JOURNAL OF DYNAMICAL AND CONTROL SYSTEMS, Vol. 4, No. 1, 1998, 59-76

SUB-RIEMANNIAN GEOMETRY: ONE-PARAMETER DEFORMATION OF THE MARTINET FLAT CASE

B. BONNARD, M. CHYBA, E. TRELAT

ABSTRACT. Consider a sub-Riemannian geometry (U, D, g), where U is a neighborhood of 0 in $\mathbb{R}^3$, D is a Martinet type distribution identified to $\ker \omega$, ω being the one-form $dz - \frac{y^2}{2} dx$, and g is a metric on D which can be written as $a(q)dx^2 + 2b(q)dxdy + c(q)dy^2$, where $q = (x, y, z)$. In a previous article [1] we proved that g can be written in a normal form where $b \equiv 0$, $a = 1 + yF(q)$, $c = 1 + G(q)$, where $G\big|_{x=y=0} = 0$. Moreover we analyzed the flat case $a = c = 1$. In this article we study the following one-parameter deformation of the flat case: $a = 1$, $c = (1 + \varepsilon y)^2$ where $\varepsilon \in \mathbb{R}$. We parametrize the set of geodesics using elliptic functions. This allows us to compute the trace of the sphere and the wave front of small radius on the plane $y = 0$. We show that the sphere of small radius is not sub-analytic. This analysis clarifies the role of one of the functional invariants in the normal form.

1. INTRODUCTION

In this article, we consider the sub-Riemannian geometry (U, D, g), where U is an open set of $\mathbb{R}^3$ containing 0, D is the distribution $\ker \omega$, ω being the Martinet one-form $dz - \frac{y^2}{2} dx$, where $q = (x, y, z)$ are coordinates in $\mathbb{R}^3$ and g is the metric on D of the form $dx^2 + cdy^2$, where $c = (1 + \varepsilon y)^2$, $\varepsilon \in \mathbb{R}$. The flat case $\varepsilon = 0$ has been studied in [1], and we proved that the sphere and the distance function *are not sub-analytic*. The aim of this article is to prove that this property is *stable*. For this we consider a specific perturbation, but the techniques we use can be applied to many situations. Moreover the results we obtain prove indeed that the generic Martinet sphere is *more complicated* than the flat one.

1991 *Mathematics Subject Classification.* 49K15, 53B21, 58F10.

Key words and phrases. Sub-Riemannian geometry, abnormal geodesics, sub-Riemannian sphere and distance.

1079-2724/98/0100-0059$15.00/0

To introduce precisely the problem, we need some definitions. An *admissible curve* is an absolutely continuous curve $\gamma : [0,T] \mapsto U \subset \mathbb{R}^3$ such that $\dot\gamma(t) \in D(\gamma(t))\backslash\{0\}$ for almost every t. Let $(\cdot,\cdot)$ be the scalar product defined by g. The length and the energy of an admissible curve γ are respectively

$$L(\gamma) = \int_0^T (\dot\gamma(t),\dot\gamma(t))^{1/2}dt \ , \ E(\gamma) = \int_0^T (\dot\gamma(t),\dot\gamma(t))dt.$$

Let q_0, $q_1 \in U$; a *minimizing curve* joining q_0 to q_1 is an admissible curve $\gamma : [0,T] \mapsto U$ joining q_0 to q_1, of minimal length. To compute the minimizers, it is convenient to use the *maximum principle* [12]. According to this principle minimizers can be selected among a restricted set of curves as follows.

Let F_1, F_2 be two real analytic vector fields on U such that $D = \text{Span}\{F_1, F_2\}$, and introduce the *control system*

$$\frac{d\gamma(t)}{dt} = \sum_{i=1}^{2} u_i(t)\ F_i(\gamma(t)),$$

where $u(t) = \big(u_1(t),\ u_2(t)\big)$ is the *control* associated to γ. Let η be a constant equal to 0 or $\frac{1}{2}$ and consider the *Hamilton function*

$$H_\eta(q,p,u) = \langle p, F(q)u\rangle - \eta(F(q)u, F(q)u),$$

where $p = (p_x, p_y, p_z)$ denotes the adjoint vector, (p,η) is nonzero, $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^3$, and $F(q)u$ stands for $\sum_{i=1}^{2} u_i\ F_i(q)$. A *bi-extremal* is an absolutely continuous curve (γ, p) defined on $[0,T]$ such that $t \mapsto (\gamma(t), p(t), u(t))$ is a solution for almost all t of the following equations:

$$\frac{d\gamma}{dt} = \frac{\partial H_\eta}{\partial p}\ (\gamma,p,u), \quad \frac{dp}{dt} = -\frac{\partial H_\eta}{\partial \gamma}\ (\gamma,p,u), \tag{1.1}$$

$$\frac{\partial H_\eta}{\partial u}\ (\gamma,p,u) = 0. \tag{1.2}$$

Its projection γ on U is called a *geodesic*. From the maximum principle, a minimizer is a geodesic.

When we study bi-extremal curves, we must distinguish between two cases. If $\eta = 0$, the bi-extremal is called *abnormal* and if $\eta \neq 0$ it is called *normal* and the associated geodesics are respectively called normal and abnormal. A geodesic is called *strictly abnormal* if it is the projection of an abnormal bi-extremal, but not a normal one.

In the normal case Eq. (1.2) can be solved as follows. Since it is linear with respect to u, the control associated to (γ, p) can be computed as an analytic mapping $\hat{u} : (\gamma, p) \mapsto \mathbb{R}^2$. Then we plug $\hat{u}$ in H_η, $\eta = \frac{1}{2}$, and we set $H_n(\gamma, p) = H_{\frac{1}{2}}(\gamma, p, \hat{u})$. Using Eqs. (1.1) and (1.2), we observe that a normal bi-extremal (γ, p) is a solution of the following *analytic differential Hamiltonian equation*:

$$\frac{d\gamma}{dt} = \frac{\partial H_n}{\partial p}(\gamma, p), \quad \frac{dp}{dt} = -\frac{\partial H_n}{\partial \gamma}(\gamma, p). \tag{1.3}$$

This computation is straightforward if F_1, F_2 are taken orthonormal. Setting $P_i = \langle p, F_i(q) \rangle$ for $i = 1, 2$, the Hamilton function takes the form

$$\frac{1}{2}(P_1^2 + P_2^2).$$

Similarly, the computation of abnormal bi-extremals is straightforward. When $\eta = 0$, constraint (1.2) takes the form

$$\langle p, F_i(q) \rangle = 0, \quad i = 1, 2, \tag{1.4}$$

and if we differentiate twice this equation with respect to time we get that abnormal geodesics are contained in the set

$$\det\big(F_1, F_2, [F_1, F_2]\big) = 0,$$

which is the plane $y = 0$ corresponding to the points where ω is not a *contact form*. Moreover they are the lines $z = z_0$. The set $y = 0$, which has an important *geometric meaning*, is called the *Martinet plane*.

Assume now that the geodesics are parametrized by *arc length* $(\dot{\gamma}, \dot{\gamma}) = 1$. We fix the initial point q_0 to 0 and let $\gamma(t, p_0)$, $p(t, p_0)$ be the solution of Eq. (1.3) starting at $t = 0$ from $(q_0 = 0, p_0)$. It is contained in the level set $H_n = \frac{1}{2}$. The *exponential mapping* is the map

$$\exp_0 : (p_0, t) \mapsto \gamma(t, p_0).$$

The point q_1 is said to be *conjugate* to $q_0 = 0$ along γ if there exists (p_0, t_1), $t_1 > 0$, such that $\gamma(\cdot) = \exp_0(p_0, \cdot)$, $q_1 = \exp_0(p_0, t_1)$, and the exponential mapping is not an immersion at (p_0, t_1). The *conjugate locus* $C(0)$ is the set of *first* conjugate points along the curves γ when we consider all the normal geodesics starting from 0. Let γ be a geodesic corresponding to a normal or an abnormal bi-extremal and starting from 0. The first point along γ where γ ceases to be minimizing is called the *cut point* and the set of such points when γ varies forms the *cut locus* $L(0)$. The *sub-Riemannian sphere* with radius $r > 0$ is the set $S(0, r)$ of points which are at sub-Riemannian distance r from 0. The wave front of length r is the set $W(0, r)$ of end-points of geodesics with length r starting from 0.

The objective of this article is to investigate the trace on the Martinet plane $y = 0$ of the sphere $S(0, r)$ and of the wave front $W(0, r)$ with small radius $r > 0$, when g is of the form $dx^2 + c\,dy^2$, $c = (1 + \varepsilon y)^2$, $\varepsilon \in \mathbb{R}$. It is a generalization of the analysis in [1] concerning the flat case. It will give us a clear explanation of the role of one of the functional invariants in the normal form where $g = adx^2 + cdy^2$. A consequence of our study is that the distance function is not sub-analytic. *Also it will show precisely that we cannot expect to regularize the sub-Riemannian sphere when we perturb the flat case.*

2. Computations of the normal geodesics

In this section we parametrize the set of normal geodesics starting from 0 and contained in a sub-domain of $U : |\varepsilon y| < 1$.

§ 2.1. Notations. Let $G_1 = \dfrac{\partial}{\partial x} + \dfrac{y^2}{2}\dfrac{\partial}{\partial z}$ and $G_2 = \dfrac{\partial}{\partial y}$. We have $D = \operatorname{Ker}\,\omega = \operatorname{Span}\{G_1, G_2\}$ and the control system is

$$\frac{dq}{dt}(t) = \sum_{i=1}^{2} u_i(t) G_i(q(t)).$$

If $(\cdot,\cdot)$ designs the scalar product associated to g, we get $(G_1, G_1) = 1$, $(G_2, G_2) = c$, $(G_1, G_2) = 0$. We introduce the following orthonormal vector fields: $F_1 = G_1$, $F_2 = \dfrac{1}{\sqrt{c}}G_2$. Moreover let $F_3 = \dfrac{\partial}{\partial z}$. If $p = (p_x, p_y, p_z)$ is the adjoint vector, we set $P_i = \langle p, F_i(q)\rangle$ for $i = 1, 2, 3$ and let P be the vector in $\mathbb{R}^3$ whose components are P_i. The Hamiltonian corresponding to normal bi-extremals is

$$H_n(p, q) = \frac{1}{2}\left[\left(p_x + p_z\frac{y^2}{2}\right)^2 + \frac{p_y^2}{c}\right] = \frac{1}{2}(P_1^2 + P_2^2).$$

§ 2.2. Geodesics equations. In the symplectic coordinates (q, p) the normal bi-extremals are solutions of the following equations:

$$\begin{aligned}
\dot{x} &= p_x + p_z\,\frac{y^2}{2}, & \dot{p}_x &= 0,\\
\dot{y} &= \frac{p_y}{c}, & \dot{p}_y &= \frac{p_y^2 c'}{2c^2} - \left(p_x + p_z\frac{y^2}{2}\right) y p_z, \qquad (2.1)\\
\dot{z} &= \frac{y^2}{2}\left(p_x + p_z\,\frac{y^2}{2}\right), & \dot{p}_z &= 0,
\end{aligned}$$

where c' is the derivative of c with respect to y. They can be written in the coordinates (q, P):

$$\begin{aligned} \dot{x} &= P_1, & \dot{P}_1 &= \frac{yP_2P_3}{\sqrt{c}}, \\ \dot{y} &= \frac{P_2}{\sqrt{c}}, & \dot{P}_2 &= -\frac{yP_1P_3}{\sqrt{c}}, \\ \dot{z} &= \frac{y^2}{2}\,P_1, & \dot{P}_3 &= 0. \end{aligned} \tag{2.2}$$

§ 2.3. Notations and symmetries. To parametrize the set of normal geodesics starting from 0 at $t = 0$, we introduce the following parameters: $P_1(0) = p_x = \sin\varphi$, $P_2(0) = p_y(0) = \cos\varphi$, $P_3 = p_z = \lambda$. Observe that Eqs. (2.1) are left invariant by the following tranformation: $X = -x$, $Y = y$, $Z = -z$, $P_X = -p_x$, $P_Y = p_y$, $P_Z = -p_z$. Hence the set of normal geodesics and the metric are left invariant by the symmetry $S : (x, y, z) \mapsto (-x, y, -z)$. Therefore in our study, we can assume $\lambda \geq 0$. Another important symmetry is the following. We can add the trivial equation $\dot{\varepsilon} = 0$ to Eqs. (2.1) to form a differential equation (2.1)$'$. This equation and the metric are left invariant by the transformation $(x,\ y,\ z,\ p_x,\ p_y,\ p_z,\ \varepsilon) \mapsto (x,\ -y,\ z,\ p_x,\ -p_y,\ p_z,\ -\varepsilon)$. In the flat case $\varepsilon = 0$, this property implies that the sphere and the wave front are left invariant by the symmetry $S' : (x, y, z) \mapsto (x, -y, z)$.

§ 2.4. Geodesics corresponding to $\lambda = 0$. Let $\widetilde{g}$ be the Riemannian metric $dx^2 + c\,dy^2$ induced by g in the plane (x, y). The geodesics of this metric are the projections on the (x, y) plane of the normal geodesics corresponding to $\lambda = 0$. The Gauss curvature of $\widetilde{g}$ is zero. The geodesics are the following. If $p_z = 0$, we have

$$P_1(t) = \sin\varphi, \quad P_2(t) = \cos\varphi,$$

and

$$dx = \tan\varphi\sqrt{c}\,dy.$$

Using $\sqrt{c} = (1 + \varepsilon y)$, we get

$$x = \left(y + \varepsilon\,\frac{y^2}{2}\right)\tan\varphi$$

when $x(0) = y(0) = 0$.

They are straight lines in the local coordinates:

$$X = x, \quad Y = y + \frac{\varepsilon y^2}{2}.$$

Nevertheless, this transformation is not induced by a local diffeomorphism preserving the distribution $\ker\omega$ (such diffeomorphisms are characterized in [1]) and ε has an invariant meaning in our problem.

Those lines are minimizing curves for the Riemannian metric $\widetilde{g}$ and hence are minimizers for g. The abnormal geodesics $t \mapsto (\pm t, 0, 0)$ corresponding to $\varphi = \pm\frac{\pi}{2}$ project onto geodesics $t \mapsto (\pm t, 0)$ and are contained in $y = 0$. A geodesic corresponding to $\lambda = 0$, $\varphi \neq \pm\frac{\pi}{2}$ does not intersect the plane $y = 0$ for any positive time. Hence we get the following proposition.

§ 2.5. Proposition. *The projections of the geodesics corresponding to* $\lambda = 0$ *in the plane* (x, y) *are straight lines in the coordinates* $X = x$, $Y = y + \varepsilon\frac{y^2}{2}$. *Such geodesics are minimizers. Among those geodesics only the abnormal geodesics* $t \mapsto (\pm t, 0, 0)$ *intersect the trace of the sphere on the Martinet plane* $y = 0$.

§ 2.6. Geodesics corresponding to $\lambda > 0$. If we use the following reparametrization:

$$d\tau = \frac{dt}{\sqrt{c}},$$

Eqs. (2.2) are equivalent to

$$\begin{aligned}
\frac{dx}{d\tau} &= \sqrt{c}\,P_1, & \frac{dP_1}{d\tau} &= yP_2P_3,\\
\frac{dy}{d\tau} &= P_2, & \frac{dP_2}{d\tau} &= -yP_1P_3,\\
\frac{dz}{d\tau} &= \sqrt{c}\frac{y^2}{2}P_1, & \frac{dP_3}{d\tau} &= 0,
\end{aligned} \tag{2.3}$$

and the characteristic equation $H_n = \frac{1}{2}$ can be written

$$\left(\frac{dy}{d\tau}\right)^2 + P_1^2(y) = 1, \tag{2.4}$$

where $P_1 = p_x + p_z\frac{y^2}{2}$ and p_x, p_z are constants. This equation describes the evolution of a particle of mass 1 and energy 1, in a potential field whose potential is P_1^2.

If $y(0) = \dot{y}(0) = 0$, the corresponding trajectories are the abnormal geodesics $t \mapsto (\pm t, 0, 0)$. If $\dot{y}(0) \neq 0$, we set $\sigma = \operatorname{sign}\dot{y}(0)$. Using the symmetry $(y,\ p_y,\ \varepsilon) \mapsto (-y,\ -p_y,\ -\varepsilon)$, we can integrate (2.4) with the assumption $\sigma = +1$. The integration is similar to the *flat case*.

Let $0 < k,\ k' < 1$ be defined by

$$k^2 = \frac{1 - \sin\varphi}{2}, \quad k^2 + k'^2 = 1, \tag{2.5}$$

and setting $\eta = \dfrac{y\sqrt{\lambda}}{2k}$, Eq. (2.4) can be written

$$\frac{\dot{\eta}^2}{\lambda} = (1 - \eta^2)(k'^2 + k^2\eta^2), \tag{2.6}$$

which has to be integrated with the initial condition $\eta(0) = 0$ and using the branch $\dot{\eta}(0) > 0$. The η variable oscillates periodically between -1 and $+1$ and can be written using *elliptic functions* (see [7]) as

$$\eta(\tau) = -\mathrm{cn}\,(K(k) + \tau\sqrt{\lambda}, k),$$

where the period is $4K$, K being the *complete elliptic integral of the first kind,*

$$K(k) = \int_0^1 \frac{d\eta}{\sqrt{(1 - \eta^2)(k'^2 + k^2\eta^2)}}. \tag{2.7}$$

Hence we get

$$y(\tau) = -\frac{2k}{\sqrt{\lambda}}\mathrm{cn}\,(K + \tau\sqrt{\lambda}, k), \tag{2.8}$$

where t and τ are related by

$$dt = (1 + \varepsilon y)d\tau.$$

Using the following formula, [7], p. 40,

$$\int \mathrm{cn}\, u\, du = \frac{1}{k}\arcsin(k\,\mathrm{sn}\, u), \tag{2.9}$$

we get

$$t = \tau - \frac{2\varepsilon}{\lambda}\,[\arcsin(k\,\mathrm{sn}(K + \tau\sqrt{\lambda})) - \arcsin k].$$

Now integrating Eqs. (2.3), we can compute $x(\tau)$ and $z(\tau)$:

$$x(\tau) = \int_0^\tau \left[\left(p_x + p_z\frac{y^2}{2}\right) + \varepsilon y\left(p_x + p_z\frac{y^2}{2}\right)\right] d\tau.$$

Using the computation of the flat case in [1], we have

$$\int_0^\tau \left(p_x + p_z\frac{y^2}{2}\right)d\tau = -\tau + \frac{2}{\sqrt{\lambda}}\big[E(K + \tau\sqrt{\lambda}) - E(K)\big].$$

Moreover,

$$\varepsilon \int_0^\tau y\left(p_x + p_z \frac{y^2}{2}\right) d\tau = \varepsilon \left[\sin\varphi \int_0^\tau y\, d\tau + \frac{\lambda}{2} \int_0^\tau y^3 d\tau\right],$$

$$\varepsilon \sin\varphi \int_0^\tau y\, d\tau = -\frac{2\varepsilon \sin\varphi}{\lambda} \left[\arcsin(k\ \mathrm{sn}(K+\tau\sqrt{\lambda})) - \arcsin k\right],$$

and

$$\frac{\lambda}{2}\,\varepsilon \int_0^\tau y^3 d\tau = -\frac{4k^3\varepsilon}{\lambda} \int_K^{K+\tau\sqrt{\lambda}} \mathrm{cn}^3 u\, du.$$

Using the formula [7], p. 87,

$$\int \mathrm{cn}^3 u\, du = \frac{1}{2k^3}\left[(2k^2-1)\arcsin(k\ \mathrm{sn}u) + k\ \mathrm{sn}u\mathrm{dn}u\right] \tag{2.10}$$

and $\mathrm{sn}K = 1$, $\mathrm{dn}K = k'$, we get

$$\frac{\lambda\varepsilon}{2}\int_0^\tau y^3\, d\tau = -\frac{2\varepsilon}{\lambda}\Big[(2k^2-1)\big[\arcsin(k\,\mathrm{sn}(K+\tau\sqrt{\lambda})) - \\ - \arcsin k\big] + k\big[\mathrm{sn}(K+\tau\sqrt{\lambda})\mathrm{dn}(K+\tau\sqrt{\lambda}) - k'\big]\Big].$$

Using $2k^2 = 1 - \sin\varphi$, we obtain

$$x(\tau) = -\tau + \frac{2}{\sqrt{\lambda}}\big(E(u) - E(K)\big) - \frac{2\varepsilon k}{\lambda}(\mathrm{sn}u\,\mathrm{dn}u - k'),$$

where $u = K + \tau\sqrt{\lambda}$.

We have the following relations:

$$\begin{aligned} E(v+2K) &= E(v) + 2E(K), \\ \mathrm{sn}(v+2K) &= -\mathrm{sn}v, \\ \mathrm{sn}(v+4K) &= \mathrm{sn}v, \\ \mathrm{dn}(v+2K) &= \mathrm{dn}v. \end{aligned}$$

In particular, if $\tau\sqrt{\lambda} = 2K$, we have

$$x(\tau) = -\tau + \frac{4}{\sqrt{\lambda}}E(K) + \frac{4\varepsilon k k'}{\lambda},$$

and if $\tau\sqrt{\lambda}$ is a multiple of $4K$, the contribution of ε vanishes and $x(\tau)$ is given by the flat case.

Similarly, we have

$$z(\tau) = \int_0^\tau \left(p_x + p_z \frac{y^2}{2}\right)\frac{y^2}{2}d\tau + \varepsilon \int_0^\tau \left(p_x + p_z \frac{y^2}{2}\right)\frac{y^3}{2}\, d\tau.$$

The first integral has been computed in the flat case:

$$\int_0^\tau \left(p_x + p_z \frac{y^2}{2}\right) \frac{y^2}{2} d\tau =$$

$$= \frac{2}{3\lambda^{3/2}} \left[(2k^2 - 1)\big(E(u) - E(K)\big) + k'^2 \tau\sqrt{\lambda} + 2k^2 \mathrm{sn} u\, \mathrm{cn} u\, \mathrm{dn} u\right],$$

where $u = K + \tau\sqrt{\lambda}$.

To compute the contribution in ε, we decompose

$$\varepsilon \int_0^\tau \left(p_x + p_z \frac{y^2}{2}\right)\frac{y^3}{2} d\tau = \varepsilon \left[\frac{\sin}{2}\varphi \int_0^\tau y^3 d\tau + \frac{\lambda}{4}\int_0^\tau y^5 d\tau\right].$$

We have

$$\varepsilon \frac{\sin\varphi}{2} \int_0^\tau y^3 d\tau = -\frac{4\varepsilon \sin\varphi k^3}{\lambda^2} \int_K^{K+\tau\sqrt{\lambda}} \mathrm{cn}^3 u\, du,$$

which is computed using (2.10).

Moreover,

$$\frac{\varepsilon\lambda}{4} \int_0^\tau y^5 d\tau = -\frac{8\varepsilon k^5}{\lambda^2} \int_K^{K+\tau\sqrt{\lambda}} \mathrm{cn}^5 u\, du.$$

To compute this quantity, we use the following formula [7], p. 87:

$$4k^2 \int \mathrm{cn}^5 u\, du = \frac{3}{2k^3}(2k^2 - 1)\big[(2k^2 - 1)\arcsin(k\, \mathrm{sn} u) +$$

$$+ k\, \mathrm{sn} u\, \mathrm{dn} u\big] + \frac{2k'^2}{k} \arcsin(k\, \mathrm{sn} u) + \mathrm{cn}^2 u\, \mathrm{sn} u\, \mathrm{dn} u.$$

Hence after simplification we obtain the following relation:

$$z(\tau) =$$

$$= \frac{2}{3\lambda^{3/2}} \left[(2k^2 - 1)(E(u) - E(K)) + k'^2 \tau\sqrt{\lambda} + 2k^2\, \mathrm{sn} u\, \mathrm{cn} u\, \mathrm{dn} u\right] -$$

$$- \frac{\varepsilon}{\lambda^2}\big[\arcsin(k\, \mathrm{sn} u) - \arcsin k + (2k^2 - 1)(k\, \mathrm{sn} u\;\; \mathrm{dn} u - kk') +$$

$$+ 2k^3 \mathrm{cn}^2 u\, \mathrm{sn} u\, \mathrm{dn} u\big],$$

where $u = K + \tau\sqrt{\lambda}$.

When $\tau\sqrt{\lambda} = 2K$, we get

$$z(\tau) = \frac{4}{3\lambda^{3/2}}\left[(2k^2-1)E(K) + k'^2 K\right] + \frac{2\varepsilon}{\lambda^2}\left[(\arcsin k + kk'(2k^2-1)\right]$$

and if $\tau\sqrt{\lambda} = 4K$, it is independent of ε and given by the flat case.

§ 2.7. Proposition. *The normal geodesics starting from* 0*, parametrized by arc-length, corresponding to* $\lambda > 0$*,* $\dot{y}(0) > 0$*, and contained in* U *are given by the following formulas*:

$$x(t) = -\tau + \frac{2}{\sqrt{\lambda}}\big(E(u) - E(K)\big) - \frac{2\varepsilon k}{\lambda}(\operatorname{sn}u \operatorname{dn}u - k'),$$

$$y(t) = -\frac{2k}{\sqrt{\lambda}} \operatorname{cn}u,$$

$$\begin{aligned} z(t) = {} & \frac{2}{3\lambda^{3/2}}\left[(2k^2-1)\big(E(u)-E(K)\big) + k'^2\tau\sqrt{\lambda} + 2k^2 \operatorname{sn}u \operatorname{cn}u \operatorname{dn}u\right] - \\ & - \frac{\varepsilon}{\lambda^2}\big[\arcsin(k\operatorname{sn}u) - \arcsin k + \\ & + (2k^2-1)(k\operatorname{sn}u\operatorname{dn}u - kk') + 2k^3\operatorname{cn}^2u\operatorname{sn}u\operatorname{dn}u\big], \end{aligned}$$

$$t = \tau - \frac{2\varepsilon}{\lambda}\left[\arcsin(k\operatorname{sn}u) - \arcsin k\right],$$

where $u = K + \tau\sqrt{\lambda}$*,* $0 < k,\ k' < 1$*,* $2k^2 = \dfrac{1-\sin\varphi}{2}$*,* $k^2 + k'^2 = 1$*, and* $\varphi \in]-\pi/2,\ +\pi/2[$.

§ 2.8. Corollary. *The first two intersections of the geodesics parametrized in the previous proposition with the set* $y = 0$ *are given by*:

If $\tau\sqrt{\lambda} = 2K$,

$$\begin{aligned} x(t) &= -\tau + \frac{4}{\sqrt{\lambda}}E(K) + \frac{4\varepsilon kk'}{\lambda}, \\ z(t) &= \frac{4}{3\lambda^{3/2}}\left[(2k^2-1)E(K) + k'^2K\right] + \frac{2\varepsilon}{\lambda^2}\big(\arcsin k + kk'(2k^2-1)\big), \\ t &= \tau + \frac{4\varepsilon}{\lambda} \arcsin k. \end{aligned}$$

If $\tau\sqrt{\lambda} = 4K$*, it corresponds to the flat case*

$$x(t) = -t + \frac{8}{\sqrt{\lambda}}E(K),$$

$$z(t) = \frac{8}{3\lambda^{3/2}}\left[(2k^2-1)E(K) + k'^2K\right],$$

$$t = \tau.$$

§ 2.9. Remark. To get the parametrization of the geodesics corresponding to $\dot{y}(0) = p_y(0) = \cos\varphi < 0$, we proceed as follows. From Subsec. 2.3 Eq. (2.1) and the metric are left invariant by the transformation $(x, y, z, p_x, p_y, p_z, \varepsilon) \mapsto (x, -y, z, p_x, -p_y, p_z, -\varepsilon)$. Hence the geodesic corresponding to $p(0) = (p_x, p_y, p_z)(0)$, $p_y(0) < 0$ is obtained by integrating equation (2.1) for the parameter $-\varepsilon$ and with the initial condition $(p_x, -p_y, p_z)(0)$. Therefore in the formulas of Proposition 2.7 we must change ε into $-\varepsilon$ to get x, z and t, and y into $-y$.

3. Trace of the sphere and of the wave front of small radius on the Martinet plane

§ 3.1. Preliminaries. In this section we analyze the intersection of the sphere and the wave front with the plane $y = 0$. Our study is localized around 0 and we choose the radius r small enough such that:

(i) each geodesic of length r is contained in U and therefore parametrized in Proposition 2.7;

(ii) we can connect the point 0 using a minimizing curve to each point at sub-Riemannian distance r from 0.

Consider now a geodesic $e : [0, T] \mapsto U$ parametrized by arc length, starting from 0 and associated to a parameter (φ, λ) of the cylinder $S^1 \times \mathbb{R}$, where $\varphi \neq \pm\frac{\pi}{2}$, $\lambda > 0$. Let $0 < t_1(\varphi, \lambda) < \cdots < t_N(\varphi, \lambda) \leq T$ be the successive times corresponding to the intersection of e with the Martinet plane $y = 0$.

If $e(t) = \big(x(t), y(t), z(t)\big)$ and $\sigma =$ sign $\dot{y}(0)$, we set

$$\Gamma_n^\sigma(\varepsilon) = \bigcup_{\substack{\varphi \neq \pm\frac{\pi}{2} \\ \lambda > 0}} \big(x(t_n(\varphi, \lambda)),\ z(t_n(\varphi, \lambda))\big).$$

If the length is fixed to r, they are denoted by $\Gamma_n^\sigma(\varepsilon, r)$. From the analysis of Sec. 2, this set has the following properties:

(i) $\Gamma_n^+(0, r) = \Gamma_n^-(0, r)$;

(ii) $\Gamma_{2n+1}^-(\varepsilon, r) = \Gamma_{2n+1}^+(-\varepsilon, r)$;

(iii) $\Gamma_{2n}^+(\varepsilon, r) = \Gamma_{2n}^-(\varepsilon, r)$.

We are going to analyze these sets near the point $(-r, 0)$.

§ 3.2. Description of $\Gamma_1^+(0, r)(k \to 1)$. In the flat case $\varepsilon = 0$, we have the following formulas (see Corollary 2.8) to parametrize $\Gamma_1^+(0, r)$:

$$\begin{aligned} \frac{x + r}{2r} &= \frac{E}{K}, \\ 6z &= r^3\left[\left(\frac{1 - 2k'^2}{E^2}\right)\left(\frac{E}{K}\right)^3 + \frac{k'^2}{K^2}\right], \end{aligned} \tag{3.1}$$

where E is the complete integral of the second kind $E(K)$. We have

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{-\frac{1}{2}} d\theta, \quad E = \int_0^{\pi/2} (1 - k^2\sin^2\theta)^{\frac{1}{2}} d\theta,$$

where $0 < k < 1$, and when $k \to 1 - 0$, $K \to \infty$, $E \to 1$ and hence $x \to -r$, $z \to 0$.

Moreover using (3.1), we can write

$$6z - r^3\left(\frac{x+r}{2r}\right)^3 = r^3\left[\left(\frac{1-2k'^2}{E^2} - 1\right)\left(\frac{E}{K}\right)^3 + \frac{k'^2}{K^2}\right]. \tag{3.2}$$

To evaluate the right-hand side of (3.2) when $k' \to 0$, we must know the asymptotic expansions of E and K. They can be obtained using the following properties, see [2] and [7]. Both E and K are solutions of a *hypergeometric equation*. When $k' \to 0$, they admit the following representation:

$$K = u_1(k'^2) \ln\left(\frac{4}{k'}\right) + u_2(k'^2),$$
$$E = u_3(k'^2) \ln(\frac{4}{k'}) + u_4(k'^2),$$

where the u_i's are analytic functions whose coefficients can be computed by recurrence; see [2] for the complete expansion. The scale of the asymptotic expansion is $k'^\alpha\left(\ln\left(\frac{1}{k'}\right)\right)^\beta$, and the first terms are given by

$$\begin{aligned} K &= \ln\left(\frac{1}{k'}\right) + \ln 4 + \frac{1}{4}k'^2 \ln\left(\frac{1}{k'}\right) + \frac{1}{4}(\ln 4 - 1)k'^2 + o(k'^3), \\ E &= 1 + \frac{1}{2}k'^2 \ln\left(\frac{1}{k'}\right) + \frac{k'^2}{2}\left(\ln 4 - \frac{1}{2}\right) + o(k'^3). \end{aligned} \tag{3.3}$$

Hence we get

$$6z - r^3\left(\frac{x+r}{2r}\right)^3 = -\frac{3}{2}r^3\frac{k'^2}{\ln^3(\frac{1}{k'})} + o\left(\frac{k'^2}{\ln^3(\frac{1}{k'})}\right). \tag{3.4}$$

We introduce the variable $X = \dfrac{x+r}{2r}$ and use the relation $X = \dfrac{E}{K}$ to expand k' and $\dfrac{1}{\ln(\frac{1}{k'})}$ in the scale $X^\alpha\left(e^{-1/X}\right)^\beta$. In the flat case this can be easily done because $\dfrac{E}{K}$ is a homographic function in the variable $\ln\left(\dfrac{4}{k'}\right)$.

We prefer to use a general method which can be applied to the nonflat case. Indeed we can write

$$X = \frac{E}{K} = \frac{1}{\ln(\frac{1}{k'})}\left(1 - \frac{\ln 4}{\ln(\frac{1}{k'})} + O\left(\frac{1}{\ln(\frac{1}{k'})}\right)^2\right);$$

therefore,

$$\frac{1}{X} = \ln\left(\frac{1}{k'}\right)\left(1 + \frac{\ln 4}{\ln(\frac{1}{k'})} + O\left(\frac{1}{\ln(\frac{1}{k'})}\right)^2\right).$$

Hence,

$$\frac{1}{X} = \ln\left(\frac{1}{k'}\right) + \ln 4 + O\left(\frac{1}{\ln(\frac{1}{k'})}\right),$$

and we get

$$k' = 4e^{-1/X}\big(1 + o(1)\big) \tag{3.5}$$

and

$$\frac{1}{\ln(\frac{1}{k'})} = X\big(1 + o(1)\big). \tag{3.6}$$

Plugging the previous estimates in (3.4), we obtain

$$z = r^3X^3\left(\frac{1}{6} - 4e^{-2/X}\right) + o\big(X^3e^{-2/X}\big). \tag{3.7}$$

§ 3.3. Description of $\Gamma_2^+(0,r)(k \to 1)$. We proceed similarly. We have

$$\frac{x+r}{2r} = \frac{E}{K},$$
$$z = \frac{r^3}{24K^3}\big[(2k^2-1)E + k'^2K\big].$$

Hence if $\dfrac{x+r}{2r} = X$, we get

$$z = r^3X^3\left(\frac{1}{24} - e^{-2/X}\right) + o\big(X^3e^{-2/X}\big). \tag{3.8}$$

§ 3.4. Description of $\Gamma_1^+(\varepsilon,r)(k \to 1)$. We generalize our computation to the nonflat case. We have

$$x + r = \frac{4}{\sqrt{\lambda}}E + \frac{4\varepsilon}{\lambda}\big(kk' + \arcsin k\big)$$

and $\tau\sqrt{\lambda} = 2K$. Hence we get

$$\frac{x+r}{2r} = \frac{\tau}{rK}E + \left(\frac{\tau}{rK}\right)^2\frac{r\varepsilon}{2}\big(kk' + \arcsin k\big). \tag{3.9}$$

We set

$$X = \frac{x+r}{2r}, \quad Y = \frac{\tau}{rK}$$

and let

$$f(k') = \frac{r\varepsilon}{2}\,(kk' + \arcsin k).$$

Since $k = \sqrt{1-k'^2}$, f is an *analytic function of* k' and moreover

$$f(k') = r\varepsilon\left(\frac{\pi}{4} + O(k'^3)\right).$$

We have

$$z = \frac{4}{3\lambda^{3/2}}\left[(2k^2-1)E + k'^2 K\right] + \frac{2\varepsilon}{\lambda^2}\left(\arcsin k + kk'(2k^2-1)\right),$$

hence we get with $\tau\sqrt{\lambda} = 2K$:

$$z = \frac{\tau^3}{6K^3}\left[(2k^2-1)E + k'^2 K\right] + \frac{\tau^4\varepsilon}{8K^4}\left(\arcsin k + kk'(2k^2-1)\right). \quad (3.10)$$

Therefore,

$$z = \frac{r^3Y^3}{6}\left[(2k^2-1)E + k'^2 K\right] + \frac{r^4Y^4\varepsilon}{8}\left(\arcsin k + kk'(2k^2-1)\right). \quad (3.11)$$

From (3.9), X and Y are related by the following equation:

$$Y^2 f + YE - X = 0. \quad (3.12)$$

Solving these equations as in the flat case, we obtain after lengthy computations the relation

$$z = r^3 g(X,\varepsilon),$$

where g is an analytic function of the parameter $r\varepsilon$ and for $\varepsilon = 0$, g is given by the right-hand side of (3.7).

Moreover, g can be decomposed into

$$g(X,\varepsilon) = v_1(X,\varepsilon) + v_2(X,\varepsilon)e^{-1/X} + v_3(X,\varepsilon)e^{-2/X} + o(X^3e^{-2/X}), \quad (3.13)$$

where $v_1(X,\varepsilon), v_2(X,\varepsilon)$, and $v_3(X,\varepsilon)$ are *analytic* functions. Precise computations show the following:

$$v_1(X,\varepsilon) = \frac{1}{6}\left(X^3 - \frac{3}{4}\varepsilon r X^4 + o(X^4)\right), \quad (3.14)$$

$$v_2(X,\varepsilon) = \varepsilon r\ o(X^5), \quad (3.15)$$

$$v_3(X,\varepsilon) = -4e^{\frac{\pi\varepsilon r}{2}}(1+\pi\varepsilon r)X^3. \quad (3.16)$$

From this analysis we deduce the following properties.

§ 3.5. **Proposition.** *Near the point* $(-r, 0)$, *the set* $\Gamma_1^+(\varepsilon, r)$ *can be represented as the graph of the function*

$$z = \frac{r^3}{6}\,(X^3 - \frac{3}{4}\varepsilon r X^4) + o(X^4)$$

where $X = \dfrac{x+r}{2r}$. *Moreover,* $\Gamma_1^-(\varepsilon, r) = \Gamma_1^+(-\varepsilon, r)$.

§ 3.6. **Proposition.** *If* r *is small enough, the set* $\Gamma_1^+(\varepsilon, r)$ *is not semi-analytic.*

Proof. Indeed we have

$$z = r^3\big(v_1(X,\varepsilon) + v_2(X,\varepsilon)e^{-1/X} + e^{-2/X}X^3\alpha\big) + o\big(X^3 e^{-2/X}\big),$$

where $\alpha = -4e^{\pi\frac{\varepsilon}{2}r}\,(1 + \pi\varepsilon r)$ and v_1, v_2 are analytic functions. For r small enough, α is nonzero.

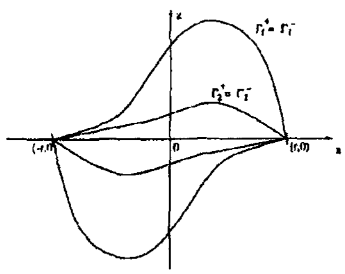
§ 3.7. **Stability.** We have proved a weak and a strong stability result.

Weak stability result. Near $(-r, 0)$, $\Gamma_1^+(\varepsilon, r)$ is the graph of a function $z = h(X)$, where h is the sum of an analytic function $h_1(X)$ and a flat function. The first nonzero term in the power series of $h_1(X)$ is $\dfrac{r^3X^3}{6}$ and is given by the flat model.

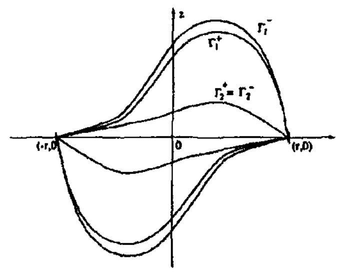
Strong stability result. We can expand the graph in the scale $X^\alpha\left(e^{-1/X}\right)^\beta$ and the coefficients of this expansion are analytic functions of the parameter ε.

§ 3.8. **Trace of the sphere and of the wave front of small radius on** $y = 0$. Let $r > 0$ be small enough. The wave front $W(0, r)$ intersected with the set $y = 0$ is the union of the sets $\Gamma_n^\sigma(\varepsilon, r)$, $n \geq 1$, with the two points $(\pm r, 0)$, corresponding to the abnormal geodesics, and all the points deduced using the symmetry $S : (x, z) \mapsto (-x, -z)$. It can be drawn with the help of Proposition 2.7 where the geodesics are parametrized by elliptic functions. A magnifying glass on the singularity $(-r, 0)$ is given by the previous asymptotic expansions. Inspection of Fig. 1 shows that the sphere $S(0, r)$ intersected with $y = 0$ is the closed curve enclosing the domain $\bigcup_{r_1 \leq r} W(0, r_1) \bigcap \{y = 0\}$ (each point of the interior of this domain is the end point of a geodesic of length $< r$). When $\varepsilon \neq 0$, both curves $\Gamma_1^\pm(\varepsilon, r)$ are distinct and the sphere $S(0, r)$ is the union of $\pm\Gamma_1^{\sigma'}(\varepsilon, r)$ with $(\pm r, 0)$, where $\sigma' = -\operatorname{sign}(\varepsilon)$. When $\varepsilon = 0$, both curves are the same. We represent below the wave front in the flat and nonflat case.

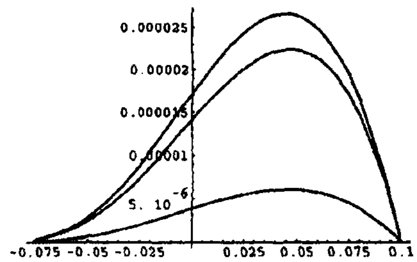
In the flat case each point P distinct from $(\pm r, 0)$ of the trace of the wave front on $y = 0$ is the endpoint of two distinct geodesics, one corresponding to $\sigma = +1$ and another corresponding to $\sigma = -1$. In particular the cut



$\varepsilon = 0$



$\varepsilon > 0$



Numerical simulations, $z > 0 : \varepsilon > 0$

Fig. 1

locus is $\pm\Gamma_1^+$. In the nonflat case $\varepsilon \neq 0$, this is no longer true. The plane $y = 0$ contains no cut point. Nevertheless each point of Γ_{2n}^+ is the end point of two distinct geodesics. It is easy to see that this property is a geometric consequence of the integrability property of the geodesic flow and can be generalized to many situations.

The wave front is ramified at the two points $(\pm r, 0)$ which are the end-points of the abnormal geodesics of length r. For the branches in the domain $z \geq 0$ the singularity is in the analytic category at $(r, 0)$ and not semi-analytic at $(-r, 0)$.

Formula (3.14) gives us the contact of $\Gamma_1^\pm$ with the line $z = 0$ and the respective position of both curves. Relation (3.8) gives us the contact of $\Gamma_2^\pm$ with the line $z = 0$.

We end this section by the following result which is a direct consequence of our analysis.

§ 3.9. Theorem. *For each* $\varepsilon \in \mathbb{R}$, the sphere of small radius is not sub-analytic.

Proof. Using Proposition 3.6, the trace of the sphere on the plane $y = 0$ is not semi-analytic. In the plane semi- and sub-analytic sets are the same. Hence the sphere is not sub-analytic.

§ 3.10. Remark: Pfaffian sets. We can write near $(-r, 0)$ the graph z as the projection in the (X, z) plane of the set in $\mathbb{R}^3$ defined by the following relations:

$$z = \sum_{i=1}^{\infty} v_i(X) w^i, \quad 0 \leq X \leq M,$$
$$w = e^{-1/X},$$

where z is an *analytic function* of the two variables X and w is a solution of the *Pfaffian equation* $X^2 dw - w dX = 0$.

Hence the graph of z near $(-r, 0)$ is a *sub-Pfaffian set.* More precisely it belongs to the class defined in [4], [9].

4. Conclusion: General case

The main motivation of this article is to show that we cannot expect to get in the general Martinet case a more regular sphere than in the flat case. In particular the persistence of the non-sub-analyticity is somehow obvious. In our analysis it is clearly identified as the following *nonproperness property*: when $k \to 1$, there exist geodesics whose y coordinate oscillates with a period $4K$ tending to infinity. Indeed y behaves like the angular velocity of an *oscillating pendulum* near a separatrix. This is also clear in the framework of the *analytic geometry.* When computing the representation of the sphere intersected with $y = 0$ as a graph $z = h(x)$, both x, z admit singularities which cannot be compensated as in the Puiseux situation. All this analysis can be generalized to many situations, and a more subtle difficulty is to control the complexity of the asymptotic expansions.

If the non-sub-analyticity is persistent it is also clear that the generic sphere is more complex. In particular we can expect to construct a one-parameter deformation of the flat case in which the coefficients of the flat part of the graph of z are *not even continuous* with respect to the parameter; see the discussion in Subsec. 3.7. If we deal with the class of metrics $g = a(y)dx^2 + c(y)dy^2$ where the set of normal geodesics is integrable, it can be proved that the graph belongs to the extension of the sub-analytic set defined in [4], [9].

Another interesting tool used in this article is elliptic functions. They are manipulated like trigonometric functions in the contact case, to represent the trace of the wave front on the Martinet plane. Also they are adequate to compute the conjugate and cut loci, and the sphere. This computation is beyond the scope of this article.

This work is only an introduction to the generic Martinet case which shall be analyzed in a forthcoming article. Unfortunatly the *complexity of the computations* becomes extreme.

In particular we must combine mathematical and numerical methods, using the package of elliptic functions in Mathematica.

References

1. A. Agrachev et al., Sub-Riemannian sphere in Martinet flat case. (to appear in *ESAIM/COCV*).

2. H. Davis, Introduction to the nonlinear differential and integral equations. *Dover, New-York*, 1962.

3. J. Dieudonné, Calcul infinitésimal. *Hermann, Paris*, 1980.

4. L. V. D Dries, A. Macintyre, and D. Marker, The elementary theory of restricted analytic fields with exponentiation. *Ann. Math.* **140** (1994), 183–205.

5. J. Ecalle, Introduction aux fonctions analysables. *Hermann, Paris*, 1992.

6. A. G. Khovanskii, Fewnomials. *Trans. Math. Mon.* **88** (1991).

7. D. F. Lawden, Elliptic functions and applications. *Springer Verlag, ew-York*, 1989.

8. E. B. Lee and L. Markus, Foundations of optimal control theory. *J. Wiley and Sons, New-York*, 1967.

9. J. M. Lion and J. P. Rolin, Théorèmes de préparation pour les fonctions logarithmo-exponentielles. *Ann. Inst. Fourier* **47** (1997), No. 3, 859–884.

10. J. Martinet, Sur les singularités des formes différentielles. *Ann. Inst. Fourier* **20** (1970), No. 1, 95–195.

11. R. Narasimhan, Introduction to the theory of analytic spaces. *Lect. Notes Math.* **25** (1966).

12. L. Pontriaguine et al., Théorie mathématique des processus optimaux. *Mir, Moscow*, 1974.

13. E. T. Whittaker and G. N. Watson, A course of modern analysis. *Cambridge Univ. Press, New-York*, 1927.

(Received 27.06.1997)

Authors' addresses:

B. Bonnard, E. Trelat
Université de Bourgogne
Laboratoire de Topologie-CNRS UMR 5584
9, avenue Alain Savary, BP 400
21004 Dijon Cedex France
E-mail: bbonnard@satie.u-bourgogne.fr

M. Chyba
Université de Genève
Section de Mathématiques
2–4 rue du Lièvre, Case 240
1211 Genève 24, Suisse