $|c|<1$. Moreover, $a_3 = -[X,Y]c$.

Proof. We give the details of the normalizations.

Normalization 1. Since $Y(0) \neq 0$, we locally identify Y with $\partial/\partial z$. The local diffeomorphisms $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ preserving 0 and Y satisfy the conditions

$$\frac{\partial\varphi_1}{\partial z} = \frac{\partial\varphi_2}{\partial z} = 0, \quad \frac{\partial\varphi_3}{\partial z} = 1.$$

Since the constraint is of order 2, $Yc = 0$ near 0 and Y is tangent to all surfaces $c = \alpha$ for sufficiently small α; hence $\frac{\partial c}{\partial z} = 0$.

Normalization 2. Since c is independent of z, using a local diffeomorphism preserving 0 and $Y = \frac{\partial}{\partial z}$, we can identify the constraint with $c = x$. Then the system can be written in the form

$$\dot{x} = X_1(q), \quad \dot{y} = X_2(q), \quad \dot{z} = X_3(q) + u,$$

and $x \leq 0$. The secondary constraint is $\dot{x} = 0$, and by assumption, a boundary arc γ_b is contained in $x = \dot{x} = 0$ and passes through 0. In the parabolic case, the affine approximation is sufficient for our analysis and the geometric model is

$$\begin{aligned} \dot{x} &= a_1 x + a_2 y + a_3 z, \\ \dot{y} &= b_0 + b_1 x + b_2 y + b_3 z, \\ \dot{z} &= c_0 + c_1 x + c_2 y + c_3 z + u, \end{aligned}$$

where γ_b is approximated by the straight line $x = 0$, $a_2 y + a_3 z = 0$.

Normalization 3. Finally, we normalize the boundary as follows. Performing a transformation of the form $z' = \alpha y + z$, in the plane $x = 0$, we can normalize the boundary arc to $x = z = 0$. By using a diffeomorphism $y' = \varphi(y)$, the boundary arc can be parametrized as $\gamma_b : t \mapsto (0, t, 0)$. The normal form follows, changing if necessary u to $-u$, and hence permuting the arcs γ_+ and γ_-. □

Theorem 4.4. *Consider the time minimization problem for the system* $\dot{q} = X(q) + uY(q)$, $q \in \mathbb{R}^3$, $|u| \leq 1$ *with the constraint* $c(q) \leq 0$. *Let* $q_0 \in \{c = 0\}$; *we assume the following*:

1. *at the point* q_0, *the vectors* X, Y, *and* $[X, Y]$ *form a frame and*
$$[X \pm Y, [X, Y]](q_0) = aX(q_0) + bY(q_0) + c[X, Y](q_0),$$
where $a < 0$;
2. *the constraint is of order* 2 *and Assumptions* C_1 *and* C_3 *hold at the point* q_0.

Then the boundary arc passing through q_0 *is small time optimal if and only if the arc* γ_- *is contained in the nonadmissible domain* $c \geq 0$. *In this case, the local time minimal synthesis with a boundary arc is of the form* $\gamma_- \gamma_+^T \gamma_b \gamma_+^T \gamma_-$, *where* γ_+^T *are arcs tangent to the boundary arc.*

Proof. The proof is straightforward and can be done using a simple reasoning visualized on the normal form. In this case $q_0 = 0$, the boundary arc is identified with $t \mapsto (0, t, 0)$ and owing to $a_3 > 0$, arcs tangent to γ_b and corresponding to $u = \pm 1$ are contained in $c \leq 0$ if $u = -1$ and in $c \geq 0$

if $u = +1$. Let B be a point of the boundary arc γ_b for sufficiently small $B = (0, y_0, 0)$. If $u = \pm 1$, we have the following approximations for arcs initiating from B:

$$x(t) = a_3(c_0 + c_2 y_0 + u)t^2/2 + o(t^2),$$
$$z(t) = (c_0 + c_2 y_0 + u)t + o(t).$$

The projections on the plane (x, z) of the arcs $\gamma_-\gamma_+\gamma_-$ and $\gamma_+\gamma_-\gamma_+$ joining 0 with B are *loops* denoted by $\tilde{\gamma}_-\tilde{\gamma}_+\tilde{\gamma}_-$ and $\tilde{\gamma}_+\tilde{\gamma}_-\tilde{\gamma}_+$ represented in Fig. 18.

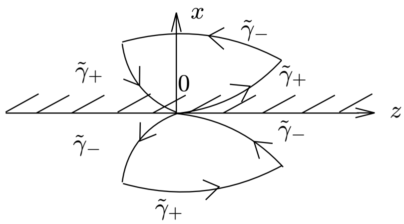


Fig. 18

The loops $\tilde{\gamma}_-\tilde{\gamma}_+\tilde{\gamma}_-$ (respectively, $\tilde{\gamma}_+\tilde{\gamma}_-\tilde{\gamma}_+$) are contained in $x \leq 0$ (respectively, $x \geq 0$). Now we can achieve the proof. Taking the original system, if the arc $\gamma_-\gamma_+\gamma_-$ joining 0 with B which is the optimal policy for the unconstrained problem is contained in $c \leq 0$, it is time minimal for the constrained case and the boundary arc is not optimal. On the contrary, we can join 0 with B by an arc $\gamma_+\gamma_-\gamma_+$ in $c \leq 0$, but this arc is time maximal. Hence clearly the boundary arc γ_b is optimal.

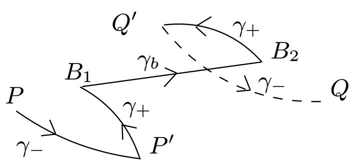


Fig. 19

In this case, the optimal synthesis easily follows. Indeed, take two points $B_1 < 0 < B_2$ of the boundary arc and consider the arcs $\gamma_-\gamma_+$ arriving at B_1 and $\gamma_+\gamma_-$ departing from B_2; this gives us the local optimal synthesis represented in Fig. 19.

□

4.2.1. *Connection with the minimum principle and geometric interpretation.* From Lemma 2.5, we have

$$\eta = \frac{\langle p, [X, [X, Y]](q)\rangle + u_b \langle p, [Y, [X, Y]](q)\rangle}{([X, Y]c)(q)},$$

where we can assume $[Y, [X, Y]] \equiv 0$. Moreover, $[X, [X, Y]] = aX + bY + c[X, Y]$ and $\langle p, Y\rangle = \langle p, [X, Y]\rangle = 0$ along the boundary. Hence, we have

$$\eta = \frac{a\langle p, X(q)\rangle}{([X, Y]c)(q)}.$$

In the normal form, $[X, Y]c = -a_3 < 0$ and $\langle p, X\rangle < 0$ by extremality. Hence the necessary condition $\eta \geq 0$ implies $a \geq 0$. In this case $\gamma_+\gamma_-\gamma_+$ is the optimal policy of the nonconstrained problem and, by Fig. 18, it is contained in $x \geq 0$. *Hence the necessary optimality condition $\eta \geq 0$ is violated if $a < 0$ in the normal form, when the boundary arc is not optimal.*

Observe also that the jumps ν_1 and ν_2 of the adjoint vectors at the entry point B_1 and at the exit point B_2 are given by the arcs extremals of the nonconstrained problem joining respectively P with B_1 and B_2 with Q. This will be used later in the multiple shooting algorithm.

4.3. **Connecting two constraints of order 2 in the parabolic case.** If there are two constraints in a small neighborhood of a point q_0, one needs to describe the transition between the two constraints. Hence we give a geometric normal form to analyze such a transition together with the optimal strategy.

Proposition 4.5. *Consider the control system $\dot{q} = X + uY$, $|u| \leq 1$, $q \in \mathbb{R}^3$, with two distinct constraints $c_i(q) \leq 0$, $i = 1, 2$. Assume that assumptions* 1 *and* 2 *of Theorem* 4.4 *hold for both constrained systems and denote by γ_b^1 and γ_b^2 the corresponding boundary arcs. Moreover, assume that the boundary arcs are optimal. Take a small neighborhood U of* 0 *containing subarcs of both γ_b^1 and γ_b^2, and assume that γ_b^1 hits the boundary $c_2 = 0$. Then there exists a geometric model of the form*

$$\begin{aligned}
\dot{x} &= a_1 x + a_3 z,\\
\dot{y} &= 1 + b_1 x + b_3 z,\\
\dot{z} &= c + u + c_1' x + c_2' y + c_3' z,
\end{aligned}$$

where the constraints are given by $c_1(q) = x$ and $c_2(q) = x + \varepsilon y$, with ε small. Moreover, the optimal policy near $q_0 = 0$ with boundary arcs is of the form $\gamma_+\gamma_-^T\gamma_b^1\gamma_-^T\gamma_b^2\gamma_-^T\gamma_+$, where the intermediate arc γ_-^T is the only arc tangent to both constraints.

Proof. The first constrained system is normalized by Lemma 4.3 as follows:

$$\begin{aligned}
\dot{x} &= a_1 x + a_3 z,\\
\dot{y} &= 1 + b_1 x + b_3 z,\\
\dot{z} &= (c+u) + c_1' x + c_2' y + c_3' z, \qquad |u| \le 1,
\end{aligned}$$

and $c_1(q) = x \le 0$. Since the constraint c_2 is of order 2, we have $\dfrac{\partial c_2}{\partial z} = 0$ and we can set $c_2(q) = d_1 x + d_2 y$, at first order. Observe that $c_2 = c_1$ if and only if $d_2 = 0$. Hence proceeding by perturbation and assuming that the arc hits the boundary of $c_2 \le 0$, we can set

$$c_2(q) = x + \varepsilon y,$$

where $\varepsilon > 0$ is small. The arcs γ_- tangent to γ_b^1 identified with $t \mapsto (0, t, 0)$ are approximated by

$$\begin{aligned}
x(t) &= a_3(c_0 + c_2 y_0 + u)t^2/2 + o(t^2),\\
y(t) &= (t + y(0)) + o(t),\\
z(t) &= (c + c_2 y_0 + u)t + o(t),
\end{aligned}$$

and to make the connection with γ_b^2 at time t, we must have

$$x(t) + \varepsilon y(t) = 0, \qquad \dot{x}(t) + \varepsilon \dot{y}(t) = 0.$$

This gives us two conditions which geometrically mean that we must construct near 0 an arc γ_- tangent to both constraints. The equations have near 0 a unique solution parametrized by $(0, y(0), 0)$, where $y(0) < 0$ is the exit point of γ_b^1 and t is the time to reach the arc γ_b^2. The estimates are

$$y(0) \sim \frac{\varepsilon}{2a_3(c+u)}, \qquad t \sim -\frac{\varepsilon}{a_3(c+u)}.$$

Also observe that from the practical point of view, this construction can be extended on each domain, where there exists an unique arc γ_- tangent to the constraints. □

4.4. **The constrained hyperbolic case.** Let γ_s^0 be the singular arc through passing $q_0 = 0$ and u_s^0 be the associated singular control. We assume that γ_s^0 is not tangent to the boundary arc. Changing if necessary u to $-u$, we can assume that the arc γ_-^0 passing through 0 is contained in the admissible domain identified with $x \le 0$ and the model is

$$\begin{aligned}
\dot{x} &= a_1 x + a_3 z + \dots,\\
\dot{y} &= 1 + b_1 x + b_3 z + \dots,\\
\dot{z} &= (c+u) + \dots, \qquad x \le 0, \quad a_3 > 0,
\end{aligned}$$

where $t \mapsto (0, t, 0)$ is the boundary arc.

We have two situations represented by the projection on the plane (x, z) in Fig. 20. Case 20(a) corresponds to $c+u_s^0 > 0$ and case 20(b) to $c+u_s^0 < 0$.

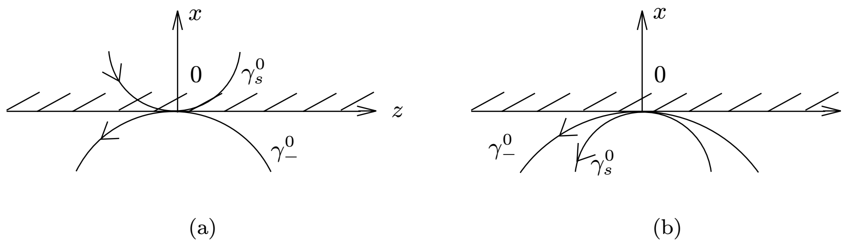


Fig. 20

In the first case, the arc γ_s^0 is not admissible and the local synthesis easily follows. Indeed, take two points $0 = (0, 0, 0)$ and $B = (0, y, 0)$, where $y > 0$, on the boundary. According to [18], the optimal synthesis joining two points in the unconstrained cases is $\gamma_-\gamma_s\gamma_-$ and is contained in the domain $x \leq 0$. Hence the boundary arc is not optimal. Moreover, a computation similar to the parabolic case shows that the necessary condition $\eta \geq 0$ along the boundary of the minimum principle is violated. Hence we have proved the following lemma.

Lemma 4.6. *With our assumption in the hyperbolic case, assume that the singular arc* γ_s^0 *through* 0 *is not admissible for the constrained problem. Then the boundary arc is not optimal and the necessary condition* $\eta \geq 0$ *in the minimum principle is violated.*

Now we consider the second case where γ_s^0 is contained in $x \leq 0$. For simplicity, we analyze only the *limit case* $|u| \leq M$ as M tends to ∞. According to the analysis in [5], the singular arc is C^0-locally optimal for the unconstrained problem and the optimal synthesis is as follows: a singular arc with jumps at the extremities along the control direction Y is identified with $\dfrac{\partial}{\partial z}$ to match the boundary conditions. To analyze the constrained case, we make the following reasoning. We denote by γ_s^T the singular arcs tangent to the boundary, which, by assumption, are contained in $x \leq 0$. Let P be a point of γ_s^T, which reaches the boundary at Q. Consider the policy represented in Fig. 21, which consists in jumping at P to another arc γ_s, reaching the boundary and then jumping to the boundary arc and following the latter.

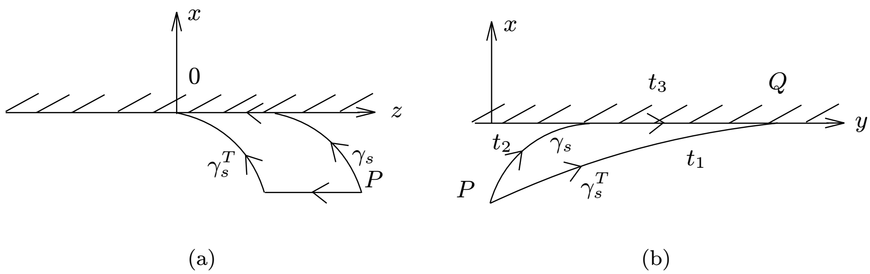


Fig. 21

If t_1, t_2, and t_3 are the times represented in Fig. 21, we have $t_2 + t_3 > t_1$ since γ_s^T is optimal for the unconstrained problem. This proves the following lemma.

Lemma 4.7. *With our assumption in the hyperbolic case, assume that the singular arc passing through $q_0 = 0$ is admissible. Then the boundary arc is optimal and for the limit case, each optimal trajectory with boundary is of the form $\mathbb{R}Y\gamma_s^T\gamma_b\gamma_s^T\mathbb{R}Y$, where $\mathbb{R}Y$ represents jumps in the control direction to match the boundary conditions.*

To study the case where $|u| \leq 1$, we first describe the local time optimal synthesis at time T for sufficiently small T, starting from $q_0 = 0$. For simplicity, we adopt the point of view of [18] and restrict ourselves on the so-called "free" nilpotent case where $[X, [X, Y]] = 0$:

$$\dot{x} = z, \quad \dot{y} = 1 - 2cx - z^2, \quad \dot{z} = c + u \tag{4.1}$$

with $|c| < 1$ and the constraint given by $x \leq 0$. Then the singular control is $u_s = 0$ and is strictly admissible. The boundary arc is $t \mapsto (0, t, 0)$ and is associated with the control $u_b = -c$. The singular arc is not tangent to the boundary arc and since we assume that γ_s^0 is admissible (see Fig. 20(a)); this means that $c < 0$.

Lemma 4.8. *Under our assumptions, the boundary arc is the optimal policy joining two points $A = (0, a, 0)$ and $B(0, b, 0)$ of the boundary.*

Proof. According to [18], extremal trajectories without boundary arc are of the form $\gamma_\pm\gamma_s\gamma_\pm$. None of these extremal trajectories allow a junction between A and B staying in the admissible domain $x < 0$. Hence the boundary arc is optimal. □

To describe the small time optimal syntheses, let us define, for sufficiently small T, the following sets:

$$\begin{aligned}
\Gamma_{bs-} &= \{\exp t_3 Z_- \exp t_2 Z_s \exp t_1 Z_b(0), t_1+t_2+t_3=T\} \cap \{x<0\},\\
\Gamma_{bs+} &= \{\exp t_3 Z_+ \exp t_2 Z_s \exp t_1 Z_b(0), t_1+t_2+t_3=T\} \cap \{x<0\},\\
\Gamma_{-s-} &= \{\exp t_3 Z_- \exp t_2 Z_s \exp t_1 Z_-(0), t_1+t_2+t_3=T\},\\
\Gamma_{-s+} &= \{\exp t_3 Z_+ \exp t_2 Z_s \exp t_1 Z_-(0), t_1+t_2+t_3=T\} \cap \{x<0\},
\end{aligned} \tag{4.2}$$

where Z_b, Z_s, and $Z_\pm$ correspond to $X+uY$ with u respectively equal to the boundary control u_b, the singular control u_s, and the regular controls $u=\pm 1$.

Lemma 4.9. *Under our assumptions, each small time optimal trajectory starting from* 0 *is of the form* $\gamma_b\gamma_s^T\gamma_\pm$ *or* $\gamma_-^0\gamma_s\gamma_\pm$. *Moreover, each of the sets defined in* (4.2) *is homeomorphic to a disk. Furthermore, we have*:

$$\begin{aligned}
\Gamma_{bs-}\cap\Gamma_{bs+} &= \Gamma_{bs} = \{\exp t_2 Z_s \exp t_1 Z_b(0), t_1+t_2=T\},\\
\Gamma_{bs-}\cap\Gamma_{-s-} &= \Gamma_{s-} = \{\exp t_2 Z_- \exp t_1 Z_s(0), t_1+t_2=T\},\\
\Gamma_{bs-}\cap\Gamma_{-s+} &= \Gamma_s \cup \Gamma_-\\
&= \{\exp t Z_s(0), t=T\} \cup \{\exp t Z_-(0), t=T\},\\
\Gamma_{bs+}\cap\Gamma_{-s-} &= \Gamma_s,\\
\Gamma_{bs+}\cap\Gamma_{-s+} &= \Gamma_{s+}\\
&= \{\exp t_2 Z_+ \exp t_1 Z_s(0), t_1+t_2=T\} \bigcap \{x<0\}.
\end{aligned} \tag{4.3}$$

These relations are depicted in Fig. 22, *by projection on the plane* (x,z).

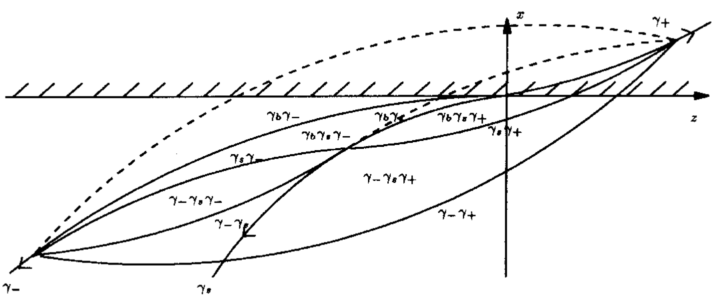


Fig. 22

Proof. According to [18], extremal trajectories without boundary arc are of the form $\gamma_\pm\gamma_s\gamma_\pm$. Similarly, after leaving a boundary arc, an extremal trajectory is of the form $\gamma_\pm\gamma_s\gamma_\pm$. Moreover, an extremal trajectory for the constrained problem cannot leave the boundary arc with an arc γ_+ which is not contained in the admissible domain. Let us also prove that if an extremal trajectory leaves a boundary arc with an arc γ_-, then there is no more switching. Indeed, writing the necessary conditions of Proposition 2.1 with our system (4.1), we obtain for the adjoint system:

$$\dot{p}_x = 2cp_y - \eta, \quad \dot{p}_y = 0, \quad \dot{p}_z = -p_x + 2zp_y.$$

Along a boundary arc $p_x = p_y = 0$, $p_y < 0$, and $\eta = 2cp_y > 0$. At the exit time t_1, we have $p_x(t_1+) = p_x(t_1-) - \nu_1 = -\nu_1$, $p_y(t_1+) = p_y(t_1-)$, $p_z(t_1+) = p_z(t_1-) = 0$, with $\nu_1 \geq 0$. Therefore, after leaving the boundary with an arc γ_-, the switching function is $\Phi(t) = -p_y(t-t_1)(t-t_1-\nu_1/p_y)$ and is strictly positive for $t > t_1$. This completes the proof of the first part of the lemma. For the second part, a straightforward computation proves assertions (4.3). □

Solving the same problem backward in time, we obtain the small time optimal syntheses to join $q_0 = 0$.

Lemma 4.10. *Under our assumptions, each small time optimal trajectory joining* 0 *is of the form* $\gamma_\pm\gamma_s^T\gamma_b$ *or* $\gamma_\pm\gamma_s\gamma_-^0$.

The small time local synthesis near the boundary follows from Lemmas 4.9 and 4.10. The computations are simplified in the "free" nilpotent case but our reasoning uses only the minimum principle and, more generally, we deduce the following proposition.

Proposition 4.11. *Consider the hyperbolic case and assume that the singular arc* γ_s^0 *passing through* $q_0 = 0$ *is not tangent to the boundary arc. If* γ_s^0 *is contained in the nonadmissible domain, the boundary arc is not optimal. If* γ_s^0 *is contained in the admissible domain, the boundary arc is locally optimal and the local optimal trajectories with boundary arcs are of the form* $\gamma_\pm\gamma_s^T\gamma_b\gamma_s^T\gamma_\pm$.

4.5. **Conclusion.** In order to classify the generic situation in dimension 3, we must analyze the elliptic case. This is beyond the scope of this article. Indeed, in the elliptic case, we must introduce second-order conditions to compute *conjugate points* along bang-bang trajectories *in the constrained case* and apply Morse theory. This will be the purpose of a forthcoming paper.

5. Control of the atmospheric arc, multiple shooting algorithm, and numerical results

5.1. **A short review on the multiple shooting algorithm.** This numerical algorithm is standard, and a classical reference is [26].

Let us consider a *single-input affine control system* in $\mathbb{R}^n$:

$$\dot{q} = X(q) + uY(q),$$

where the control satisfies the constraint $|u| \leq 1$ and the state q is submitted to a scalar constraint $c(q) \leq 0$. The problem is to minimize the cost

$$J(u) = \int_0^{t_f} \varphi(q(t))dt$$

among all trajectories satisfying $q(0) \in M_0, q(t_f) \in M_1$, where M_0 and M_1 are submanifolds in $\mathbb{R}^n$.

Now applying the *minimum principle* stated in Sec. 2.3.1, one comes to a *boundary value problem* of the following form:

$$\dot{z}(t) = F(z(t), t) = \begin{cases} F_0(t, z(t)) & \text{if } \ t_0 \leq t < t_1, \\ F_1(t, z(t)) & \text{if } \ t_1 \leq t < t_2, \\ & \dots, \\ F_s(t, z(t)) & \text{if } \ t_s \leq t \leq t_f, \end{cases} \tag{5.1}$$

where $z = (q, p) \in \mathbb{R}^{2n}$ and $t_1, t_2, \dots, t_s \in [t_0, t_f]$ are:

- either *switching times*, i.e., times at which the shooting function $\Phi(t) = \langle p(t), Y(q(t)) \rangle$ vanishes, and hence the control $u(t)$ may pass, e.g., from -1 to $+1$;
- or *junction times*, i.e., times at which the trajectory joins a boundary arc;
- or *contact times*, i.e., times at which the trajectory only touches the boundary.

Moreover, the following conditions hold at these points:

$$\forall j \in \{1, \dots, s\} \quad r_j(t_j, z(t_j^-), z(t_j^+)) = 0. \tag{5.2}$$

Among these conditions, we have:

- *continuity conditions* on the state and costate at switching points;
- at junction and contact points: *continuity conditions* on the state, *jump conditions* on the costate, and *conditions on the constraint c*. For example, if one joins a boundary arc *of order p*, then the following p conditions hold at that point:

$$c = \dot{c} = \cdots = c^{(p-1)} = 0.$$

Finally, we have the *boundary conditions*:

$$r_{s+1}(t_f, z(t_0), z(t_f)) = 0, \tag{5.3}$$

which contain:

- the initial and final conditions on the state;
- the initial and final conditions on the costate, given by the minimum principle (for instance if the component $q_i(t_0)$ of the state is free, then the corresponding component of the costate vector is equal to zero at time t_0);
- if the final time t_f is not fixed, then the Hamiltonian vanishes at time t_f.

Remark. A priori t_f is unknown. On the other hand, in the multiple shooting method the *number s of switchings has to be fixed* and must be deduced from a geometric analysis of the problem.

The multiple shooting method consists in subdividing the interval $[t_0, t_f]$ in N subintervals, where the value of $z(t)$ at the beginning of each subinterval is unknown. More precisely, let $t_0 < \sigma_1 < \cdots < \sigma_k < t_f$ be a *fixed* subdivision of the interval $[t_0, t_f]$. At each point σ_j the function z is *continuous*. We can consider σ_j as a fixed switching point at which the following conditions hold:

$$z(\sigma_j^+) = z(\sigma_j^-), \quad \sigma_j = \sigma_j^* \text{ fixed.}$$

Now introduce the *nodes*:

$$\{x_1, \dots, x_m\} = \{t_0, t_f\} \cup \{\sigma_1, \dots, \sigma_k\} \cup \{t_1, \dots, t_s\}. \tag{5.4}$$

We arrive at the following *boundary value problem*:

$$\begin{aligned} &\dot{z}(t) = F(t, z(t)) = \begin{cases} F_1(t, z(t)) & \text{if } x_1 \le t < x_2, \\ F_2(t, z(t)) & \text{if } x_2 \le t < x_3, \\ & \dots, \\ F_{m-1}(t, z(t)) & \text{if } x_{m-1} \le t \le x_m, \end{cases} \\ &\forall j \in \{2, \dots, m-1\} \quad r_j(x_j, z(x_j^-), z(x_j^+)) = 0, \\ &r_m(x_m, z(x_1), z(x_m)) = 0, \end{aligned} \tag{5.5}$$

where $x_1 = t_0$ is fixed and $x_m = t_f$.

Remark. The stability of the method can be improved by increasing the number of nodes. Indeed, the principle of the method is to overcome the unstability of a simple shooting method, where the influence of inaccurate initial data can grow exponentially with the length $t_f - t_0$ (see [26]).

Set $z_j^+ = z(x_j^+)$. Let $z(t, x_{j-1}, z_{j-1}^+)$ denote the solution of the Cauchy problem

$$\dot{z}(t) = F(t, z(t)), \quad z(x_{j-1}) = z_{j-1}^+.$$

We have

$$z(x_j^-) = z(x_j^-, x_{j-1}, z_{j-1}^+).$$

The interior and boundary conditions can be rewritten as

$$\begin{aligned} \forall j \in \{2, \dots, m-1\} \quad r_j(x_j, z(x_j^-, x_{j-1}, z_{j-1}^+), z_j^+) = 0, \\ r_m(x_m, z_1^+, z(x_m^-, x_{m-1}, z_{m-1}^+)) = 0. \end{aligned} \tag{5.6}$$

Now we set

$$Z = (z_1^+, x_m, z_2^+, x_2, \dots, z_{m-1}^+, x_{m-1})^T \in \mathbb{R}^{(2n+1)(m-1)},$$

where $z \in \mathbb{R}^{2n}$. Then the previous conditions hold if

$$G(Z) = \begin{pmatrix} r_m(x_m, z_1^+, z(x_m^-, x_{m-1}, z_{m-1}^+)) \\ r_2(x_2, z(x_2^-, x_1, z_1^+), z_2^+) \\ \vdots \\ r_{m-1}(x_m, z(x_{m-1}^-, x_{m-2}, z_{m-2}^+), z_{m-1}^+) \end{pmatrix} = 0. \tag{5.7}$$

The problem is now reduced to *determine a zero* of the function G which is defined on a vector space whose dimension is proportional to the number of switching points and points of the subdivision. The equation $G = 0$ can be solved iteratively by using a Newton-type method.

We refer to [11, 17] for more details on numerical methods. Our algorithm is written in FORTRAN, and simulations were performed with MATHLAB.

5.2. **The atmospheric re-entry problem.**

5.2.1. *The model.* Let 0 be the center of the planet, $K = NS$ is the axis of rotation, Ω is the angular velocity. We denote by $E = (e_1, e_2, e_3)$ with $e_3 = K$ an inertial frame with center 0. The reference frame is the quasi-inertial frame $R_1 = (I, J, K)$ with origin 0, rotating around K, with angular speed Ω and I is chosen to intersect the Greenwich meridian. Let r_T be the radius of the planet, G be the center of mass of the shuttle. We denote by (r, l, L) the spherical coordinates of G, where $r \geq r_T$ is the distance OG, $h = r - r_T$ is the altitude, l is the longitude, and L is the latitude. We denote by $R_1' = (e_r, e_l, e_L)$ a moving frame with center G, where e_r is the local vertical, (e_l, e_L) is the local horizontal plane, and e_L points to the north. The spherical coordinates have a singularity at the poles.

Let $\xi : t \mapsto (x(t), y(t), z(t))$ be the trajectory of G measured in the quasi-inertial frame attached to the planet and $\overrightarrow{v} = \dot{x}I + \dot{y}J + \dot{z}K$ be the relative velocity. The vector $\overrightarrow{v}$ is represented by its module v and two angles:

- γ: *path inclination* which is the angle with respect to the horizontal plane;
- χ: *azimuth angle* which is the angle of the projection of $\overrightarrow{v}$ in the horizontal plane measured with respect to the axis e_L.

We denote by (i, j, k) the orthonormal frame, where $i = \overrightarrow{v}/v$, j is the unitary vector in the plane (i, e_r) perpendicular to i and oriented by $j \cdot e_r > 0$, and $k = i \wedge j$.

The system is written in the coordinates $(r, v, \gamma, L, l, \chi)$. The forces acting on the vehicle are the gravitational force $\overrightarrow{P} = m\overrightarrow{g}$ and aerodynamic force which decomposes into a *drag force* $\overrightarrow{D}$ opposite to the relative velocity and a *lift force* $\overrightarrow{L}$ perpendicular to the velocity. Since $q = (r, L, l)$ are measured in a quasi inertial frame, we have additional Coriolis and centripetal forces. The aerodynamic forces have simple expressions in the frame (i, j, k) and from [20] and [12] the equations of the system are:

$$
\begin{aligned}
\frac{dr}{dt} &= v \sin\gamma, \\
\frac{dv}{dt} &= -g\sin\gamma - \frac{1}{2}\rho\frac{SC_D}{m}v^2 \\
&\quad + \Omega^2 r\cos L(\sin\gamma\cos L - \cos\gamma\sin L\cos\chi), \\
\frac{d\gamma}{dt} &= \cos\gamma\left(-\frac{g}{v} + \frac{v}{r}\right) + \frac{1}{2}\rho\frac{SC_L}{m}v\cos\mu \\
&\quad + 2\Omega\cos L\sin\chi + \Omega^2\frac{r}{v}\cos L(\cos\gamma\cos L + \sin\gamma\sin L\cos\chi), \\
\frac{dL}{dt} &= \frac{v}{r}\cos\gamma\cos\chi, \\
\frac{dl}{dt} &= \frac{v}{r}\frac{\cos\gamma\sin\chi}{\cos L}, \\
\frac{d\chi}{dt} &= \frac{1}{2}\rho\frac{SC_L}{m}\sin\mu\frac{v}{\cos\gamma} + \frac{v}{r}\cos\gamma\tan L\sin\chi \\
&\quad + 2\Omega(\sin L - \tan\gamma\cos L\cos\chi) + \Omega^2\frac{r}{v}\frac{\sin L\cos L\sin\chi}{\cos\gamma},
\end{aligned}
\tag{5.8}
$$

where μ is the *bank angle*, S is the reference area and C_L and C_D are respectively the lift and drag coefficients depending on the *angle of attack* α (incidence) and the Mach number. The air density is ρ and we take an exponential model: $\rho(r) = \rho_0 e^{-\beta r}$. For the atmospheric arc, the angle of attack α is a constant and the *control is the bank angle* μ. We set $u_1 = \cos\mu$ and $u_2 = \sin\mu$.

5.2.2. *Optimal control.* The problem is to steer the vehicle from an initial manifold M_0 to a terminal manifold M_1. To be more precise, the terminal time t_f is free and the boundary conditions are given in Table 1.

The state constraints are of the form $c_i(q) \leq 0$ for $i = 1, 2, 3$:

- constraint on the *thermal flux*:

$$\varphi = c_q\sqrt{\rho}v^3 \leq \varphi^{\max},$$

  where c_q is a constant;

	Initial conditions	Terminal conditions
altitude h	119.82 km	15 km
velocity v	7404.95 ms^{-1}	445 ms^{-1}
flight angle γ	-1.84 deg	free
latitude L	0	10.99 deg
longitude l	free	166.48 deg
azimuth χ	free	free

Table 1. Boundary conditions

- constraint on the *normal acceleration*

$$\gamma_n = \gamma_{n_0} \rho v^2 \leq \gamma_n^{\max};$$

- constraint on the *dynamic pressure*

$$\frac{1}{2} \rho v^2 \leq P^{\max}.$$

They are approximated in Fig. 23 in the flight domain in terms of the *drag* $d = \dfrac{1}{2} \dfrac{SC_D}{m} \rho v^2$ and v.

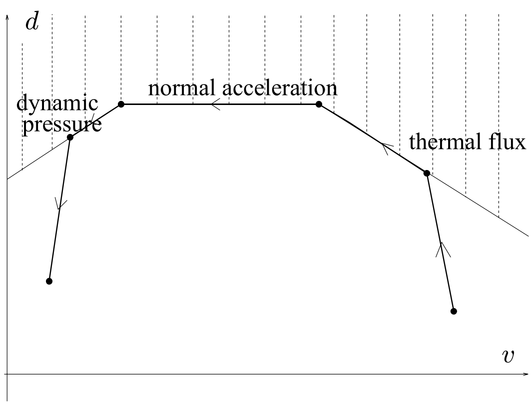


Fig. 23. Constraints: Harpold and Graves strategy

The optimal control problem is to minimize the total amount of the thermal flux:

$$J(\mu) = \int_0^{t_f} c_q \sqrt{\rho} v^3 dt. \tag{5.9}$$

If we introduce the new time parameter

$$ds = \varphi dt, \tag{5.10}$$

our optimal problem is a *time minimizing problem.*

5.2.3. *Harpold and Graves strategy* [14]. If we use the approximation $\dot{v} = -d$, the cost can be written as

$$J(\mu) = K \int_{v_f}^{v_0} \frac{v^2}{\sqrt{d}} dv, \quad K > 0,$$

and the optimal strategy is to maximize during the flight the drag d. This is the policy described in [14] which reduces the problem to the problem of finding a system trajectory to track the boundary of the domain in the following order: thermal flux → normal acceleration → dynamic pressure (see Fig. 23).

5.2.4. *Properties and structure of the system.* The problem is to minimize time for a system of the form

$$\frac{dq}{dt} = X(q) + u_1 Y_1(q) + u_2 Y_2(q),$$

where $u_1 = \cos\mu$, $u_2 = \sin\mu$, and $q = (r, v, \gamma, L, l, \chi)$. If we set $q_1 = (r, v, \gamma)$ and $q_2 = (L, l, \chi)$, the system can be decomposed as follows:

$$\dot{q}_1 = F_1(q_1, u_1) + O(\Omega), \quad \dot{q}_2 = F_2(q, u_2).$$

The first system governing the *longitudinal motion* has the form

$$\begin{aligned} \frac{dr}{dt} &= (v \sin\gamma)\psi, \\ \frac{dv}{dt} &= -\left(g \sin\gamma + \frac{1}{2}\rho \frac{SC_D}{m} v^2\right)\psi + o(\Omega), \\ \frac{d\gamma}{dt} &= \left(\cos\gamma\left(-\frac{g}{v} + \frac{v}{r}\right) + \frac{1}{2}\rho\frac{SC_L}{m} v \cos\mu + 2\Omega \cos L \sin\chi\right)\psi + o(\Omega) \end{aligned} \tag{5.11}$$

and the second system governing the *lateral motion* has the form

$$\begin{aligned} \frac{dL}{dt} &= \left(\frac{v}{r}\cos\gamma\cos\chi\right)\psi, \\ \frac{dl}{dt} &= \left(\frac{v}{r}\frac{\cos\gamma\sin\chi}{\cos L}\right)\psi, \\ \frac{d\chi}{dt} &= \left(\frac{1}{2}\rho\frac{SC_L}{m}\sin\mu\frac{v}{\cos\gamma} + \frac{v}{r}\cos\gamma\tan L\sin\chi\right)\psi \\ &\quad + 2\Omega(\sin L - \tan\gamma\cos L\cos\chi)\psi + o(\Omega), \end{aligned} \tag{5.12}$$

where $\psi = 1/\varphi$ and the centripetal force $o(\Omega)$ is neglected.

For the control of the atmospheric arc, one of the main problems in the flight domain is to avoid violation of the thermal flux constraint in the first part of the arc and this requires a careful analysis of the longitudinal motion. Omitting the Coriolis and centripetal terms, (5.11) is a scalar input affine system of the form

$$\dot{q} = X + uY, \quad |u| \leq 1, \qquad \text{where } q = (r, v, \gamma) \in \mathbb{R}^3,$$

and the state constraints are of the form $c_i(q) \leq 0$, $i = 1, 2, 3$.

5.2.5. *Lie bracket properties of the longitudinal motion and constraints.* Consider the system $\dot{q} = X + uY$, $q = (x, y, z) \in \mathbb{R}^3$, where

$$X = \psi \left(v \sin\gamma \frac{\partial}{\partial r} - (g \sin\gamma - k\rho v^2) \frac{\partial}{\partial v} + \cos\gamma \left(-\frac{g}{v} + \frac{v}{r} \right) \frac{\partial}{\partial \gamma} \right),$$

$$Y = \psi k' \rho v \frac{\partial}{\partial \gamma},$$

which describes the longitudinal motion when the rotation of the Earth is neglected ($\Omega = 0$) and $g = g_0/r^2$ is assumed to be constant. The following results obtained by computations are crucial.

Lemma 5.1. *In the flight domain, where* $\cos\gamma \neq 0$*, we have*:

1. X*,* Y*, and* $[X, Y]$ *are linearly independent*;
2. $[Y, [X, Y]] \in \text{span}\{Y, [X, Y]\}$;
3. $[X, [X, Y]](q) = a(q)X(q) + b(q)Y(q) + c(q)[X, Y](q)$ *with* $a < 0$.

Lemma 5.2. *Assuming that* C_D *and* C_L *are constants, the constraints are of order* 2 *and Assumption* C_1*, "*YXc *does not vanish on the boundary," is satisfied in the flight domain.*

5.2.6. *Application of the classification to the space shuttle.* The constraints are of order 2 and Assumption C_1 is satisfied. In the part of the flight domain, where the boundary arc is admissible and nonsaturating (Assumption C_3), the arc γ_- violates the constraints along the boundary. Hence we have proved the following assertion (see Theorem 4.4).

Corollary 5.3. *Assume that* $\Omega = 0$ *and consider the longitudinal motion in the re-entry problem. Then in the flight domain, where Assumption* C_3 *is satisfied, a boundary arc is locally optimal and the small time optimal synthesis with fixed boundary conditions on* (r, v, γ) *is of the form* $\gamma_- \gamma_+^T \gamma_b \gamma_+^T \gamma_-$.

5.3. Extremals of the problem.

Preliminaries. First, we consider the problem without constraint on the state. The Hamiltonian is

$$H(q,p,u) = \langle p, X(q)\rangle + u_1 \langle p, Y_1(q)\rangle + u_2 \langle p, Y_2(q)\rangle + p^0\varphi,$$

where $u = (u_1, u_2)$, $u_1 = \cos\mu$, $u_2 = \sin\mu$, and

- $p = (p_r, p_v, p_\gamma, p_L, p_l, p_\chi)$ is the vector dual to the state q;
- p^0 is the dual component of the flux.

If the trajectories are parametrized by $ds = \varphi(q)dt$, then the optimal problem is a time minimization problem. The control domain is $u_1^2 + u_2^2 = 1$ and the optimal control problem is *not convex* and can be relaxed by taking $u_1^2 + u_2^2 \leq 1$ in order to ensure the existence of optimal solutions. According to the minimum principle, the optimal controls have to minimize $u \mapsto H(q,p,u)$ over $u_1^2 + u_2^2 = 1$. Hence outside the switching surface Σ: $\langle p, Y_1\rangle = \langle p, Y_2\rangle = 0$, an extremal control is given by

$$\begin{aligned} u_1 = \cos\mu &= -\frac{\langle p, Y_1\rangle}{\sqrt{\langle p, Y_1\rangle^2 + \langle p, Y_2\rangle^2}} = -\frac{\cos\gamma\, p_\gamma}{\sqrt{\cos^2\gamma\, p_\gamma^2 + p_\chi^2}}, \\ u_2 = \sin\mu &= -\frac{\langle p, Y_2\rangle}{\sqrt{\langle p, Y_1\rangle^2 + \langle p, Y_2\rangle^2}} = -\frac{p_\chi}{\sqrt{\cos^2\gamma\, p_\gamma^2 + p_\chi^2}}. \end{aligned} \tag{5.13}$$

The corresponding extremals are called *regular* and the extremals contained in the switching surface are called *singular*. Owing to the existence of singularities, the behavior of regular extremals is complex and the analysis is outlined in [7].

The system parametrized by s can be written as

$$\frac{dq}{ds} = \tilde{X}(q) + u_1\tilde{Y}_1(q) + u_2\tilde{Y}_2(q), \tag{5.14}$$

where $\tilde{X} = \varphi^{-1}X$, $\tilde{Y}_1 = \varphi^{-1}Y_1$, and $\tilde{Y}_2 = \varphi^{-1}Y_2$. We set $\tilde{F}(q,\mu) = \tilde{X}(q) + \cos\mu\,\tilde{Y}_1(q) + \sin\mu\,\tilde{Y}_2(q)$.

Definitions. We denote by $E : u(\cdot) \mapsto q(t, q_0, u)$ the endpoint mapping of system (5.14) and by $E' : \mu(\cdot) \mapsto q(t, q_0, \mu)$ the endpoint mapping associated with $\dot{q} = \tilde{F}(q,\mu)$. Observe that if e is the mapping $\mu(\cdot) \mapsto (\cos\mu, \sin\mu)$, we have $E' = E \circ e$. If we endow the set of inputs with the L^∞-norm topology, both mappings E and E' are Fréchet differentiable and have inputs where the Fréchet derivatives are singular, i.e., not surjective. The following result is standard.

Lemma 5.4. *The regular extremals are the singularities of E' and the singular extremals are the singularities of E.*

Hence regular extremals are the singular trajectories of the system $\dot{q} = \tilde{F}(q,\mu)$. They are depend on system (5.14) and the constraint $u_1^2 + u_2^2 = 1$. The reduced Hamiltonian takes the form

$$\tilde{H}(q,p,\mu) = \langle p, \tilde{F}(q,\mu)\rangle.$$

The extremals contained in $\tilde{H} = 0$ are called *exceptional*. The optimality status of singular trajectories in the time optimal control problem can be investigated using the Morse theory developed in [5] under generic assumptions (see also [24], [2], and [30]). This requires the computation of the second order derivative of E' to evaluate the conjugate points. This computation is simplified along singular trajectories corresponding to constant controls. The algorithm is given in [4].

The computations of singular extremals which depend only upon the system can be reduced to a calculation of [7] if we assume that $\Omega = 0$. Indeed, in this case the axis NS is arbitrary and the system decomposes into a system in dimension 5 of the form $\dot{q}' = \tilde{X}' + u_1 \tilde{Y}_1' + u_2 \tilde{Y}_2'$, where $q' = (r, v, \gamma, L, \chi)$ and $\dot{l} = F(q')$. From [7] we obtain the following assertion.

Lemma 5.5. *For simplicity,we assume that $g = g_0$. The singular trajectories of $\dot{q}' = \tilde{X}' + u_1 \tilde{Y}_1' + u_2 \tilde{Y}_2'$ are located at $\chi = k\pi$ with $u_2 = 0$ and are the singular trajectories of the single input system $\dot{q}' = \tilde{X}' + u_1 \tilde{Y}_1'$.*

The computation in the single input case is standard. Now, varying the axis NS, we generate all the singular extremals for the full system, assuming $\Omega = 0$. If the Earth rotation is not neglected, the poles are fixed.

5.4. **Reduction procedure.** In order to implement the multiple shooting algorithm, we design a quasi-optimal trajectory based on the following reasoning.

Sling effect. According to our numerical data, every trajectory starting from our initial conditions violates the constraint on the thermal flux if Ω is taken as 0. Hence the Coriolis force which dominates at the beginning the centripetal force has to be used in the first part of the trajectory to compensate the gravitation and to track the boundary arc. Moreover, the Coriolis component in the longitudinal motion is given by $F_c = 2\Omega \cos L \sin\chi$ and is maximized for $L = 0$ and $\chi = \pi/2$. Therefore, since $L(0) = 0$, we choose $\chi(0) \simeq \pi/2$.

Embedding procedure. If $\Omega = 0$, taking into account the structure of the system, we observe that if we relax the boundary conditions on χ, L, and l, then the adjoint vector is such that $p_\chi \equiv p_L \equiv p_l \equiv 0$, and the problem is reduced to an optimal control problem for an affine single input control system in dimension 3 describing the longitudinal motion. Hence the boundary arcs are small time optimal and the local optimal synthesis has been computed in Sec. 4 for fixed boundary conditions. Since at the end γ is free, a quasi-optimal trajectory is of the form

$$\gamma_- \gamma_+^T \gamma_{\text{flux}} \gamma_+^T \gamma_{\text{acc}} \gamma_+^T,$$

where $\gamma_\pm$ are arcs associated with $u_1 = \cos\mu = \pm 1$ and $u_2 = \sin\mu = 0$ and γ_{flux} and γ_{acc} are boundary arcs corresponding to the constraint on the thermal flux and on the normal acceleration, respectively, where the constraint on the dynamic pressure is not active. The terminal latitude is adjusted using a small variation of $\chi(0)$ near $\pi/2$.

This strategy is only an *approximation of the optimal policy* for two reasons. First of all, since Ω cannot be neglected in the first part of the trajectory, our policy is not extremal. But it can be verified numerically that an extremal policy is such that $|\cos\mu| \simeq 1$, $\sin\mu \simeq 0$, since $|p_\gamma| \gg |p_\chi|$, except for during short durations corresponding to switchings. Second, the transfer time should be supposed sufficiently small to ensure optimality. Otherwise we must estimate the conjugate and cut points (see [4] for details).

5.5. **Conclusion.** Having chosen such a policy, the exact switching times are computed using our multiple shooting algorithm, *implemented without using the extremal system.* This is realized in the next section.

5.6. **Numerical simulations and results.** Switching times and initial values of latitude, longitude, and azimuth should be determined by the multiple shooting method. More precisely,

- the first switching time, from γ_- to γ_+, allows one to adjust the entry in the iso-flux phase, which is characterized by $\varphi = \varphi^{\max}$, $\dot\varphi = 0$;
- the third switching time, from γ_{flux} to γ_+, is used to adjust the entry in the isonormal acceleration phase;
- the fifth switching time, from γ_{acc} to γ_+, permits to adjust the final velocity $v(t_f)$;
- the initial azimuth $\chi(0)$ is used to adjust the terminal latitude $L(t_f)$.

On the other hand, the final time is determined by the final altitude.

Results are drawn in Figs. 24–26.

References

1. A. Agrachev and R. Gamkrelidze, Symplectic methods for optimization and control. In: *Geometry of Feedback and Optimal Control, Dekker, New York* (1998), 19–77.
2. A. Agrachev and A. Sarychev, On abnormal extremals for Lagrange variational problems. *J. Math. Systems, Estimation, Control* **8** (1998), No. 1, 87–118.
3. O. Bolza, Calculus of Variations. *Chelsea Publishing Co. New York* (1973).

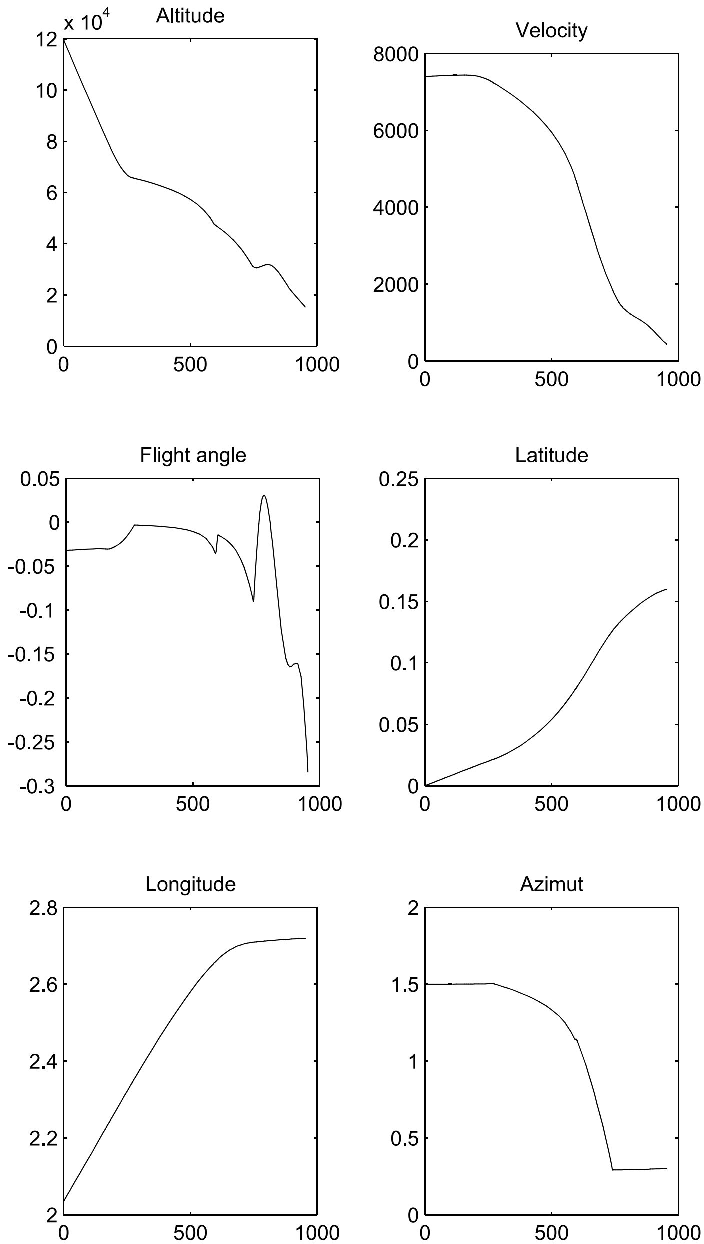


Fig. 24. State coordinates

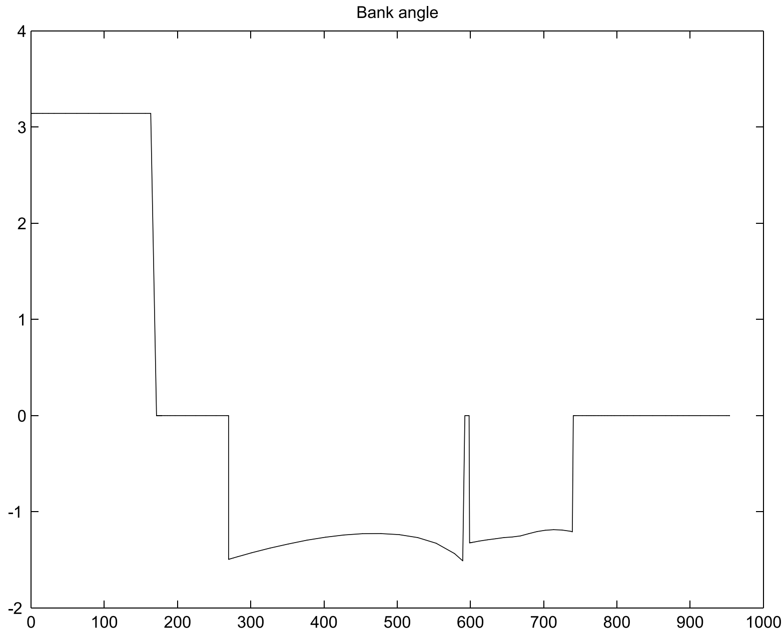


Fig. 25. Bank angle

4. B. Bonnard, L. Faubourg, G. Launay, and E. Trélat, Optimal control of the atmospheric arc of a space shuttle and numerical simulations by multiple shooting technique. (to appear in *J. Optim. Theory Appl.*).
5. B. Bonnard and I. Kupka, Théorie des singularités de l'application entrée/sortie et optimalité des trajectoires singulières dans le problème du temps minimal. *Forum Math.* **5** (1993), No. 2, 111–159.
6. B. Bonnard, G. Launay, and M. Pelletier, Classification générique de synthèses temps minimales avec cible de codimension un et applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), No. 1, 55–102.
7. B. Bonnard and E. Trélat, Une approche géométrique du contrôle optimal de l'arc atmosphérique de la navette spatiale. *ESAIM Control Optim. Calc. Var.* **7** (2002), 179–222 (electronic).
8. A. Bressan, The generic local time-optimal stabilizing controls in dimension 3. *SIAM J. Control Optim.* **24** (1986), No. 2, 177–190.
9. A. Bressan and B. Piccoli, A generic classification of time-optimal planar stabilizing feedbacks. *SIAM J. Control Optim.* **36** (1998), No. 1, 12–32.
10. A. E. Bryson (Jr.) and Yu Chi Ho, Applied optimal control. *Hemisphere Publishing Corp. Washington, D. C.* (1975). *Optimization, estimation, and control. Revised printing.*

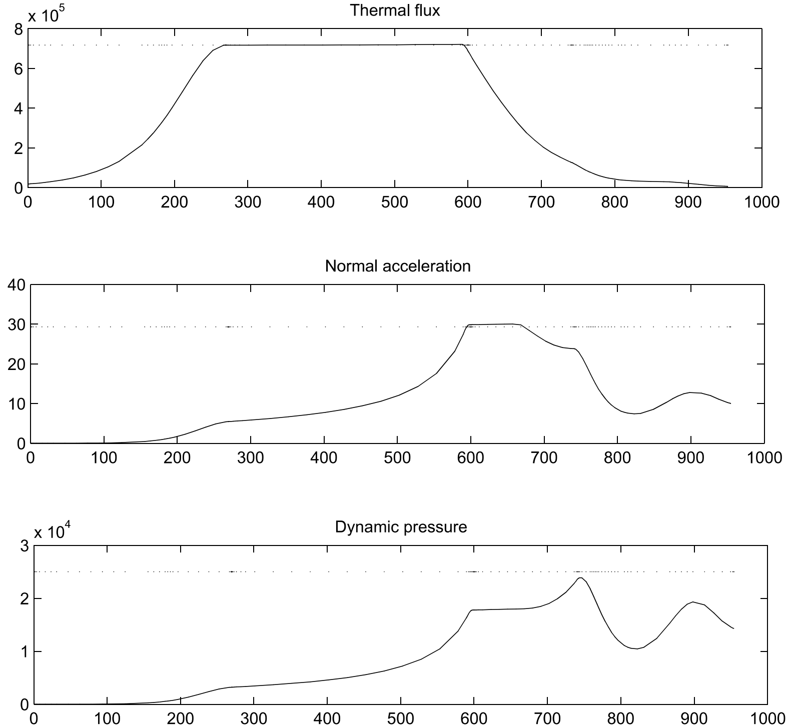


Fig. 26. State constraints

11. R. Bulirsch, Die Mehrzielmethode zur numerischen Lösung von nichtlinearen Randwertproblemen und Aufgaben der optimalen Steuerung. *Deutsches Zentrum für Luft- und Raumfahrt* (DLR), *Oberpfaffenhofen, Germany, Report of the Carl-Cranz-Gesellschaft* (1971).
12. Mécanique spatiale. *Cepadues Eds* (1995).
13. J. M. Coron and L. Praly, Guidage en rentrée atmosphérique. *Rapport 415 CNES*, Octobre 2000.
14. J. Harpold and C. Graves, Shuttle entry guidance. *J. Astaunautical Sciences* **27** (1979), 239–268.
15. A. D. Ioffe and V. M. Tikhomirov, Theory of extremal problems. *North-Holland Publishing Co., Amsterdam* (1979).
16. D. H. Jacobson, M. M. Lele, and J. L. Speyer, New necessary conditions of optimality for control problems with state-variable inequality constraints. *J. Math. Anal. Appl.* **35** (1971), 255–284.

17. P. Hiltmann, Numerische Lösung von Mehrpunkt-Randwertproblemen und Aufgaben der optimalen Steuerung mit Steuerfunktionen über endlichdimensionalen Räumen. *Dissertation, Fakultät für Mathematik und Informatik, Technische Universität München* (1990).
18. A. J. Krener and H. Schättler, The structure of small-time reachable sets in low dimensions. *SIAM J. Control Optim.* **27** (1989), No. 1, 120–147.
19. H. Maurer, On optimal control problems with bounded state variables and control appearing linearly. *SIAM J. Control Optim.* **15** (1977), No. 3, 345–362.
20. A. Miele, Recent advances in the optimization and guidance of aeroassociated orbital transfers. *Acta Astronautica* **38** (1996), No. 10, 747–768.
21. H. J. Pesch, A practical guide to the solution of real-life optimal control problems. *Control and Cybernet.* **23** (1994), Nos. 1/2.
22. B. Piccoli, Classification of generic singularities for the planar time-optimal synthesis. *SIAM J. Control Optim.* **34** (1996), No. 6, 1914–1946.
23. L. Pontriaguine, V. Boltianski, R. Gamkrelidze, and E. Michtchenko, Théorie mathématique des processus optimaux. *Éditions Mir, Moscow* (1974).
24. A. Sarychev, The index of the second variation of a control system. *Math. USSR Sb.* **41** (1982), No. 3, 338–401.
25. H. Schättler, The local structure of time optimal trajectories in dimension 3. *SIAM J. Control Optim.* **26** (1988), No. 4, 899–918.
26. J. Stoer and R. Bulirsch, Introduction to numerical analysis. *Springer-Verlag, Berlin* (1980).
27. H. J. Sussmann, The structure of time-optimal trajectories for single-input systems in the plane: the C^∞ nonsingular case. *SIAM J. Control Optim.* **25** (1987), No. 2, 433–465.
28. ———, Envelopes, high-order optimality conditions and Lie brackets. In: *Proc. 28th IEEE Conf. Decision and Control, Vols.* 1–3 (1989), 1107–1112.
29. E. Trélat, Some properties of the value function and its level sets for affine control systems with quadratic cost. *J. Dynam. Control Systems* **6** (2000), No. 4, 511–541.
30. E. Trélat, Asymptotics of accessibility sets along an abnormal trajectory. *ESAIM Control Optim. Calc. Var.* **6** (2001), 387–414.

(Received June 11, 2002, received in revised form October 16, 2002)

Authors' addresses:

B. Bonnard
U. de Bourgogne. Département de mathématiques
LAA0. BP 47870. 21078 Dijon (France)
E-mail: bbonnard@u-bourgogne.fr

L. Faubourg
U. de Bourgogne. Département de mathématiques
LAA0. BP 47870. 21078 Dijon (France)
E-mail: lfaubour@u-bourgogne.fr

G. Launay
U. de Bourgogne. Département de mathématiques
LAA0. BP 47870. 21078 Dijon (France)

E. Trélat
U. de Paris sud. Laboratoire ANEDP
Math UMR 8628. bat 425. 91405 Orsay (France)
E-mail: emmanuel.trelat@math.u-psud.fr