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A variational method using fractional order Hilbert spaces for tomographic reconstruction of blurred and noised binary images

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Abstract

We provide in this article a refined functional analysis of the Radon operator restricted to axisymmetric functions, and show that it enjoys strong regularity properties in fractional order Hilbert spaces. This study is motivated by a problem of tomographic reconstruction of binary axially symmetric objects, for which we have available one single blurred and noised snapshot. We propose a variational approach to handle this problem, consisting in solving a minimization problem settled in adapted fractional order Hilbert spaces. We show the existence of solutions, and then derive first order necessary conditions for optimality in the form of optimality systems.

Keywords: Radon operator; Fractional order Hilbert spaces; Minimization

1. Introduction

Our study is motivated by a physical experiment led at the CEA[1] that consists in reconstructing a three-dimensional binary axially symmetric object from a single X-ray radiography which

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is moreover blurred and noised. The behavior of some heavy material is studied during an implosion process, and a single radiography is performed during the implosion. At some specific moment, a very brief flash of X-rays is fired from a punctual source through the object and arrives at a detector. Since the object is very dense, X-rays must be of high energy, and many drawbacks appear in practice, causing a high level of blur and noise on the radiograph.

We stress on the fact that we have available only one radiography and thus, in turn, classic methods of tomographic reconstruction used in medicine, optics, geophysics, etc, which are requiring the knowledge of many projections of the object (taken from different angles), do not apply to our context. Furthermore, the objects under consideration are composed of one homogeneous medium, and of some holes. In the mathematical modeling of the problem, this feature turns into a binary constraint which is difficult to handle, and only few results exist in that direction.

It is assumed that, during the implosion, the shape of the object remains axially symmetric, so that, in theory, a single snapshot is enough to reconstruct the whole object. Moreover, since the source is quite far from the object, it is assumed that X-rays are parallel and orthogonal to the symmetry axis of the object. It follows that the Radon transform has a nice expression, derived hereafter. Recall that the aim of radiography is to measure the attenuation of X-rays through the object. Every point of the radiograph, determined by cartesian coordinates (y, z), corresponds to a measure of this attenuation, and the Radon transform of the object is defined by the projection operator

$$(H_0\bar{u})(y, z) = \int_{\mathbb{R}} \bar{u}(x, y, z)\, dx, \tag{1}$$

where the function $\bar{u}$ (with compact support) denotes the density of the object, and x is a coordinate along the rays. Since the objects under consideration are bounded and axially symmetric, we make use of cylindrical coordinates (r, θ, z), where the z-axis corresponds to the symmetry axis. Then, setting $\bar{u}(x, y, z) = u(\sqrt{x^2 + y^2}, z)$ and $H_0 u = H_0\bar{u}$, we arrive at

$$(H_0 u)(y, z) = 2 \int_{|y|}^{+\infty} u(r, z) \frac{r}{\sqrt{r^2 - y^2}}\, dr, \tag{2}$$

for all $y, z \in \mathbb{R}$. In the sequel we adopt the following notations and conventions. We assume that the set of density functions is the set of bounded variation functions on $\mathbb{R}^+ \times \mathbb{R}$, having a compact support contained in the subset $\Omega = [0, a) \times (-a, a)$ of $\mathbb{R}^2$, where $a > 0$ is fixed, and taking their values in the binary set $\{0, 1\}$. In particular, the upper bound of the integral in (2) can be set to a. Notice that, for every density function u, the function $H_0 u$ is of compact support contained in $\Omega_1 = (-a, a)^2$.

It has been shown in [1] that H_0 extends to a linear continuous operator from $L^2(\Omega)$ to $L^2(\Omega_1)$. However, inverting the operator H_0 requires more differentiability, and it turns out that H_0^{-1} cannot be extended to a continuous operator from any space $L^p(\Omega_1)$ to any space $L^q(\Omega)$.[2] This property illustrates the fact that the problem is ill-posed, and the operator is bad-conditioned.

[2] It can however be extended to a continuous linear operator from the Sobolev space $W^{1,2}(\Omega_1)$ to $L^2(\Omega)$.

Hence, applying the inverse operator to the radiography causes significant errors and leads to a bad reconstruction of the object.

Moreover, as mentioned formerly, due to many drawbacks in the physical experiment, the resulting radiography may be strongly blurred and noised, and actually what we observe on the radiography is

$$v_d = BH_0u + \tau,$$

that is, the projection of the density of the object, which is moreover blurred and noised. Here, B is a linear operator representing the effect of the blur. Usually, it is assumed in practice that B is the convolution with a positive symmetric kernel K with compact support and such that $\int K\,d\mu = 1$, and that τ is an additive Gaussian white noise of zero mean. In the sequel, we set $H = BH_0$.

To deal with this ill-posed problem, we have proposed in [1] a regularization process based on a variational approach. More specifically, let $BV(\Omega)$ denote the space of bounded variation functions, defined as the space of functions $u \in L^1(\Omega)$ whose distributional gradient Du is a finite vector Radon measure, satisfying

$$\int_\Omega u \operatorname{div}\varphi\, dx = -\langle Du, \varphi\rangle = -\int_\Omega \varphi \cdot d(Du) = -\int_\Omega \varphi \cdot \sigma_u\, d|Du|,$$

for every $\varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^2)$, where $\mathcal{C}_c^1(\Omega, \mathbb{R}^2)$ denotes the space of continuously differentiable vector functions of compact support contained in Ω, and where $\sigma_u : \Omega \to \mathbb{R}^2$ is a $|Du|$-measurable function satisfying $|\sigma_u| = 1$ almost everywhere on Ω. The total variation of $u \in BV(\Omega)$ is then defined as the total variation of the Radon measure Du, that is, by

$$\Phi(u) = \sup\left\{ \int_\Omega u(x) \operatorname{div}\varphi(x)\, dx \;\middle|\; \varphi \in \mathcal{C}_c^1\big(\Omega, \mathbb{R}^2\big),\ \|\varphi\|_{L^\infty} \leqslant 1 \right\} = \int_\Omega |Du| = |Du|(\Omega).$$

Endowed with the norm $\|u\|_{BV} = \|u\|_{L^1} + \Phi(u)$, the space $BV(\Omega)$ is a Banach space.

Since $\Omega = [0, a) \times (-a, a)$ is bounded and $\partial\Omega$ is Lipschitz, functions of $BV(\Omega)$ have a trace of class L^1 on the subset

$$\Gamma = \{a\} \times (-a, a) \cup [0, a) \times \{-a\} \cup [0, a) \times \{a\} \tag{3}$$

of $\partial\Omega$, and the trace mapping $T : BV(\Omega) \to L^1(\Gamma)$ is linear and bounded (see [12]). The space $BV_0(\Omega)$ is then defined as the kernel of T. It is the space of bounded variation functions on Ω vanishing on Γ, and since T is bounded, it is a Banach space, endowed with the induced norm.

Let v_d be the projected image (observed data), and let $\alpha > 0$. Assume that $v_d \in L^2(\Omega_1)$. Since $H = BH_0$ is a linear continuous operator from $L^2(\Omega)$ to $L^2(\Omega_1)$, we have considered in [1] the problem of minimizing the functional

$$u \longmapsto \frac{1}{2}\|Hu - v_d\|^2_{L^2(\Omega_1)} + \alpha\Phi(u)$$

over all functions $u \in BV(\Omega)$ satisfying $u(x) \in \{0, 1\}$ almost everywhere on Ω. Solutions of that minimization problem can then be proposed as a tomographic reconstruction in our problem.

Using a penalization procedure to tackle the nonconvex constraint, we have proposed some numerical methods that however do not provide very satisfactory results, due to the fact that we do not take into account the deep regularity properties of the projection operator.

The Radon transform and its regularity properties have been investigated in a large number of works (see e.g. [5,4,6,10,13–19,22–24] and the references therein), where range characterizations of the Radon transform and their potential applications to tomography are described. Regularity properties are in general derived in the spaces L^p; however, as mentioned above, in our tomography problem the use of Lebesgue spaces does not lead to satisfactory practical results, which incites to derive stronger regularity features, taking into account the specific expression of the Radon transform, so as to propose a minimization problem settled with a more adapted norm.

In the present article, we provide a refined functional analysis of the Radon projection operator H_0 defined by (2), and show that it enjoys strong regularity properties in fractional order Hilbert spaces (Section 2). In turn, we propose in Section 3 a modified minimization problem settled in adapted fractional order Hilbert spaces. We show the existence of solutions, and, using a penalization procedure to deal with the nonconvex binarity constraint, we derive first order necessary conditions for optimality in the form of optimality systems. Since many properties of fractional order Hilbert spaces are used throughout the article, and that not all of them are so standard, we provide Appendix A, gathering different equivalent definitions and characterizations of those spaces, defined on $\mathbb{R}^n$ or on some bounded subset, in particular in terms of Fourier transform and fractional Laplacian. The development of algorithms based on the theoretical results of this article will be the subject of investigation of a next work.

2. Functional analysis of the projection operator

2.1. Preliminaries

Recall that the densities of the objects under consideration are represented by bounded variation functions defined on the set $\Omega = [0, a) \times (-a, a)$, having a compact support contained in Ω, and taking their values in $\{0, 1\}$.

For every function $u \in BV(\Omega)$, the projection operator is defined by

$$(H_0u)(y, z) = 2 \int_{|y|}^{a} u(r, z) \frac{r}{\sqrt{r^2 - y^2}} \, dr,$$

for $|y| < a$ and $|z| < a$. Note that $(H_0u)(y, z) = (H_0u)(-y, z)$, for almost all $y, z \in \mathbb{R}$. Notice that, for every $u \in BV(\Omega)$ having a compact support contained in Ω, extending u by 0 outside Ω, the function H_0u has a compact support as well, contained in $\Omega_1 = (-a, a)^2$. In this section we investigate the regularity of H_0u.

First of all, observe that, for y fixed, the function $z \mapsto (H_0u)(y, z)$ is a bounded variation function on $(-a, a)$, and a stronger regularity property cannot be expected for such functions u. However, since the function $(y, z) \mapsto H_0(y, z)$ is a kind of convolution of the function u with respect to the variable y, more regularity is expected with respect to this variable.

Before stating the main result, we first recall a definition of fractional order Hilbert spaces.

Let U be an open subset of $\mathbb{R}^n$. For $k \in \mathbb{N}$, the Hilbert space $H^k(U)$ is defined as the space of all functions of $L^2(U)$, whose partial derivatives up to order k, in the sense of distributions, can be identified with functions of $L^2(U)$. Endowed with the norm

$$\|f\|_{H^k(U)} = \left(\sum_{|\beta| \leqslant k} \|D^\beta f\|^2_{L^p(U)} \right)^{1/2},$$

$H^k(U)$ is a Hilbert space. For $k = 0$, there holds $H^0(U) = L^2(U)$.

For $s \in (0, 1)$, the fractional order Hilbert space $H^s(U)$ is defined as the space of all functions $f \in L^2(U)$ such that

$$\iint_{U \times U} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}} \, dx \, dy < +\infty.$$

Endowed with the norm

$$\|f\|_{H^s(U)} = \left(\|f\|^2_{L^2(U)} + \iint_{U \times U} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{1/2},$$

$H^s(U)$ is a Hilbert space.

It is possible to define the Hilbert spaces $H^s(U)$ in other equivalent ways. In particular, the relations with the Fourier transform or with the fractional Laplacian operator are surveyed in Appendix A. These characterizations will be used repeatedly throughout the article.

2.2. *Functional properties of the projection operator*

The next theorem is our first main result.

Theorem 1. *For every $u \in BV(\Omega)$, the function $(z, y) \mapsto (H_0 u)(y, z)$ belongs to the Banach space $BV(\Omega_1) \cap L^1(-a, a; H^s(-a, a))$, for every $s \in [0, 1)$. Moreover, for every $s \in [0, 1)$, there exists $C > 0$ such that, for every $u \in BV(\Omega)$, there holds*

$$\|H_0 u\|_{BV(\Omega_1)} + \|H_0 u\|_{L^1(-a,a; H^s(-a,a))} \leqslant C \|u\|_{BV(\Omega)}; \tag{4}$$

in other words, the operator

$$H_0 : BV(\Omega) \longrightarrow BV(\Omega_1) \cap L^1\big(-a, a; H^s(-a, a)\big)$$

is linear and continuous. For every $s \in [0, 1)$, the operator H_0 is linear and continuous as well for the following spaces:

- $H_0 : BV_0(\Omega) \longrightarrow BV_0(\Omega_1) \cap L^1(-a, a; H^s(-a, a))$;
- $H_0 : L^1(-a, a; BV(0, a)) \longrightarrow BV(\Omega_1) \cap L^1(-a, a; H^s(-a, a))$;
- $H_0 : L^1(-a, a; BV_0(0, a)) \longrightarrow BV_0(\Omega_1) \cap L^1(-a, a; H^s(-a, a))$.

Moreover, for $s = 1/2$, the statements above can be strengthened by replacing $H^s(-a,a)$ by the Lions–Magenes space[3] $H_{00}^{1/2}(-a,a)$.

In the above statement, the Banach space $L^1(-a,a;BV(0,a))$ is endowed with the norm

$$\int_{-a}^{a} \|u(\cdot,z)\|_{BV(0,a)}\,dz.$$

The Banach space $L^1(-a,a;BV_0(0,a))$ is a closed subspace of $L^1(-a,a;BV(0,a))$ and thus is endowed with the induced norm. Recall that the space $BV_0(\Omega)$ is the space of bounded variation functions of Ω vanishing on the subset Γ defined by (3). The space $BV_0(0,a)$ is defined similarly as the space of bounded variation functions on $[0,a)$ vanishing at a.

The Banach space $L^1(-a,a;H^s(-a,a))$ is endowed with the norm

$$\int_{-a}^{a} \|v(\cdot,z)\|_{H^s(-a,a)}\,dz.$$

In the inequality (4), the function H_0u is considered as a function of (z,y) instead of (y,z). The result means in particular that, for almost every $z \in (-a,a)$, the function $y \mapsto (H_0u)(y,z)$ belongs to $H^s(-a,a)$ for every $s \in [0,1)$, and the resulting function of z is of class L^1.

Similarly, every $u \in L^1(-a,a;BV(0,a))$ is considered as a function of (z,r) instead of (r,z); this means that, for almost every $z \in (-a,a)$, the function $r \mapsto u(r,z)$ belongs to $BV(0,a)$, and the resulting function of z is of class L^1 on $(-a,a)$.

Remark 1. It actually follows from the proof below (see Lemma 3 and Remark 3) that $BV(\Omega)$ (resp., $BV_0(\Omega)$) is continuously embedded in $L^1(-a,a;BV(0,a))$ (resp., $L^1(-a,a;BV_0(0,a))$).

Remark 2. Theorem 1 and Remark 1 hold as well for the blurred projection operator $H = BH_0 = K \star H_0$.

Proof of Theorem 1. Let us first prove that H_0 is linear and continuous from $L^1(\Omega)$ into $L^1(\Omega_1)$.

Lemma 1. *For every $u \in L^1(\Omega)$, there holds $\|H_0u\|_{L^1(\Omega_1)} \leqslant 2\pi a\|u\|_{L^1(\Omega)}$.*

Proof of Lemma 1. For every $z \in (-a,a)$, one has

$$\int_{-a}^{a} \left|(H_0u)(y,z)\right| dy \leqslant 2\int_{-a}^{a}\int_{|y|}^{a} \left|u(r,z)\right| \frac{r}{\sqrt{r^2-y^2}}\,dr\,dy,$$

[3] The Lions–Magenes space $H_{00}^{1/2}(-a,a)$ is the subset of functions $f \in H^{1/2}(-a,a)$ such that $\rho^{-1/2}f \in L^2(-a,a)$, where the function ρ is defined on $(-a,a)$ by $\rho(y) = a - |y|$. General definitions and properties of the Lions–Magenes space are recalled in Appendix A.2.2.

and, using Fubini's Theorem and the fact that $\int_{-r}^{r} \frac{r}{\sqrt{r^2-y^2}}\,dy = r\pi$, one arrives at

$$\int_{-a}^{a} \left|(H_0u)(y,z)\right| dy \leqslant 2\pi a \int_{0}^{a} \left|u(r,z)\right| dr.$$

Integrating with respect to z, the result follows. □

We next prove that H_0 is linear and continuous from $BV(\Omega)$ into $BV(\Omega_1)$.

Lemma 2. *There exists $C_0 > 0$ such that $\|H_0u\|_{BV(\Omega_1)} \leqslant C_0\|u\|_{BV(\Omega)}$, for every $u \in BV(\Omega)$.*

Proof of Lemma 2. Using Lemma 1, it suffices to prove the existence of a constant $C_0 > 0$ such that

$$\int_{\Omega_1} (H_0u)(y,z)\,\mathrm{div}\,\xi(y,z)\,dy\,dz \leqslant C_0\|u\|_{BV(\Omega)}\|\xi\|_{L^\infty(\Omega_1)},$$

for every $u \in BV(\Omega)$ and every $\xi = (\xi_1,\xi_2) \in C_c^1(\Omega_1,\mathbb{R}^2)$. Using Fubini's Theorem, one has

$$\begin{aligned}
&\int_{\Omega_1} (H_0u)(y,z)\,\mathrm{div}\,\xi(y,z)\,dy\,dz \\
&\quad = 2\int_{-a}^{a}\int_{-a}^{a}\int_{|y|}^{a} u(r,z)\frac{r}{\sqrt{r^2-y^2}}\left(\frac{\partial\xi_1}{\partial y}(y,z)+\frac{\partial\xi_2}{\partial z}(y,z)\right) dr\,dy\,dz \\
&\quad = 2\int_{-a}^{a}\int_{0}^{a} u(r,z)\int_{-r}^{r} \frac{r}{\sqrt{r^2-y^2}}\left(\frac{\partial\xi_1}{\partial y}(y,z)+\frac{\partial\xi_2}{\partial z}(y,z)\right) dy\,dr\,dz \\
&\quad = \int_{-a}^{a}\int_{0}^{a} u(r,z)\,\mathrm{div}\,\varphi(r,z)\,dr\,dz \\
&\quad = \int_{\Omega} u(r,z)\,\mathrm{div}\,\varphi(r,z)\,dr\,dz
\end{aligned}$$

where the function $\varphi = (\varphi_1,\varphi_2)$ is defined on $[0,a]\times[-a,a]$ by

$$\varphi_1(r,z) = 2\int_{0}^{r}\int_{-\tau}^{\tau} \frac{\tau}{\sqrt{\tau^2-y^2}}\frac{\partial\xi_1}{\partial y}(y,z)\,dy\,d\tau,$$

$$\varphi_2(r,z) = 2\int_{-r}^{r} \frac{r}{\sqrt{r^2-y^2}}\xi_2(y,z)\,dy.$$

An easy computation shows that

$$\varphi_1(r,z) = 2\int_{-r}^{r} \frac{y}{\sqrt{r^2-y^2}}\xi_1(y,z)\,dy.$$

The function φ is of class C^1, but is not of compact support contained in Ω. Hence, we must take into account the trace of u on $\partial\Omega$. Recall that, since Ω is bounded and $\partial\Omega$ is Lipschitz, functions of $BV(\Omega)$ have a trace on $\partial\Omega$ of class L^1, and we denote by $T_{\partial\Omega} : BV(\Omega) \to L^1(\partial\Omega)$ the corresponding bounded linear trace mapping (see [12]). Using Green's formula, one has

$$\int_{\Omega} u(r,z)\operatorname{div}\varphi(r,z)\,dr\,dz = -\int_{\Omega} \varphi\cdot d(Du) + \int_{\partial\Omega} (\varphi.\nu)T_{\partial\Omega}u\,d\lambda,$$

where ν denotes the outer unit normal on $\partial\Omega$, and λ denotes the standard one-dimensional Lebesgue measure (note that $\partial\Omega$ is made of four segments). The first integral is bounded by

$$\left|\int_{\Omega} \varphi\cdot d(Du)\right| = \left|\int_{\Omega} \varphi\cdot\sigma_u\,d|Du|\right| \leqslant \|\varphi\|_{L^\infty}|Du|(\Omega) \leqslant \|\varphi\|_{L^\infty(\Omega)}\|u\|_{BV(\Omega)},$$

and the second integral is bounded by

$$\left|\int_{\partial\Omega} (\varphi.\nu)T_{\partial\Omega}u\,d\lambda\right| \leqslant C_T\|\varphi\|_{L^\infty(\partial\Omega)}\|u\|_{BV(\Omega)},$$

where $C_T > 0$ is the norm of the trace operator $T_{\partial\Omega}$. Clearly, there exists $C_1 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\partial\Omega)} \leqslant C_1\|\xi\|_{L^\infty(\Omega_1)}.$$

The proof follows. □

Lemma 3. *Let $a < b$ and $c < d$ be real numbers, let $O = (a,b)\times(c,d)$, and let $g \in BV(O)$. For almost every $x \in (a,b)$, the marginal function $g^x : y \mapsto g(x,y)$ is of bounded variation on (c,d). Moreover, $|Dg|(O) \geqslant \int_a^b |Dg^x|(c,d)\,dx$.*

Remark 3. It follows from this lemma that $BV(O)$ is continuously embedded in the space $L^1(a,b;BV(c,d))$. This fact justifies the end of Remark 1.

Proof. The proof of this lemma is actually contained in [6] (see also [12, Theorem 2, p. 220]), however since this result is used repeatedly in the proof of the theorem, we provide a proof for the convenience of the reader.

First of all, since $g \in L^1(O)$, it follows from Fubini's Theorem that $g^x \in L^1(c,d)$ for almost every $x \in (a,b)$. Recall that $W^{1,1}(O)$ is dense in $BV(O)$ in the sense of the intermediate convergence, that is, there exists a sequence of functions $g_k \in W^{1,1}(O)$ such that g_k converges to g in $L^1(O)$ and $|Dg_k|(O) \to |Dg|(O)$ (see e.g. [12]). Note that, since $g_k \in W^{1,1}(O)$, there holds $Dg_k = \nabla g_k$ and $|Dg_k|(O) = \int_O \|\nabla g_k(x,y)\|\,dx\,dy$.

From this result, we deduce two properties.

First, we infer that there exists a subsequence $(g^x_{\varphi(k)})$ of the sequence of functions g^x_k: $y \mapsto g_k(x,y)$ that converges to $g^x : y \mapsto g(x,y)$ in $L^1(c,d)$, for almost every $x \in (a,b)$ (with φ independent on x). Indeed, since g_k converges to g in $L^1(O)$, denoting $h_k(x) = \int_c^d |g_k(x,y) - g(x,y)|\,dy$, it follows from Fubini's Theorem that

$$\int_a^b h_k(x)\,dx = \int_a^b \int_c^d \big|g_k(x,y) - g(x,y)\big|\,dy\,dx \to 0,$$

i.e., h_k converges to 0 in $L^1(a,b)$. Therefore, there exists a subsequence of (h_k) converging almost everywhere to 0 on (a,b). In other words, a subsequence of (g^x_k) converges to g^x in $L^1(c,d)$, for almost every $x \in (a,b)$.

Second, we infer that

$$\liminf_{k\to+\infty}\big|Dg^x_k\big|(c,d) < +\infty,$$

for almost every $x \in (a,b)$. Indeed, we have $|Dg_k|(O) \to |Dg|(O)$, and

$$|Dg_k|(O) = \int_O \big\|\nabla g_k(x,y)\big\|\,dx\,dy \geqslant \int_a^b \int_c^d \left|\frac{\partial g_k}{\partial y}(x,y)\right|dy\,dx = \int_a^b \big|Dg^x_k\big|(c,d)\,dx.$$

Note that the latter equality holds because, since $g_k \in W^{1,1}(O)$, it follows from Fubini's Theorem that, for almost every $x \in (a,b)$, the function g^x_k belongs to $W^{1,1}(c,d)$, and thus in particular its total variation is $|Dg^x_k|(c,d) = \int_c^d |\frac{\partial g_k}{\partial y}(x,y)|\,dy$. From Fatou's Lemma, the function $x \mapsto \liminf_{k\to+\infty} |Dg^x_k|(c,d)$ is measurable on (a,b), and

$$\int_a^b \liminf_{k\to+\infty}\big|Dg^x_k\big|(c,d)\,dx \leqslant \liminf_{k\to+\infty}\int_a^b \big|Dg^x_k\big|(c,d)\,dx \leqslant |Dg|(O). \tag{5}$$

It follows that $\liminf_{k\to+\infty} |Dg^x_k|(c,d) < +\infty$ for almost every $x \in (a,b)$.

From these two points, we can achieve the proof of the lemma, as follows. Let $\psi \in C^1_c((c,d),\mathbb{R})$ such that $\|\psi\|_{L^\infty} \leqslant 1$. Then, for almost every $x \in (a,b)$,

$$\int_c^d g^x(y)\psi'(y)\,dy = \lim_{k\to+\infty}\int_c^d g^x_{\varphi(k)}(y)\psi'(y)\,dy \leqslant \liminf_{k\to+\infty}\big|Dg^x_k\big|(c,d) < +\infty$$

and therefore $g^x \in BV(c,d)$. Moreover, integrating this inequality on $[a,b]$ and using (5) leads to $\int_a^b |Dg^x|(c,d)\,dx \leqslant |Dg|(O)$. $\square$

In the sequel, we denote by $\mathcal{F}_y v$ the Fourier transform of an integrable function $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with respect to the first variable, that is,

$$(\mathcal{F}_y v)(\xi, z) = \int_{\mathbb{R}} v(y, z) e^{-2i\pi y\xi}\, dy,$$

for all $\xi, z \in \mathbb{R}$.

Recall, for every $u \in L^1(\Omega)$, the function $H_0 u$ is of compact support contained in Ω_1. In the lemma below, and in the sequel, $\tilde{u}$ (resp. $\widetilde{H_0 u}$) denotes the extension by 0 to $\mathbb{R}^2$ of the function u (resp. $H_0 u$). Similarly, we denote by $\widetilde{H_0}$ the operator defined by $\widetilde{H_0} u = \widetilde{H_0 u}$, for every $u \in L^1(\Omega)$.

Lemma 4. *There holds*

$$(\mathcal{F}_y \widetilde{H_0 u})(\xi, z) = 2\pi \int_0^a r \tilde{u}(r, z) J_0(2\pi \xi r)\, dr, \tag{6}$$

for every $u \in L^1(\Omega)$, every $\xi \in \mathbb{R}$ and almost every $z \in \mathbb{R}$, where J_0 is the Bessel function of the first kind defined by

$$J_0(x) = \frac{2}{\pi} \int_0^1 \frac{\cos(tx)}{\sqrt{1-t^2}}\, dt. \tag{7}$$

The adjoint of $\mathcal{F}_y \widetilde{H_0}$ (with L^2 as a pivot space) is given by

$$\big((\mathcal{F}_y \widetilde{H_0})^* v\big)(r, z) = 2\pi r \int_{\mathbb{R}} v(\xi, z) J_0(2\pi \xi r)\, d\xi, \tag{8}$$

for every $v \in L^1(\mathbb{R}^2)$, every $r \in [0, a)$ and almost every $z \in (-a, a)$.

Proof. Applying Fubini's Theorem, we compute, for every $\xi \in \mathbb{R}$ and almost every $z \in (-a, a)$,

$$\begin{aligned}
(\mathcal{F}_y \widetilde{H_0 u})(\xi, z) &= \int_{-a}^{a} H_0 u(y, z) e^{-2i\pi y\xi}\, dy \\
&= 2 \int_{-a}^{a} \int_{|y|}^{a} u(r, z) \frac{r}{\sqrt{r^2 - y^2}} e^{-2i\pi y\xi}\, dr\, dy \\
&= 2 \int_0^a \int_{-r}^{r} u(r, z) \frac{r}{\sqrt{r^2 - y^2}} e^{-2i\pi y\xi}\, dy\, dr
\end{aligned}$$

$$= 2\int_0^a ru(r,z)\left(\int_{-1}^1 \frac{1}{\sqrt{1-t^2}} e^{-2i\pi rt\xi}\,dt\right)dr$$

$$= 2\int_0^a ru(r,z)\hat{\nu}(r\xi)\,dr$$

where

$$\nu(t) = \frac{1}{\sqrt{1-t^2}}\mathbf{1}_{[-1,1]}(t),$$

and $\hat{\nu}$ is the Fourier transform of the function ν. The function $\hat{\nu}$ can be computed using the Bessel function of the first kind J_0 defined by (7) (see [2]). Since ν is even, its Fourier transform is

$$\hat{\nu}(\omega) = 2\int_0^1 \frac{\cos(2\pi\omega t)}{\sqrt{1-t^2}}\,dt = \pi J_0(2\pi\omega),$$

and the formula (6) follows. Let us now compute the adjoint of $\mathcal{F}_y\widetilde{H_0}$, with L^2 as a pivot space. For every $v \in L^1(\mathbb{R}^2)$ and every $u \in L^\infty(\Omega)$, we have

$$\left\langle (\mathcal{F}_y\widetilde{H_0})^* v, u\right\rangle = \langle v, \mathcal{F}_y\widetilde{H_0 u}\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} v(\xi,z)\mathcal{F}_y\widetilde{H_0 u}(\xi,z)\,d\xi\,dz$$

$$= 2\pi\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^a r\tilde{u}(r,z)v(\xi,z)J_0(2\pi\xi r)\,dr\,d\xi\,dz$$

$$= 2\pi\int_0^a\int_{\mathbb{R}} r\tilde{u}(r,z)\int_{\mathbb{R}} v(\xi,z)J_0(2\pi\xi r)\,d\xi\,dz\,dr$$

and hence $(\mathcal{F}_y\widetilde{H_0})^* v(r,z) = 2\pi r\int_{\mathbb{R}} v(\xi,z)J_0(2\pi\xi r)\,d\xi$. □

To prove the theorem, we next make use of the asymptotic properties of the Bessel functions J_0 and J_1, where the function J_1 is defined by

$$J_1(x) = \frac{x}{\sqrt{\pi}\,\Gamma(3/2)}\int_0^1 \cos(tx)\sqrt{1-t^2}\,dt.$$

Recall that

$$\left|J_0(x)\right| \leqslant 1, \qquad \left|J_1(x)\right| \leqslant \frac{1}{\sqrt{2}}, \tag{9}$$

$$J_0'(x) = -J_1(x), \qquad \frac{d}{dx}\big(xJ_1(x)\big) = xJ_0(x), \tag{10}$$

for every $x \in \mathbb{R}$, and

$$\left|J_1(x)\right| \leqslant \frac{1}{\sqrt{x}} \tag{11}$$

as $x \to +\infty$ (see e.g. [2]).

Lemma 5. *There exists $C_2 > 0$ (only depending on a) such that, for every $u \in L^1(-a, a; BV(0, a))$, there holds*

$$\left|(\mathcal{F}_y \widetilde{H_0 u})(\xi, z)\right| \leqslant \frac{C_2}{(1+\xi^2)^{3/4}}\big(\left|u_z(a)\right| + \|u_z\|_{L^1(0,a)} + |Du_z|(0, a)\big), \tag{12}$$

for every $\xi \in \mathbb{R}$ and almost every $z \in (-a, a)$.

In the above statement, recall that $u \in L^1(-a, a; BV(0, a))$ is seen as a function of (z, r); in particular, for almost every $z \in (-a, a)$, the function $r \mapsto u_z(r) = u(r, z)$ is of bounded variations on $[0, a)$, and its total variation is denoted $|Du_z|(0, a)$. Also, note that $u_z(a)$ exists for almost every $z \in (-a, a)$.

Proof of Lemma 5. Using the formula (6) and the estimate (9), it is first clear that

$$\left|(\mathcal{F}_y \widetilde{H_0 u})(\xi, z)\right| \leqslant 2\pi a \|u_z\|_{L^1(0,a)}, \tag{13}$$

for every $\xi \in \mathbb{R}$ and almost every $z \in (-a, a)$.

From (10), there holds

$$\frac{d}{dr}\big(2\pi\xi r J_1(2\pi\xi r)\big) = (2\pi\xi)^2 r J_0(2\pi\xi r),$$

and, using Green's formula (integration by parts), one gets, for every $\xi \neq 0$ and almost every $z \in (-a, a)$ (such that $u_z(a)$ exists),

$$\begin{aligned}(\mathcal{F}_y \widetilde{H_0 u})(\xi, z) &= \frac{1}{2\pi\xi^2}\int_0^a u_z(r)(2\pi\xi)^2 r J_0(2\pi\xi r)\, dr \\ &= \frac{a}{\xi} J_1(2\pi\xi a) u_z(a) - \frac{1}{\xi}\int_{[0,a]} r J_1(2\pi\xi r)\, d(Du_z)\end{aligned} \tag{14}$$

and hence, using (11), it follows that

$$\left|(\mathcal{F}_y \widetilde{H_0 u})(\xi, z)\right| \leqslant \frac{1}{|\xi|^{3/2}}\sqrt{\frac{a}{2\pi}}\big(\left|u_z(a)\right| + |Du_z|(0, a)\big) \tag{15}$$

as $|\xi| \to +\infty$.

The estimate (12) finally follows from (13) and (15). □

We are now in a position to estimate $\|H_0u\|_{L^1(-a,a;H^s(-a,a))}$. Using (12), it first follows that, for almost every $z \in (-a,a)$, the H^s norm of the function $y \mapsto (\widetilde{H_0u})(y,z)$ is estimated by[4]

$$\big\|(\widetilde{H_0u})(\cdot,z)\big\|_{H^s(\mathbb{R})} = \bigg(\int_{\mathbb{R}} \big(1+\xi^2\big)^s \big|(\mathcal{F}_y H_0u)(\xi,z)\big|^2\,d\xi\bigg)^{1/2}$$

$$\leqslant C_2\big(\big|u_z(a)\big| + \|u_z\|_{L^1(0,a)} + |Du_z|(0,a)\big)\bigg(\int_{\mathbb{R}} \big(1+\xi^2\big)^{s-3/2}\,d\xi\bigg)^{1/2}.$$

The integral $\int_{\mathbb{R}}(1+\xi^2)^{s-3/2}\,d\xi$ converges if and only if $2s-3<-1$, that is, $s<1$. It follows that, for almost every $z \in (-a,a)$, the function $y \mapsto (\widetilde{H_0u})(y,z)$ belongs to $H^s(\mathbb{R})$, for every $s \in [0,1)$.

Now, for almost every $z \in (-a,a)$, the function $y \mapsto (H_0u)(y,z)$ is the restriction to $(-a,a)$ of the function $y \mapsto (\widetilde{H_0u})(y,z)$ (which is by definition equal to 0 outside $(-a,a)$). It then follows from the characterization of fractional Hilbert spaces on a subset by the quotient norm (see Appendix A.2.1) that this function belongs to $H^s(-a,a)$, for every $s \in [0,1)$, and that, up to some constant, $\|(H_0u)(\cdot,z)\|_{H^s(-a,a)} \leqslant \|(\widetilde{H_0u})(\cdot,z)\|_{H^s(\mathbb{R})}$, for almost every $z \in (-a,a)$. Hence, for every $s \in [0,1)$, there exists $C_3>0$ such that, for every $u \in L^1(-a,a;BV(0,a))$, there holds

$$\big\|H_0u(\cdot,z)\big\|_{H^s(-a,a)} \leqslant C_3\big(\big|u_z(a)\big| + \|u_z\|_{L^1(0,a)} + |Du_z|(0,a)\big), \tag{16}$$

for almost every $z \in (-a,a)$.

As a byproduct, note that the function $y \mapsto H_0u(y,z)$, defined on $(-a,a)$, can be extended (by 0) to a function of $H^s(\mathbb{R})$, for every $s \in [0,1)$, for almost every $z \in (-a,a)$. It follows from [25, Lemma 37.1] (see results recalled in Appendix A.2.1) that the function $y \mapsto \rho(y)^{-s}H_0u(y,z)$ belongs to $L^2(-a,a)$, where ρ denotes the distance to the boundary of $(-a,a)$, that is, $\rho(y) = a-|y|$ for every $y \in (-a,a)$. In turn, for $s=1/2$, the function $y \mapsto H_0u(y,z)$ belongs to the Lions–Magenes space $H_{00}^{1/2}(-a,a)$ (see Appendix A.2.2), for almost every $z \in (-a,a)$.

Integrating (16) with respect to z leads to

$$\|H_0u\|_{L^1(-a,a;H^s(-a,a))} \leqslant C_3\bigg(\int_{-a}^{a} \big|u(a,z)\big|\,dz + \|u\|_{L^1(\Omega)} + \int_{-a}^{a} |Du_z|(0,a)\,dz\bigg). \tag{17}$$

This inequality implies the remaining items of the theorem.

Indeed, let us first consider functions $u \in BV(\Omega)$. It has already been mentioned that the trace operator is continuous from $BV(\Omega)$ into $L^1(\partial\Omega)$, hence it follows that

$$\int_{-a}^{a} \big|u(a,z)\big|\,dz \leqslant C_4\|u\|_{BV(\Omega)} \tag{18}$$

[4] Here, we use the definition of the H^s norm in terms of Fourier transform, recalled in Appendix A.1.1.

for some constant $C_4 > 0$. Moreover, from Lemma 3, there holds

$$\int_{-a}^{a} |Du_z|(0,a)\,dz \leqslant |Du|(\Omega). \tag{19}$$

The estimate (4) follows from (17), (18), (19), and Lemma 2.

The other items follow similarly. This ends the proof of the theorem. □

Theorem 1 states a strong functional property of the projection operator, which is however not very suitable in view of a variational approach. In order to derive necessary conditions for optimality, it would be better to establish functional properties of H_0 in some Hilbert spaces. This is the object of the next section.

2.3. *Hilbertian functional properties of the projection operator*

We have already mentioned that we handle functions of bounded variation on Ω that take their values in $\{0,1\}$ almost everywhere. Denote by $BV(\Omega,\{0,1\})$ the set of such functions. First of all, notice that such functions belong to $L^1(-a,a;BV([0,a),\{0,1\}))$, as already mentioned in Remarks 1 and 3; they also share the following property.

Lemma 6. *For every $u \in BV(\Omega,\{0,1\})$, the function $(z,r) \mapsto u(r,z)$ belongs to the Banach space $L^1(-a,a;H^s(0,a))$, for every $s \in [0,1/2)$.*

Proof of Lemma 6. Let $u \in BV(\Omega,\{0,1\})$. As mentioned above, from Lemma 3, the function $u_z : r \mapsto u_z(r) = u(r,z)$ is of bounded variation on $[0,a)$, for almost every $z \in (-a,a)$. Since u_z takes its values in $\{0,1\}$, its set of discontinuities is finite. It follows that, for almost every $z \in (-a,a)$, there exist an integer n_z and real numbers $(\alpha_i)_{1\leqslant i\leqslant n_z}$, $(\beta_i)_{1\leqslant i\leqslant n_z}$ satisfying

$$0 \leqslant \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_{n_z} < \beta_{n_z} \leqslant a,$$

such that

$$u_z(r) = \sum_{i=1}^{n_z} \mathbf{1}_{[\alpha_i,\beta_i]}(r), \tag{20}$$

for almost every $r \in [0,a)$. Note that the total variation of the function u_z is $\int_{[0,a)} |Du_z| = 2n_z$. From Lemma 3, there holds

$$\int_{\Omega} |Du| \geqslant 2\int_{-a}^{a} n_z\,dz,$$

and hence the function $z \mapsto n_z$ belongs to $L^1(-a,a)$.

The function u_z is extended by 0 outside $[0, a)$, into a function $\tilde{u}_z \in L^1(\mathbb{R})$. Using (20), one easily computes, for almost every $z \in (-a, a)$, the Fourier transform of $\tilde{u}_z$ as

$$(\mathcal{F}\tilde{u}_z)(\xi) = \sum_{i=1}^{n_z} \frac{\sin(\pi(\beta_i - \alpha_i)\xi)}{\pi\xi} e^{-i\pi(\beta_i+\alpha_i)\xi},$$

for every $\xi \in \mathbb{R}$. In particular, there holds

$$\left|(\mathcal{F}\tilde{u}_z)(\xi)\right| \leqslant \sum_{i=1}^{n_z} |\beta_i - \alpha_i| \leqslant a, \tag{21}$$

for every $\xi \in \mathbb{R}$, and

$$\left|(\mathcal{F}\tilde{u}_z)(\xi)\right| \leqslant \frac{n_z}{\pi|\xi|}, \tag{22}$$

for every $\xi \in \mathbb{R} \setminus \{0\}$. Using the definition of the H^s norm in terms of Fourier transform (recalled in Appendix A.1.1), and using (21) and (22), one has the estimate

$$\begin{aligned}
\|\tilde{u}_z\|^2_{H^s(\mathbb{R})} &= \int_{\mathbb{R}} \left(1+\xi^2\right)^s \left|\mathcal{F}\tilde{u}_z(\xi)\right|^2 d\xi \\
&= \int_{|\xi|\geqslant 1} \left(1+\xi^2\right)^s \left|\mathcal{F}\tilde{u}_z(\xi)\right|^2 d\xi + \int_{|\xi|\leqslant 1} \left(1+\xi^2\right)^s \left|\mathcal{F}\tilde{u}_z(\xi)\right|^2 d\xi \\
&\leqslant \int_{|\xi|\geqslant 1} \left(1+\xi^2\right)^s \frac{n_z^2}{\pi^2\xi^2} d\xi + \int_{|\xi|\leqslant 1} \left(1+\xi^2\right)^s a^2\, d\xi
\end{aligned}$$

which is convergent if $s < 1/2$. Hence $\tilde{u}_z \in H^s(\mathbb{R})$, for almost every $z \in (-a, a)$ and every $s \in [0, 1/2)$. Since u_z is the restriction of $\tilde{u}_z$ to $(0, a)$, it follows, using the definition of $H^s(0, a)$ in terms of quotient norm (see Appendix A.2.1), that $u_z \in H^s(0, a)$, for almost every $z \in (-a, a)$ and every $s \in [0, 1/2)$. Moreover, there exists a constant $C > 0$, depending only on s and a, such that

$$\|u_z\|_{H^s(0,a)} \leqslant C n_z.$$

Since the function $z \mapsto n_z$ belongs to $L^1(-a, a)$, we infer that the function $(z, r) \mapsto u(r, z)$ belongs to $L^1(-a, a; H^s(0, a))$, for $s \in [0, 1/2)$. $\square$

To comply with the variational approach that we propose next, it would be better to deal with Hilbert spaces and, for instance, to replace L^1 with L^2 in the previous statements. Unfortunately, we have the following negative remark.

Remark 4. There exist some functions $u \in BV(\Omega, \{0, 1\})$ such that the function $(z, r) \mapsto u(r, z)$ does not belong to $L^2(-a, a; BV(0, a))$.

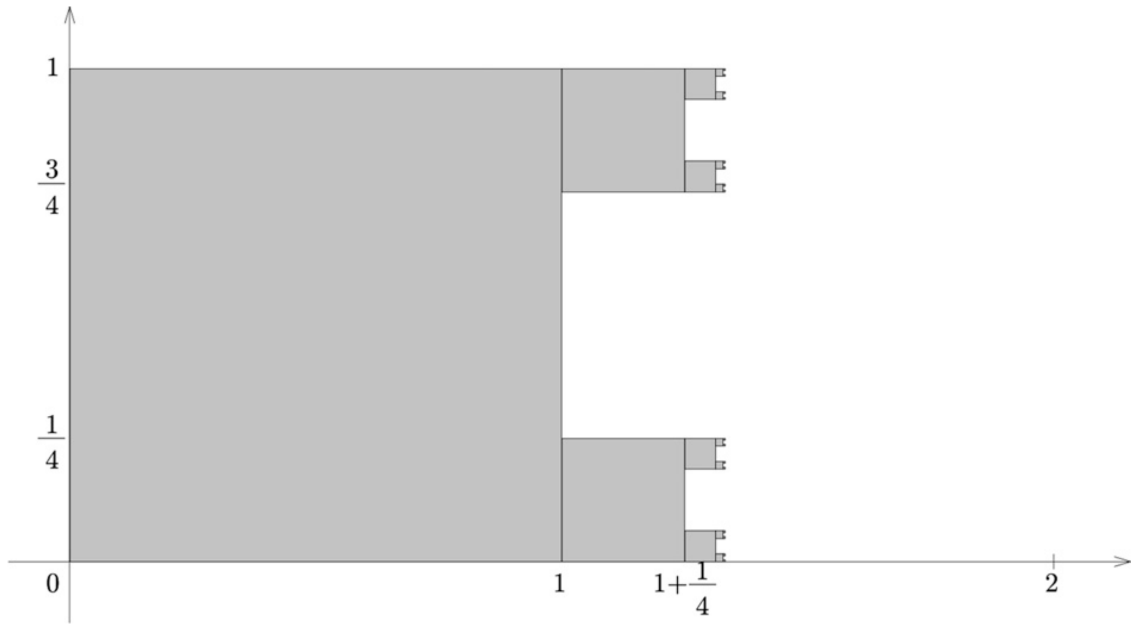

Fig. 1. Fractal object.

Let us provide an example of such a function.[5] Consider in the plane, endowed with coordinates (x, y), the unit square $[0, 1] \times [0, 1]$. We append to this square, on its right side, the two smaller squares

$$\left[1, 1+\frac{1}{4}\right] \times \left[1, \frac{1}{4}\right] \quad \text{and} \quad \left[1, 1+\frac{1}{4}\right] \times \left[\frac{3}{4}, 1\right].$$

Then, we apply a similar appending procedure to each of these latter squares, and so forth, iteratively. We obtain a fractal object (see Fig. 1). Then, we claim that the function u defined on $[0, 2] \times [0, 1]$ as the characteristic function of this fractal domain is of bounded variation.

Indeed, the L^1 norm of u is the sum of the areas of all squares, that is

$$\sum_{k=0}^{+\infty} 2^k \left(\frac{1}{4^k}\right)^2 = \sum_{k=0}^{+\infty} \frac{1}{2^k} < +\infty.$$

To prove that $u \in BV([0, 2] \times [0, 1], \{0, 1\})$, it suffices to show that the marginal functions $u_x : y \mapsto u(x, y)$ and $u_y : x \mapsto u(x, y)$ are of bounded variation (see [12, Theorem 2, p. 220]). This property is obvious, since for any y the marginal functions u_y have at most one jump, and for every x the marginal functions u_x have a finite number n_x of jumps. More precisely,

$$n_x = \begin{cases} 2^k & \text{if } 1 + \sum_{i=1}^{k-1} \frac{1}{4^i} \leqslant x \leqslant 1 + \sum_{i=1}^{k} \frac{1}{4^i}, \quad \forall k \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

[5] This example has been indicated to us by Simon Masnou.

There holds

$$\int_0^2 n_x\,dx = \sum_{k=1}^{+\infty} \frac{2^k}{4^k} = \sum_{k=1}^{+\infty} \frac{1}{2^k} < +\infty,$$

that is, the function $x \mapsto n_x$ belongs to $L^1(0,2)$, as expected (see Lemma 6 and its proof), however it does not belong to $L^2(0,2)$ since

$$\int_0^2 n_x^2\,dx = \sum_{k=1}^{+\infty} \frac{(2^k)^2}{4^k} = \sum_{k=1}^{+\infty} 1 = +\infty.$$

This example shows that the functions considered in our framework, belonging to $BV(\Omega,\{0,1\})$, do not necessarily belong to the Banach space $L^2(-a,a;BV(0,a))$.

In what follows, we are however going to work within this latter space. More precisely, since our functions vanish on the set Γ defined by (3), we are going to work within the space $L^2(-a,a;BV_0(0,a))$. Indeed, this Banach space is better suited for our tomography problem, because we are able to prove that the projection operator H_0 is linear and continuous from $L^2(-a,a;BV_0(0,a))$ into the Hilbert space $L^2(-a,a;H^s(-a,a))$, for every $s \in [0,a)$ (see Theorem 2 further). The Hilbert space $L^2(-a,a;H^s(-a,a))$ is then far more suitable for numerical purposes than the Banach space $L^1(-a,a;H^s(-a,a))$, and the use of a scalar product makes easier the derivation of an optimality system (first order necessary conditions for optimality).

Hence, although the space $L^2(-a,a;BV_0(0,a))$ differs from the usual space $BV_0(\Omega)$, it happens to be relevant in our problem. Actually, in practice, the binary functions considered in our imaging process belong to the space $L^\infty(-a,a;BV_0(0,a))$ (and thus, a fortiori, to $L^2(-a,a;BV_0(0,a))$). Indeed, in practice the functions $z \mapsto n_z$ are uniformly bounded with respect to $z \in (-a,a)$. Concretely, this means that the number of jumps of our binary images is uniformly bounded in every direction.

The next theorem, which is our second main result, provides nice functional properties of the projection operator H_0 in this new framework. We first give some notations. Denote by

$$X = L^2\bigl(-a,a;BV_0(0,a)\bigr), \tag{23}$$

the set of all functions $u \in L^2(\Omega)$ such that the function $(z,r) \mapsto u(r,z)$ belongs to $L^2(-a,a;BV_0(0,a))$. Recall that $BV_0(0,a)$ is the closed subset of the set of functions $f \in BV(0,a)$ vanishing at a; the total variation, which is a semi-norm, is a norm on $BV_0(0,a)$. It follows that the space X is a closed subspace of the Banach space $L^2(-a,a;BV(0,a))$, and can be endowed with the norm

$$\|u\|_X = \left(\int_{-a}^{a} \bigl(|Du_z|(0,a)\bigr)^2\,dz\right)^{1/2}. \tag{24}$$

Recall that, for $s \geqslant 0$, $H_0^s(-a,a)$ is defined as the closure in $H^s(-a,a)$ of the set of all smooth functions having a compact support contained in $(-a,a)$. Note that, for $s \in [0,1/2]$, there holds $H_0^s(-a,a) = H^s(-a,a)$ (see Appendix A). For every $s \in [0,1)$, $s \neq 1/2$, denote by

$$Y_s = L^2\big(-a,a; H_0^s(-a,a)\big), \tag{25}$$

the set of all functions $v \in L^2(\Omega_1)$ such that the function $(z,y) \mapsto v(y,z)$ belongs to $L^2(-a,a; H_0^s(-a,a))$. It is a closed subspace of $L^2(-a,a; H^s(-a,a))$, and, endowed with the norm

$$\|v\|_{Y_s} = \left(\int_{-a}^{a} \big\| v(\cdot,z) \big\|^2_{H^s(-a,a)}\, dz \right)^{1/2}, \tag{26}$$

Y_s is a Hilbert space. For $s = 1/2$, define, similarly, the Hilbert space

$$Y_{1/2} = L^2\big(-a,a; H_{00}^{1/2}(-a,a)\big). \tag{27}$$

Theorem 2. *For every $s \in [0,1)$, the operator H_0 is linear and continuous from X into Y_s.*

Remark 5. Theorem 2 holds as well for the blurred projection operator $H = BH_0 = K \star H_0$.

Proof of Theorem 2. Using the reasoning of the proof of Theorem 1, the inequality (16) implies that, for every $u \in L^2(-a,a; BV_0(0,a))$,

$$\big\| H_0 u(\cdot,z) \big\|_{H^s(-a,a)} \leqslant C_3\big(\|u_z\|_{L^1(0,a)} + |Du_z|(0,a) \big),$$

for almost every $z \in (-a,a)$. Integrating with respect to z the square of this inequality leads to the result. □

3. Variational approach for tomographic reconstruction

3.1. Minimization problem in fractional Sobolev spaces

Let v_d be the projected image (observed data), and let $\alpha > 0$. As explained in Section 1, we have proposed in [1] an approach for tomographic reconstruction based on the consideration of the minimization problem

$$\begin{cases} \min F(u), \quad \text{with } F(u) = \dfrac{1}{2}\|Hu - v_d\|^2_{L^2(\Omega_1)} + \alpha \Phi(u), \\ u \in BV(\Omega), \\ u(r,z) \in \{0,1\} \quad \text{a.e. on } \Omega. \end{cases}$$

Note that the pointwise constraint, $u(r,z) \in \{0,1\}$ a.e. on Ω, is a very hard constraint. The constraint set is not convex and its interior is empty for most usual topologies.

Based on the functional analysis of Section 2, we propose here to define the functional to be minimized in another way, according to the regularity properties proved in Theorems 1 and 2.

In view of deriving an optimality system and taking benefit of some Hilbertian structure, we use the results of Theorem 2 and Remark 5, according to which the blurred projection operator H is linear and continuous from X into Y_s, for every $s \in [0, 1)$. Assuming that $v_d \in Y_s$ (observed data), we are led to consider the minimization problem

$$(\mathcal{P}_1^s) \quad \begin{cases} \min F_1^s(u), \quad \text{with } F_1^s(u) = \frac{1}{2}\|Hu - v_d\|_{Y_s}^2 + \alpha\Phi(u), \\ u \in BV(\Omega) \cap X, \\ \|u\|_X \leqslant \eta, \\ u(r,z) \in \{0,1\} \quad \text{a.e. on } \Omega, \end{cases}$$

where $s \in [0, 1)$ and $\alpha > 0$ are fixed parameters, and $\eta > 0$ is a fixed (large) real number. The constraint $\|u\|_X \leqslant \eta$ happens to be necessary in order to derive an existence theorem. For η large enough, this constraint is however not active[6] and does not change anything to the above minimization problem.

The parameter $\alpha > 0$ is the weight of the total variation. This term is a usual regularization term in image processing. In our framework there is however another possibility, namely, we can replace this term with a regularization term involving the norm of X. In that case, we rather consider the minimization problem

$$(\mathcal{P}_2^s) \quad \begin{cases} \min F_2^s(u), \quad \text{with } F_2^s(u) = \frac{1}{2}\|Hu - v_d\|_{Y_s}^2 + \frac{\alpha}{2}\|u\|_X^2, \\ u \in X, \\ u(r,z) \in \{0,1\} \quad \text{a.e. on } \Omega. \end{cases}$$

Note that, in the latter minimization problem, the use of $\eta > 0$ is not needed.

Theorem 3. *For all $s \in [0, 1)$, $\alpha > 0$ and $\eta > 0$, the minimization problems $(\mathcal{P}_1^s)$ and $(\mathcal{P}_2^s)$ admit at least one solution.*

Proof. Let $s \in [0, 1)$, $\alpha > 0$ and $\eta > 0$ be fixed. Let (u_n) be a minimizing sequence in $BV(\Omega) \cap X$ of the minimization problem $(\mathcal{P}_1^s)$, satisfying $\|u_n\|_X \leqslant \eta$ and $u_n(x) \in \{0, 1\}$ almost everywhere on Ω. Then, the sequences $(\|Hu_n - v_d\|_{Y_s})_{n\in\mathbb{N}}$, $(\Phi(u_n))_{n\in\mathbb{N}}$ and $(\|u_n\|_X)_{n\in\mathbb{N}}$ are bounded, and we deduce three things.

First, the sequence $(Hu_n)_{n\in\mathbb{N}}$ is bounded in the Hilbert space Y_s, and therefore, up to a subsequence it converges weakly to some $w \in Y_s$; in particular, using the continuous embedding of $BV(-a, a)$ in $L^2(-a, a)$, it converges as well to w, up to some subsequence, for the weak topology of $L^2(\Omega_1)$.

Second, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $BV(\Omega) \cap L^\infty(\Omega)$, and hence, up to a subsequence, it converges to some $u \in BV(\Omega) \cap L^\infty(\Omega)$ for the weak-star topology, i.e., up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges strongly to some $u \in L^1(\Omega)$ and $(Du_n)_{n\in\mathbb{N}}$ converges to Du in the space of Radon measures for the weak-star topology.

---

[6] As explained formerly, the slices of the binary images treated in practice have a finite number of connected components, uniformly with respect to the slice; this means that there exists $R > 0$ such that $\|u\|_{L^\infty(-a,a;BV_0(0,a))} \leqslant \eta$, for every u in the set under consideration, and this implies the constraint considered here.

Third, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X (by η). The space X inherits of the weak compactness properties of L^2 and BV; hence, the sequence $(u_n)_{n\in\mathbb{N}}$ converges, up to a subsequence, to some $\bar{u}\in X$ for the weak-star topology, and the lower semi-continuity of the norm yields the inequality $\|\bar{u}\|_X \leqslant \eta$. In particular, $(u_n)_{n\in\mathbb{N}}$ converges weakly to $\bar{u}$ in $L^2(\Omega)$, and thus $\bar{u}=u$.

From Theorem 2, the operator H is linear and continuous from X into Y_s. Hence, up to a subsequence, $(Hu_n)_{n\in\mathbb{N}}$ converges weakly to Hu in Y_s. It follows that $w=Hu$.

Since Φ is lower semi-continuous with respect to the weak-star topology, there holds

$$\Phi(u)\leqslant \liminf_{n\to\infty}\Phi(u_n).$$

Using the weak convergence, up to a subsequence, of $(Hu_n)_{n\in\mathbb{N}}$ to Hu in Y_s, we infer that

$$\inf F_1^s=\lim_{n\to+\infty}\left(\frac{1}{2}\|Hu_n-v_d\|_{Y_s}^2+\alpha\Phi(u_n)\right)\geqslant \|Hu-v_d\|_{Y_s}^2+\alpha\Phi(u)=F_1^s(u),$$

and hence $F_1^s(u)=\inf F_1^s$. Finally, (u_n) converges to u in $L^1(\Omega)$, and thus, converges almost everywhere (up to a subsequence) to u. Hence, the pointwise constraint $u(x)\in\{0,1\}$ is satisfied almost everywhere, and therefore u is a solution of $(\mathcal{P}_1^s)$.

For the minimization problem $(\mathcal{P}_2^s)$, the proof is similar but simpler. □

3.2. *Penalization of the binarity constraint*

Due to the binarity constraint, $u(r,z)\in\{0,1\}$ a.e. on Ω, which is not convex and is of empty interior for most usual topologies, the minimization problems $(\mathcal{P}_1^s)$ and $(\mathcal{P}_2^s)$ are not directly tractable for numerical purposes. To deal with the binarity constraint, we propose as in [1] a penalization method. Let $\varepsilon>0$, $\beta\geqslant 0$, and let $\bar{u}_i^s$ be a solution of $(\mathcal{P}_i^s)$, for $i=1,2$. Define

$$F_{i,\varepsilon}^s(u)=F_i^s(u)+\frac{1}{2\varepsilon}\|u-u^2\|_{L^2(\Omega)}^2+\frac{\beta}{2}\|u-\bar{u}_i^s\|_{L^2(\Omega)}^2,$$

for $i=1,2$, and consider the penalized minimization problems

$$(\mathcal{P}_{1,\varepsilon}^s)\quad \begin{cases}\min F_{1,\varepsilon}^s(u),\\ u\in BV(\Omega)\cap X,\\ \|u\|_X\leqslant\eta,\\ \|u\|_{L^\infty(\Omega)}\leqslant\eta,\end{cases}$$

and

$$(\mathcal{P}_{2,\varepsilon}^s)\quad \begin{cases}\min F_{2,\varepsilon}^s(u),\\ u\in X,\\ \|u\|_{L^\infty(\Omega)}\leqslant\eta.\end{cases}$$

Actually in what follows η is chosen large enough, and the resulting constraint should not be active. This constraint is however necessary in order to derive existence results, but does not affect the numerical process. In the sequel we do not mention the dependence of the penalized problems $(\mathcal{P}_{i,\varepsilon}^s)$, $i=1,2$ with respect to the parameter η that can be chosen as large as desired but is fixed. A contrario the penalization parameter ε will tend to 0. The term $\frac{\beta}{2}\|u-\bar{u}_i^s\|^2_{L^2(\Omega)}$ is an additional penalization term permitting to focus, from the theoretical point of view, on a particular solution $\bar{u}_i^s$ of $(\mathcal{P}_i^s)$. In practice, the solution $\bar{u}_i^s$ is of course not known and we choose $\beta=0$.

Theorem 4. *The penalized minimization problem $(\mathcal{P}^s_{i,\varepsilon})$ has at least one solution $u^s_{i,\varepsilon}$, for $i=1,2$.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence of the minimization problem $(\mathcal{P}^s_{1,\varepsilon})$. First, in particular, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $BV(\Omega)$ and in $L^\infty(\Omega)$, and thus, up to a subsequence, converges to some $u^s_{1,\varepsilon}\in BV(\Omega)$ for the weak-star topology, and hence for the strong $L^1(\Omega)$ topology. Since it is moreover bounded (by η) in $L^\infty(\Omega)$, it follows from the Lebesgue dominated convergence theorem that the convergence holds for the strong topology of $L^p(\Omega)$, for every $p\in[1,+\infty)$. In addition, since every closed ball of $L^\infty(\Omega)$ is compact for the L^∞ weak star topology, we infer that $\|u^s_{1,\varepsilon}\|\leqslant\eta$.

The rest of the proof is similar to the proof of Theorem 3.

For the minimization problem $(\mathcal{P}^s_{2,\varepsilon})$, the proof is similar but simpler. □

Theorem 5.

1. *Every cluster point u_1^* in $BV(\Omega)\cap X$ of the family $(u^s_{1,\varepsilon})$ at $\varepsilon=0$ is a solution of $(\mathcal{P}^s_1)$, and every cluster point u_2^* in X of the family $(u^s_{2,\varepsilon})$ at $\varepsilon=0$ is a solution of $(\mathcal{P}^s_2)$. If moreover $\beta>0$ then $u_i^*=\bar{u}_i^s$, for $i=1,2$.*
2. *There holds $\lim_{\varepsilon\to0}F^s_{i,\varepsilon}(u_{i,\varepsilon})=\inf F_i^s$, and $\lim_{\varepsilon\to0}\int_\Omega|Du^s_{1,\varepsilon}|=\int_\Omega|Du_1^*|$.*

Proof. Since $\bar{u}_1^s\in BV(\Omega)$ is a solution of $(\mathcal{P}_1^s)$, one has $F^s_{1,\varepsilon}(u^s_{1,\varepsilon})\leqslant F^s_{1,\varepsilon}(\bar{u}^s_1)=F^s_1(\bar{u}^s_1)=\inf F^s_1$, for every $\varepsilon>0$. Therefore, the family $(u^s_{1,\varepsilon})$ is bounded in $BV(\Omega)$, and $\|u^s_{1,\varepsilon}-(u^s_{1,\varepsilon})^2\|_{L^2(\Omega)}$ tends to 0 as ε tends to 0. Let u_1^* be a (strong) cluster point of $(u^s_{1,\varepsilon})$ in $BV(\Omega)\cap X$ of the family $(u^s_{1,\varepsilon})$ at $\varepsilon=0$. Using the same reasoning as in the proof of Theorem 4, it follows that u_1^* is a (strong) cluster point of $(u^s_{1,\varepsilon})$ in $L^p(\Omega)$, for every $p\in[1,+\infty)$. Then,

$$\big\|u_1^*-\big(u_1^*\big)^2\big\|_{L^2(\Omega)}\leqslant\liminf_{\varepsilon\to0}\big\|u^s_{1,\varepsilon}-\big(u^s_{1,\varepsilon}\big)^2\big\|_{L^2(\Omega)}=0,$$

so that $u_1^*(1-u_1^*)=0$ almost everywhere on Ω, which is the binarity constraint. Since $F_1^s(u_{1,\varepsilon})+\beta\|u^s_{1,\varepsilon}-\bar{u}^s_1\|^2_2\leqslant F^s_{1,\varepsilon}(u^s_{1,\varepsilon})\leqslant\inf F^s_1$, one gets $F^s_1(u_1^*)\leqslant F^s_1(u_1^*)+\beta\|u_1^*-\bar{u}^s_1\|^2_2\leqslant\inf F^s_1$. Therefore u_1^* is a solution of $(\mathcal{P}^s_1)$. In addition, if $\beta>0$, then $u_1^*=\bar{u}^s_1$. Finally, since $\inf F^s_1=F^s_1(u^*)\leqslant F^s_{1,\varepsilon}(u^*)\leqslant\liminf F^s_{1,\varepsilon}(u^s_{1,\varepsilon})\leqslant\inf F^s_1$, (since $F^s_{1,\varepsilon}(u^s_{1,\varepsilon})\leqslant\inf F^s_1$,) it follows that $\lim_{\varepsilon\to0}F^s_{1,\varepsilon}(u^s_{1,\varepsilon})=\inf F^s_1$. Moreover, writing

$$\begin{aligned}
&\liminf_{\varepsilon\to0}F_1^s\big(u^s_{1,\varepsilon}\big)+\limsup_{\varepsilon\to0}\frac{1}{\varepsilon}\big\|u^s_{1,\varepsilon}-\big(u^s_{1,\varepsilon}\big)^2\big\|_2^2\\
&\quad\leqslant\lim_{\varepsilon\to0}\bigg(F_1^s\big(u^s_{1,\varepsilon}\big)+\frac{1}{\varepsilon}\big\|u^s_{1,\varepsilon}-\big(u^s_{1,\varepsilon}\big)^2\big\|_2^2\bigg)=F_1^s\big(u_1^*\big)\\
&\quad\leqslant\liminf_{\varepsilon\to0}F_1^s\big(u^s_{1,\varepsilon}\big),
\end{aligned}$$

it follows that $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\|u^s_{1,\varepsilon}-(u^s_{1,\varepsilon})^2\|_2^2=0$, and then that $\lim_{\varepsilon\to 0}F_1^s(u^s_{1,\varepsilon})=F_1^s(u_1^*)$. The conclusion follows then from the continuity properties of H.

The proof is similar for the family $(u^s_{2,\varepsilon})$. □

3.3. Optimality system of the penalized minimization problems

This section is devoted to the derivation of first order necessary conditions for the penalized minimization problems $(\mathcal{P}^s_{i,\varepsilon})$, $i=1,2$. Throughout the section, let $s\in[0,1)$, $\alpha>0$, $\eta>0$, $\varepsilon>0$ and $\beta\geqslant 0$ be fixed. Denote by

$$V_i=\begin{cases} BV(\Omega)\cap X & \text{if } i=1,\\ X & \text{if } i=2,\end{cases}$$

$$J_i(u)=\begin{cases} \Phi(u) & \text{if } i=1,\\ \frac{1}{2}\|u\|_X^2 & \text{if } i=2,\end{cases}$$

and

$$\mathcal{B}_\eta=\begin{cases} \{u\in BV(\Omega)\cap X \mid \|u\|_X\leqslant\eta \text{ and } \|u\|_{L^\infty(\Omega)}\leqslant\eta\} & \text{if } i=1,\\ \{u\in X\mid \|u\|_{L^\infty(\Omega)}\leqslant\eta\} & \text{if } i=2.\end{cases}$$

For $i=1,2$, for every $u\in V_i\cap\mathcal{B}_\eta$, define

$$G^s(u)=\frac{1}{2}\|Hu-v_d\|^2_{Y_s},$$

and

$$G^s_{i,\varepsilon}(u)=G^s(u)+\frac{1}{2\varepsilon}\big\|u-u^2\big\|^2_{L^2(\Omega)}+\frac{\beta}{2}\big\|u-\bar{u}^s_i\big\|^2_{L^2(\Omega)}.$$

With these notations, one has

$$F^s_i(u)=G^s(u)+\alpha J_i(u),$$
$$F^s_{i,\varepsilon}(u)=G^s_{i,\varepsilon}(u)+\alpha J_i(u),$$

for $i=1,2$, and the penalized minimization problem $(\mathcal{P}^s_{i,\varepsilon})$ is the problem of minimizing the functional $F^s_{i,\varepsilon}$ over all functions $u\in\mathcal{B}_\eta\cap V_i$. Denote by $u^s_{i,\varepsilon}$ a solution of the minimization problem $(\mathcal{P}^s_{i,\varepsilon})$, for $i=1,2$.

The functional $F^s_{i,\varepsilon}$ is not differentiable, due to the term J_i that involves a total variation. However the functional $G^s_{i,\varepsilon}$ is differentiable, for $i=1,2$, and

$$\nabla G^s_{i,\varepsilon}(u)=\nabla G^s(u)+q_\varepsilon(u)+\beta\big(u-\bar{u}^s_i\big),$$

for the pivot space L^2, where

$$q_\varepsilon(u)=\frac{(u-u^2)(1-2u)}{\varepsilon}, \tag{28}$$

for every $u\in V_i$.

In what follows, we first provide an expression of the differential of G^s in terms of Fourier transform (Section 3.3.1), and then using the fractional Laplacian operator or the fractional Dirichlet Laplacian operator (Section 3.3.2). Then, using results of nonsmooth analysis to deal with the nonsmooth character of the functional J_i, we derive first order necessary conditions in terms of optimality systems for both penalized minimization problems $(\mathcal{P}^s_{i,\varepsilon})$, $i = 1, 2$ (Sections 3.3.3 and 3.3.4).

3.3.1. Computation of ∇G^s in terms of Fourier transform

Recall that $G^s(u) = \frac{1}{2}\|Hu - v_d\|^2_{Y_s}$, with Y_s defined by (25) and (27). Using the results recalled in Appendix A (Appendices A.2.1 and A.2.2), for every $s \in [0, 1)$, for every $u \in Y_s$, the function $Hu - v_d$ can be extended by 0 to a function of $L^2(-a, a; H^s(\mathbb{R}))$, and its norm can be computed in terms of Fourier transform, by

$$G^s(u) = \frac{1}{2}\int_{-a}^{a}\int_{\mathbb{R}} \left|\mathcal{F}_y(Hu - v_d)(\xi, z)\right|^2 \left(1 + \xi^2\right)^s d\xi\, dz$$

$$= \frac{1}{2}\left\|\mathcal{F}_y(Hu - v_d)\right\|^2_{L^2(\omega_s)},$$

where $L^2(\omega_s)$ is the weighted Hilbertian space of all complex valued functions f defined on $\mathbb{R} \times (-a, a)$ such that

$$\int_{\mathbb{R}\times(-a,a)} \left|f(\xi, z)\right|^2 \omega_s(\xi)\, d\xi\, dz < +\infty,$$

where

$$\omega_s(\xi) = \left(1 + \xi^2\right)^s. \tag{29}$$

Setting $w_d = H^{-1}(v_d)$, we get finally

$$\nabla G^s(u) = (\mathcal{F}_y H)^* \omega_s (\mathcal{F}_y H)(u - w_d), \tag{30}$$

with L^2 as a pivot space.

To make this expression more explicit, we next compute the Fourier transform of the blurred projection operator H. Recall that $\tilde{v}$ denotes the extension by 0 to $\mathbb{R}^2$ of any function v and that, from Lemma 4,

$$(\mathcal{F}_y \widetilde{H_0 u})(\xi, z) = 2\pi \int_0^a r u(r, z) J_0(2\pi\xi r)\, dr,$$

for every $u \in L^1(\Omega)$, every $\xi \in \mathbb{R}$ and almost every $z \in (-a, a)$, where J_0 is the Bessel function of the first kind, and

$$\left((\mathcal{F}_y \widetilde{H_0})^* v\right)(r, z) = 2\pi r \int_{\mathbb{R}} v(\xi, z) J_0(2\pi\xi r)\, d\xi,$$

for every $v \in L^1(\mathbb{R}^2)$, every $r \in [0,a)$ and almost every $z \in (-a,a)$. As explained in the introduction, the radiography is blurred, and the blur is modeled by a linear operator B writing as a convolution with a positive symmetric kernel K (in practice, a Gaussian kernel) with compact support.

Lemma 7. *The Fourier transform of the blurred projection operator $H = BH_0$ with respect to the first variable is*

$$(\mathcal{F}_y B\widetilde{H_0 u})(\xi,z) = (\mathcal{F}_y K)(\xi,\cdot) \star_2 (\mathcal{F}_y \widetilde{H_0 u})(\xi,\cdot)(z), \tag{31}$$

for every $u \in L^1(\Omega)$, every $\xi \in \mathbb{R}$ and almost every $z \in (-a,a)$, where the notation $\star_2$ stands for the convolution product with respect to the second variable. Its adjoint (with L^2 as a pivot space) is

$$\big((\mathcal{F}_y B\widetilde{H_0})^* v\big)(r,z) = (\mathcal{F}_y \widetilde{H_0})^* (\mathcal{F}_y g \star_2 v)(r,z), \tag{32}$$

for every $v \in L^1(\mathbb{R}^2)$, every $r \in [0,a)$ and almost every $z \in (-a,a)$.

Proof. Since $B\widetilde{H_0 u} = K \star (\widetilde{H_0 u})$, one computes

$$\begin{aligned}
(\mathcal{F}_y B\widetilde{H_0 u})(\xi,z) &= \iiint_{\mathbb{R}^3} K(y-x,z-s)(\widetilde{H_0 u})(x,s)e^{-2i\pi\xi y}\,dy\,dx\,ds \\
&= \iint_{\mathbb{R}^2} (\widetilde{H_0 u})(x,s)e^{-2i\pi\xi x}\left(\int_{\mathbb{R}} K(y-x,z-s)e^{-2i\pi\xi(y-x)}\,dy\right)dx\,ds \\
&= \iint_{\mathbb{R}^2} (\mathcal{F}_y K)(\xi,z-s)(\widetilde{H_0 u})(x,s)e^{-2i\pi\xi x}\,dx\,ds \\
&= \int_{\mathbb{R}} (\mathcal{F}_y K)(\xi,z-s)(\mathcal{F}_y H_0 u)(\xi,s)\,ds \\
&= (\mathcal{F}_y K)(\xi,\cdot) \star_2 (\mathcal{F}_y H_0 u)(\xi,\cdot)(z),
\end{aligned}$$

for every $u \in L^1(\Omega)$, every $\xi \in \mathbb{R}$ and almost every $z \in (-a,a)$. Let us now compute the adjoint. For every $v \in L^1(\mathbb{R}^2)$ and every $u \in L^\infty(\Omega)$, we have

$$\begin{aligned}
&\big\langle (\mathcal{F}_y B\widetilde{H_0})^* v, u\big\rangle = \langle v, \mathcal{F}_y B\widetilde{H_0 u}\rangle \\
&\quad= \iint_{\mathbb{R}^2} v(\xi,z)(\mathcal{F}_y B\widetilde{H_0 u})(\xi,z)\,d\xi\,dz \\
&\quad= \iiint_{\mathbb{R}^3} v(\xi,z)(\mathcal{F}_y K)(\xi,z-s)(\mathcal{F}_y \widetilde{H_0 u})(\xi,s)\,ds\,d\xi\,dz \\
&\quad= 2\pi \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-a}^{a}\int_{0}^{a} v(\xi,z)(\mathcal{F}_y K)(\xi,z-s)\,r u(r,s) J_0(2\pi\xi r)\,dr\,ds\,d\xi\,dz
\end{aligned}$$

$$= \int_0^a \int_{-a}^a 2\pi r u(r,s) \left(\iint_{\mathbb{R}^2} v(\xi,z)(\mathcal{F}_y K)(\xi, z-s) J_0(2\pi\xi r)\, d\xi\, dz \right) ds\, dr,$$

and hence, we infer that

$$(\mathcal{F}_y B \widetilde{H_0})^* v(r,s) = 2\pi r \int_{\mathbb{R}} J_0(2\pi\xi r) \left(\int_{\mathbb{R}} (\mathcal{F}_y K)(\xi, z-s) v(\xi,z)\, dz \right) d\xi.$$

Since the kernel K of B is symmetric, it follows that

$$(\mathcal{F}_y B \widetilde{H_0})^* v(r,s) = 2\pi r \int_{\mathbb{R}} J_0(2\pi\xi r) \big((\mathcal{F}_y K)(\xi,\cdot) \star_2 v(\xi,\cdot) \big)(s)\, d\xi.$$

The formula (32) then follows from (8). $\square$

3.3.2. Computation of ∇G^s in terms of fractional Laplacian

By definition, there holds

$$G^s(u) = \frac{1}{2} \int_{-a}^a \left\| (Hu - v_d)(\cdot, z) \right\|^2_{H_0^s(-a,a)} dz,$$

for every $u \in Y_s$, and every $s \in [0,1)$, $s \neq 1/2$. For $s = 1/2$, $H_0^s(-a,a)$ is replaced with $H_{00}^{1/2}(-a,a)$. Using the results of Appendix A.2.3, it follows that, for every $f \in H_0^s(-a,a)$ whenever $s \in [0,1)$, $s \neq 1/2$, or $f \in H_{00}^{1/2}(-a,a)$ whenever $s = 1/2$, the norm of f within these spaces is equivalent to

$$\left(\|f\|^2_{L^2(U)} + \left\| (-\Delta)^{s/2} f \right\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2},$$

where f is extended by 0 outside $(-a,a)$ (notice that $(-\Delta)^{s/2} f$ is not of compact support). Here, $(-\Delta)^\alpha$ is the fractional Laplacian operator on $\mathbb{R}^n$, defined, using the Fourier transform $\mathcal{F}f$ of f, by $(-\Delta)^\alpha f = \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F} f)$ (see Appendix A.1.2). It follows easily that

$$\nabla G^s(u) = R_{\Omega_1} \big(\mathrm{id} + (-\Delta)^s \big) (\widetilde{Hu} - \tilde{v}_d), \tag{33}$$

with L^2 as a pivot space, where $\widetilde{Hu} - \tilde{v}_d$ is the extension of $Hu - v_d$ by 0 outside $(-a,a)$, and R_{Ω_1} is the restriction to Ω_1.

Another possibility is to express the differential of G^s using the fractional Dirichlet Laplacian operator A defined in Appendix A.2.4. We get, similarly,

$$\nabla G^s(u) = \big(\mathrm{id} + A^s \big) (Hu - v_d), \tag{34}$$

with L^2 as a pivot space.

3.3.3. Optimality system of $(\mathcal{P}^s_{1,\varepsilon})$

In this case the regularization term is the total variation semi-norm. First of all, recall the next result of [8] that will be used to tackle the nonsmooth character of the total variation, and then to derive optimality conditions for the penalized problem $\mathcal{P}^s_{1,\varepsilon}$.

Theorem 6. *(See [8, Theorem 2.3].) Let A be a Borelian subset of $\mathbb{R}^n$. Let $\bar{u} \in K \cap BV(A)$ be the solution of*

$$\begin{cases} \min \mathcal{J}(u) + \alpha \int_A |Du|, \\ u \in K \cap BV(A), \end{cases}$$

where K is a closed convex subset of $L^p(A)$ and $\mathcal{J}$ is continuous and Gâteaux differentiable from $L^p(A)$ to $\mathbb{R}$ $(1 \leqslant p < +\infty)$, and either K is bounded or $\mathcal{J}$ is coercive. Then, there exists $\bar{\lambda} \in (\mathcal{M}(A)^n)'$ (the dual space of Radon measures) such that

$$\big\langle \mathcal{J}'(\bar{u}) - \alpha\, div\, \bar{\lambda}, u - \bar{u} \big\rangle \geqslant 0, \tag{35}$$

for every $u \in K \cap BV(A)$, and

$$\langle \bar{\lambda}, \mu - D\bar{u} \rangle + \int_\Omega |D\bar{u}| \leqslant \int_\Omega |\mu|, \tag{36}$$

for every $\mu \in (\mathcal{M}(A))^n$, where $D : BV(A) \to (\mathcal{M}(A))^n$ and

$$\langle div\, \bar{\lambda}, u \rangle = -\langle \bar{\lambda}, Du \rangle, \tag{37}$$

for every $u \in BV(A)$.

This result cannot be applied to the original problem $(\mathcal{P}^s_1)$ since the set of constraints is not convex, but can be used to handle the penalized problem $(\mathcal{P}^s_{1,\varepsilon})$. The proof is similar to the one of [8], and yields the existence of $\lambda^\varepsilon \in (\mathcal{M}(\Omega)^2)'$ such that

$$\big\langle \nabla G^s_{1,\varepsilon}\big(u^\varepsilon\big) - \alpha \operatorname{div} \lambda^\varepsilon,\ u - u^\varepsilon \big\rangle \geqslant 0, \tag{38}$$

for every $u \in \mathcal{B}_\eta$, and

$$\big\langle \lambda^\varepsilon, \mu - Du^\varepsilon \big\rangle + \int_\Omega \big|Du^\varepsilon\big| \leqslant \int_\Omega |\mu|, \tag{39}$$

for every $\mu \in (\mathcal{M}(\Omega))^2$. Considering $\mu = Dv$ with $v \in BV(\Omega)$ in (39) leads to

$$\big\langle \lambda^\varepsilon, D\big(v - u^\varepsilon\big)\big\rangle + \int_\Omega \big|Du^\varepsilon\big| \leqslant \int_\Omega |Dv|,$$

for every $v \in BV(\Omega)$, that is,

$$\Phi(v) \geqslant \Phi\big(u^\varepsilon\big) - \big\langle \mathrm{div}\lambda^\varepsilon, v - u^\varepsilon\big\rangle,$$

which is equivalent to $\mu^\varepsilon \in \partial\Phi(u^\varepsilon)$, where $\mu^\varepsilon = -\mathrm{div}\,\lambda^\varepsilon$. We thus get the following result.

Theorem 7. *Let $u^s_{1,\varepsilon}$ be a solution of $(\mathcal{P}^s_{1,\varepsilon})$. Then there exist $\lambda^\varepsilon \in (\mathcal{M}(\Omega)^2)'$, $q^\varepsilon = q^\varepsilon(u^s_{1,\varepsilon}) \in L^\infty(\Omega)$ defined by* (28), *and $\mu^\varepsilon = -div\,\lambda^\varepsilon$ such that*

$$\big\langle \nabla G^s\big(u^s_{1,\varepsilon}\big) + q^\varepsilon + \alpha\mu^\varepsilon, u - u^s_{1,\varepsilon}\big\rangle \geqslant 0, \tag{40}$$

for every $u \in \mathcal{B}_\eta$, and

$$\mu^\varepsilon \in \partial\Phi\big(u^s_{1,\varepsilon}\big). \tag{41}$$

3.3.4. Optimality system of $(\mathcal{P}^s_{2,\varepsilon})$

In this case,

$$J_2(u) = \frac{1}{2}\|u\|^2_X = \frac{1}{2}\int_{-a}^{a}\big(|Du_z|(0,a)\big)^2\,dz = \frac{1}{2}\int_{-a}^{a}\big(\varphi(u_z)\big)^2\,dz,$$

for every $u \in X = L^2(-a,a;BV_0(0,a))$. Here and in the sequel, the notation $\varphi(f)$ is used to denote the total variation of a function $f \in BV(0,a)$. There holds $X' = L^2(-a,a;(BV_0(0,a))')$. For every $\lambda \in X'$, viewed as function of $z \in (-a,a)$ of class L^2 with values in $(BV_0(0,a))'$, denote $\lambda_z = \lambda(z) \in (BV_0(0,a))'$, for almost every $z \in (-a,a)$. The duality product between X and X' is defined by

$$\langle\lambda, v\rangle_{X',X} = \int_{-a}^{a}\langle\lambda_z, v_z\rangle_{BV_0',BV_0}\,dz,$$

for every $\lambda \in X'$ and every $v \in X$.

Lemma 8. *The functional J_2 is convex and locally Lipschitzian on X.*

Proof. The convexity is obvious. To establish the local Lipschitzian property, we use the fact that the total variation φ is Lipschitzian and the Cauchy–Schwarz inequality, getting the estimate

$$\begin{aligned}\big|J_2(v) - J_2(u)\big| &\leqslant \frac{1}{2}\int_{-a}^{a}\big(\varphi(u_z) + \varphi(v_z)\big)\big|\varphi(u_z) - \varphi(v_z)\big|\,dz \\ &\leqslant \big(\|u\|_X + \|v\|_X\big)\|u - v\|_X \\ &\leqslant \big(2\|u\|_X + \rho\big)\|u - v\|_X,\end{aligned}$$

for all $u, v \in X$ such that $\|u - v\|_X \leqslant \rho$. □

It follows from this lemma that J_2 is subdifferentiable and that the classical subdifferential and the generalized Clarke subdifferential of J_2 coincide (see [9]). Moreover, the Clarke generalized directional derivative and the classical directional derivative coincide as well.

Recall that the penalization problem $(\mathcal{P}^s_{2,\varepsilon})$ consists in minimizing $G^s_{2,\varepsilon}(u) + \frac{\alpha}{2} J_2(u)$ over all functions $u \in \mathcal{B}_\eta$. This problem is equivalent to the minimization problem

$$\min_{u\in X} G^s_{2,\varepsilon}(u) + \frac{\alpha}{2} J_2(u) + \chi_{\mathcal{B}_\eta}(u),$$

where $\chi_{\mathcal{B}_\eta}$ is defined by $\chi_{\mathcal{B}_\eta}(u) = 0$ whenever $u \in \mathcal{B}_\eta$, and $\chi_{\mathcal{B}_\eta}(u) = +\infty$ else.

A necessary condition for $u^s_{2,\varepsilon}$ to be an optimal solution to $(\mathcal{P}^s_{2,\varepsilon})$ is

$$0 \in \partial\big(G^s_{2,\varepsilon} + \alpha J_2 + \chi_{\mathcal{B}_\eta}\big)\big(u^s_{2,\varepsilon}\big),$$

and hence, using the standard rules of the subdifferential calculus,

$$0 \in \nabla G^s_{2,\varepsilon}\big(u^s_{2,\varepsilon}\big) + \alpha \partial J_2\big(u^s_{2,\varepsilon}\big) + \partial \chi_{\mathcal{B}_\eta}\big(u^s_{2,\varepsilon}\big), \tag{42}$$

since the considered functions are convex and locally Lipschitzian (see [9]).

Lemma 9. *Let $\lambda \in X'$. Then, for every $u \in X$, $\lambda \in \partial J_2(u)$ if and only if $\lambda_z \in \varphi(u_z)\partial\varphi(u_z)$, for almost every $z \in (-a, a)$.*

Proof. The statement of this lemma is natural, and the proof is quite easy, however it is not possible to apply directly results of [9]. We next provide a proof for the convenience of the reader.

Since J_2 is a proper convex locally Lipschitzian function, $\partial J_2(u)$ is nonempty for every $u \in X$, and

$$\partial J_2(u) = \big\{\lambda \in X' \;\big|\; J_2'(u; v) \geqslant \langle \lambda, v\rangle_{X',X}\ \forall v \in X\big\},$$

where $J_2'(u; v)$ denotes the directional derivative at u in the direction v. In addition,

$$J_2'(u; v) = \sup\big\{\langle \lambda, v\rangle_{X',X} \;\big|\; \lambda \in \partial J_2(u)\big\}.$$

We next compute $J_2'(u; v)$, for all $u, v \in X$. One has

$$\begin{aligned}\frac{J_2(u+tv) - J_2(u)}{t} &= \frac{1}{2}\int_{-a}^{a} \frac{\varphi(u_z + tv_z)^2 - \varphi(u_z)^2}{t}\, dz \\ &= \frac{1}{2}\int_{-a}^{a} \big(\varphi(u_z + tv_z) + \varphi(u_z)\big)\left(\frac{\varphi(u_z + tv_z) - \varphi(u_z)}{t}\right) dz.\end{aligned} \tag{43}$$

By definition of the subdifferential, there holds

$$\frac{\varphi(u_z + tv_z) - \varphi(u_z)}{t} \geqslant \langle \mu, v_z\rangle_{BV_0', BV_0},$$

for every $\mu \in \partial\varphi(u_z)$, every $v_z \in BV_0(0,a)$, and almost every $z \in (-a,a)$. Since

$$\varphi(u_z + tv_z) - \varphi(u_z) \leqslant |t|\varphi(v_z),$$

we get

$$\left|\frac{\varphi(u_z + tv_z) - \varphi(u_z)}{t}\right| \leqslant \max\left(\varphi(v_z), \left|\langle \lambda_z, v_z\rangle_{BV_0', BV_0}\right|\right),$$

for every $t \in (0,1]$ and almost every $z \in (-a,a)$. Note that

$$\left|\langle \mu, v_z\rangle_{BV_0', BV_0}\right| \leqslant \|\mu\|_{BV_0'}\varphi(v_z),$$

and

$$\|\mu\|_{BV_0'} = \sup_{\|v\|_{BV_0} \leqslant 1} \langle \mu, v\rangle_{BV_0', BV_0},$$

for almost every $z \in (-a,a)$. Since $\mu \in \partial\varphi(u_z)$, it follows that

$$\langle \mu, v\rangle_{BV_0', BV_0} \leqslant \varphi(u_z + v) - \varphi(u_z) \leqslant \varphi(v),$$

for every $v \in BV_0(0,a)$. Hence, $\|\mu\|_{BV_0'} \leqslant 1$ and $|\langle \mu, v_z\rangle_{BV_0', BV_0}| \leqslant \varphi(v_z)$. Therefore,

$$\left|\frac{\varphi(u_z + tv_z) - \varphi(u_z)}{t}\right| \leqslant \varphi(v_z), \tag{44}$$

for every $t \in (0,1]$ and almost every $z \in (-a,a)$, and we infer that

$$\left|\varphi'(u_z; v_z)\right| \leqslant \varphi(v_z). \tag{45}$$

Moreover,

$$0 \leqslant \varphi(u_z + tv_z) + \varphi(u_z) \leqslant 2\varphi(u_z) + t\varphi(v_z),$$

hence, using (44),

$$\left|\left(\varphi(u_z + tv_z) + \varphi(u_z)\right)\left(\frac{\varphi(u_z + tv_z) - \varphi(u_z)}{t}\right)\right| \leqslant \left(2\varphi(u_z) + \varphi(v_z)\right)\varphi(v_z).$$

The function $z \mapsto (2\varphi(u_z) + \varphi(v_z))\varphi(v_z)$ is integrable on $(-a,a)$, since the functions $z \mapsto \varphi(u_z)^2$, $z \mapsto \varphi(v_z)^2$ and $z \mapsto \varphi(u_z)\varphi(v_z)$ are integrable (indeed, $u, v \in X$). Using (45), we infer that the function

$$z \mapsto \varphi(u_z)\varphi'(u_z; v_z) \tag{46}$$

is integrable on $(-a, a)$, for every $v \in X$. Therefore, applying the Lebesgue theorem to (43), we get

$$\lim_{t \to 0} \frac{J_2(u + tv) - J_2(u)}{t} = \int_{-a}^{a} \varphi(u_z)\varphi'(u_z; v_z)\, dz.$$

Finally

$$J_2'(u; v) = \int_{-a}^{a} \varphi(u_z)\varphi'(u_z; v_z)\, dz \geqslant \int_{-a}^{a} \big\langle \varphi(u_z)\lambda_z, v_z\big\rangle_{BV_0', BV_0}\, dz, \tag{47}$$

for every $\lambda_z \in \partial\varphi(u_z)$, since $\langle \lambda_z, v_z\rangle_{BV_0', BV_0} \leqslant \varphi'(u_z; v_z)$.

For every $u \in X$, define

$$\mathcal{E}(u) = \big\{\mu \in X' \;\big|\; \mu : z \mapsto \varphi(u_z)\lambda_z \text{ with } \lambda_z \in \partial\varphi(u_z) \text{ for a.e. } z \in (-a, a)\big\}.$$

We claim that $\mathcal{E}(u) = \partial J(u)$. Indeed, let us first prove that $\mathcal{E}(u) \subset \partial J(u)$. For every $\mu \in \mathcal{E}(u)$, there holds

$$\langle \mu, v\rangle_{X', X} = \int_{-a}^{a} \langle \mu_z, v_z\rangle_{BV_0', BV_0}\, dz = \int_{-a}^{a} \varphi(u_z)\langle \lambda_z, v_z\rangle_{BV_0', BV_0}\, dz,$$

for every $v \in X$. Since $\lambda_z \in \partial\varphi(u_z)$, using (47) we get that

$$\langle \mu, v\rangle_{X', X} \leqslant J'(u; v),$$

for every $v \in X$. This implies that $\mu \in \partial J(u)$ (see [9]), and the inclusion follows. The proof of the converse inclusion readily follows the one of [9, p. 77], and is thus skipped. The key point is the measurability of the function $z \mapsto \varphi(u_z)\varphi'(u_z; v_z)$, for every $v \in X$, which follows in particular from (46).

We have thus proved that every $\mu \in \partial J(u)$ is such that $\mu : z \mapsto \varphi(u_z)\lambda_z$, with $\lambda_z \in \partial\varphi(u_z)$. The lemma follows. □

We are now in a position to derive the optimality system of $(\mathcal{P}^s_{2,\varepsilon})$.

Theorem 8. *Let $u^s_{2,\varepsilon}$ be a solution of $(\mathcal{P}^s_{2,\varepsilon})$. Then, there exist $\mu^\varepsilon \in X'$, $q^\varepsilon = q^\varepsilon(u^s_{2,\varepsilon}) \in L^\infty(\Omega)$ defined by* (28)*, such that*

$$\big\langle \nabla G^s\big(u^s_{2,\varepsilon}\big) + q^\varepsilon + \alpha\mu^\varepsilon, u - u^s_{2,\varepsilon}\big\rangle_{X', X} \geqslant 0, \tag{48}$$

for every $u \in \mathcal{B}_\eta$, and

$$\mu^\varepsilon_z \in \varphi\big(\big(u^s_{2,\varepsilon}\big)_z\big)\partial\varphi\big(\big(u^s_{2,\varepsilon}\big)_z\big), \tag{49}$$

for almost every $z \in (-a, a)$.

Appendix A. Fractional order Hilbert spaces

In this appendix we gather different definitions and characterizations of fractional order Hilbert spaces, on $\mathbb{R}^n$ and on bounded subsets, not all of them being so standard. The main references are [3,7,11,20,21,25,26].

Let U be an open bounded subset of $\mathbb{R}^n$. For $k \in \mathbb{N}$, the Hilbert space $H^k(U)$ is defined as the space of all functions of $L^2(U)$, whose partial derivatives up to order k, in the sense of distributions, can be identified with functions of $L^2(U)$. Endowed with the norm

$$\|f\|_{H^k(U)} = \left(\sum_{|\beta| \leqslant k} \|D^\beta f\|^2_{L^p(U)} \right)^{1/2}$$

$H^k(U)$ is a Hilbert space. For $k = 0$, there holds $H^0(U) = L^2(U)$.

For $s \in (0,1)$, the fractional order Hilbert space $H^s(U)$ is defined as the space of all functions $f \in L^2(U)$ such that

$$\iint_{U \times U} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < +\infty.$$

Endowed with the norm

$$\|f\|_{H^s(U)} = \left(\|f\|^2_{L^2(U)} + \iint_{U \times U} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2},$$

$H^s(U)$ is a Hilbert space.

For a positive noninteger real number $s > 0$, denote by $[s]$ the floor of s, and let $\alpha \in (0,1)$ such that $s = [s] + \alpha$. The fractional order Hilbert space $H^s(U)$ is defined as the space of all functions $f \in L^2(U)$, whose partial derivatives of order $[s]$, in the sense of distributions, can be identified with functions of $H^\alpha(U)$. Endowed with the norm

$$\|f\|_{H^s(U)} = \left(\|f\|^2_{H^{[s]}(U)} + \sum_{|\beta| = [s]} \iint_{U \times U} \frac{|D^\beta f(x) - D^\beta f(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy \right)^{1/2},$$

$H^s(U)$ is a Hilbert space.

Let $\mathcal{D}(U)$ denote the space of C^∞ functions on U, having a compact support contained in U. For every $s \geqslant 0$, define $H^s_0(U)$ as the closure of $\mathcal{D}(U)$ in $H^s(U)$. The space $H^s_0(U)$ is a closed subspace of $H^s(U)$ and thus inherits of its Hilbertian structure. There holds $H^s_0(U) = H^s(U)$ if and only if $0 \leqslant s \leqslant 1/2$ (see [21, Theorem 11.1]), or whenever $U = \mathbb{R}^n$. The space $H^{-s}(U)$ is defined as the dual of $H^s_0(U)$.

It is possible to define the fractional order Hilbert spaces $H^s(U)$ in other equivalent ways, in particular, using the Fourier transform or using the fractional Laplacian operator. The situation is quite simple for $U = \mathbb{R}^n$ but is more intricate for a bounded domain U.

A.1. Other characterizations for $U = \mathbb{R}^n$

A.1.1. Fourier transform

Another possible definition of $H^s(\mathbb{R}^n)$ goes by using the Fourier transform $\mathcal{F}$, as follows. For every $s > 0$, define

$$H^s(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) \mid (1 + |\xi|^2)^{s/2} \mathcal{F}f \in L^2(\mathbb{R}^n) \},$$

endowed with the norm

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s \left|\mathcal{F}(f)(\xi)\right|^2 d\xi \right)^{1/2}.$$

Note that $f \in H^s(\mathbb{R}^n)$ if and only if $(\mathrm{id} - \Delta)^{s/2} f \in L^2(\mathbb{R}^n)$, where the operator $(\mathrm{id} - \Delta)^{s/2}$ is defined by its symbol $(1 + |\xi|^2)^{s/2}$, or, in other words, is defined using the Fourier transform by $(\mathrm{id} - \Delta)^{s/2} f = \mathcal{F}^{-1}((1 + |\xi|^2)^{s/2} \mathcal{F}f)$.

Note that, for $s < 0$, one has

$$H^s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid (1 + |\xi|^2)^{s/2} \mathcal{F}f \in L^2(\mathbb{R}^n) \},$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}^n$.

A.1.2. Fractional Laplacian operator

Define the fractional Laplacian operator $(-\Delta)^\alpha$, using the Fourier transform $\mathcal{F}f$ of f, by $(-\Delta)^\alpha f = \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F}f)$. This definition actually makes sense for $\alpha \in (-n/2, 1]$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Note that $(-\Delta)^\alpha f \notin \mathcal{S}$ since $|\xi|^{2\alpha}$ introduces a singularity at the origin in its Fourier transform; however, $(-\Delta)^\alpha f$ is of class C^∞ (see e.g. [7]). Clearly, $(-\Delta)^1 = -\Delta$, $(-\Delta)^0 = \mathrm{id}$, and $(-\Delta)^{\alpha_1} \circ (-\Delta)^{\alpha_2} = (-\Delta)^{\alpha_1 + \alpha_2}$. Moreover, the operator $(-\Delta)^\alpha$ is selfadjoint on $L^2(\mathbb{R}^n)$.

An easy computation shows that, for every $\alpha \in (0, 1)$, there exists a constant $C_{n,\alpha}$ such that, for every $f \in \mathcal{S}(\mathbb{R}^n)$, $(-\Delta)^\alpha f(x)$ coincides with the principal value of the singular integral

$$C_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+2\alpha}} \, dy.$$

Actually, $C_{n,\alpha}$ is a positive constant such that

$$\int_{\mathbb{R}^n} \frac{1 - \cos(\xi . y)}{|y|^{n+2\alpha}} \, dy = \frac{|\xi|^{2\alpha}}{C_{n,\alpha}},$$

for every $\xi \in \mathbb{R}^n$.

Note that the above singular integral is well defined whenever $0 < \alpha < 1/2$, and in that case it is not necessary to consider the principal value; for $1/2 \leqslant \alpha < 1$, the singularity is near $x = y$.

To avoid the use of principal values, one has the other equivalent expression (obvious to obtain with a change of variable)

$$(-\Delta)^\alpha f(x) = -\frac{1}{2} C_{n,\alpha} \int\limits_{\mathbb{R}^n} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+2\alpha}}\, dy.$$

Then, for every $s \in (0,1)$ and every $f \in H^s(\mathbb{R}^n)$, one easily gets

$$\begin{aligned} \|f\|^2_{H^s(\mathbb{R}^n)} &= \|f\|^2_{L^2(\mathbb{R}^n)} + \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &= \|f\|^2_{L^2(\mathbb{R}^n)} + \frac{2}{C_{n,s}} \big\|(-\Delta)^{s/2} f\big\|^2_{L^2(\mathbb{R}^n)}, \end{aligned}$$

and therefore $H^s(\mathbb{R}^n)$ can be equivalently defined as the space of all functions of $L^2(\mathbb{R}^n)$ such that the distribution $(-\Delta)^{s/2} f$ can be identified with a function of $L^2(\mathbb{R}^n)$.

The relation of the Hilbert spaces H^s with the domains of the fractional Laplacian operator is the following. The operator $-\Delta$, defined by Fourier transform, is a selfadjoint positive operator on $L^2(\mathbb{R}^n)$, of domain $D(-\Delta) = H^2(\mathbb{R}^n)$. The fractional operator $(-\Delta)^s$ has been defined above by Fourier transform, for $s > 0$. Using the interpolation theory and results from [20,21], one can establish that

$$D\big((-\Delta)^s\big) = H^{2s}\big(\mathbb{R}^n\big),$$

for every $s \in [0,1]$.

A.2. *Other characterizations on a bounded domain*

The situation on a bounded subset U of $\mathbb{R}^n$ is more delicate.

A.2.1. *Quotient norm and extensions*

The space $H^s(U)$ can be as well defined as the set of restrictions of functions of $H^s(\mathbb{R}^n)$ to U, with the quotient norm

$$\|f\|_{H^s(U)} = \inf\big\{\|\tilde f\|_{H^s(\mathbb{R}^n)} \;\big|\; \tilde f \in H^s\big(\mathbb{R}^n\big),\ \tilde f_{|U} = f\big\}.$$

If U is bounded with a smooth boundary, then, for every $f \in L^2(U)$ such that

$$\iint\limits_{U\times U} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < +\infty,$$

there exists an extension $\tilde f \in L^2(\mathbb{R}^n)$ of f (defined by symmetry, locally at the boundary of the domain) for which

$$\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\tilde f(x)-\tilde f(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < +\infty$$

(see [25, Lemma 36.1]). This extension by symmetry is a specific one. Concerning extensions outside U, note that, for U bounded with smooth boundary, for every $s \in \mathbb{R}$ (H^s with $s < 0$ is defined further), there exists a continuous linear extension mapping $P_U \in L(H^s(U), H^s(\mathbb{R}^n))$, satisfying $P_U f_{|U} = f$ a.e. for every $f \in H^s(U)$, and a continuous restriction mapping $R_U \in L(H^s(\mathbb{R}^n), H^s(U))$, such that $R_U P_U = \mathrm{id}_{H^s(U)}$. Hence, $H^s(U)$ can be as well defined using the equivalent norm

$$\left(\|f\|^2_{L^2(U)} + \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|P_U f(x) - P_U f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}.$$

It would be more interesting (in view of using the fractional Laplacian, see further) to extend f by 0 outside U and to use the above double integral on $\mathbb{R}^n \times \mathbb{R}^n$. The extension by 0 is however more delicate. Denoting $\tilde{f}$ the extension of f by 0 outside U, the linear mapping $f \in H^s(U) \mapsto \tilde{f} \in H^s(\mathbb{R}^n)$ is well defined and continuous if and only if $0 \leqslant s < 1/2$. This means that, for $0 \leqslant s < 1/2$, the space $H^s(U)$ can be as well defined using the equivalent norm

$$\left(\|f\|^2_{L^2(U)} + \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2},$$

where f is extended by 0 outside U (but this fact is not true for $s \geqslant 1/2$ because of phenomena on the boundary).

Actually, the following result holds true [25, Lemma 37.1]. Denoting $\tilde{f}$ the extension of f by 0 outside U, for U bounded with Lipschitz boundary, for $0 < s < 1$, $\tilde{f} \in H^s(\mathbb{R}^n)$ if and only if $f \in H^s(U)$ and $\rho^{-s} f \in L^2(U)$, where ρ denotes the distance to the boundary of U.

The following extension result holds for the space $H^s_0(U)$. Denoting $\tilde{f}$ the extension of f by 0 outside U, for $s \geqslant 0$, the linear mapping $f \in H^s_0(U) \mapsto \tilde{f} \in H^s(\mathbb{R}^n)$ is well defined and continuous if and only if $s \notin \mathbb{N} + 1/2$.[7]

It follows that, for instance, for every $s \in (0,1)$, $s \neq 1/2$, the space $H^s_0(U)$ can be as well defined using the equivalent norm

$$\left(\|f\|^2_{L^2(U)} + \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2},$$

where f is extended by 0 outside U (but this fact is not true for $s = 1/2$). Actually, for $s = 1/2$, the extension by 0 is linear and continuous for a subspace of $H^{1/2}(U)$, which is next defined as $H^{1/2}_{00}(U)$.

[7] For instance, $1 \in H^{1/2}_0(U) = H^{1/2}(U)$ and the extension by 0 is piecewise smooth and discontinuous, hence is not in $H^{1/2}(\mathbb{R})$. Indeed, although functions of $H^{1/2}$ are not continuous in general, piecewise smooth functions that are discontinuous at one point do not belong to $H^{1/2}$. For example, consider f, a Heaviside function that is multiplied by some smooth plateau function; then, $f' = \delta_0 + \psi$ with ψ smooth, hence $i\xi(\mathcal{F}u)(\xi) = 1 + (\mathcal{F}\psi)(\xi)$, so that $|\mathcal{F}f|$ behaves like $1/|\xi|$ at infinity, and hence $(1+|\xi|^2)^{1/4}|\mathcal{F}f| \notin L^2(\mathbb{R})$, i.e., $f \notin H^{1/2}(\mathbb{R})$ (see [25, Chapter 33]).

A.2.2. The Lions–Magenes space $H_{00}^{1/2}(U)$

The Lions–Magenes space $H_{00}^{1/2}(U)$ is defined as the set of functions $f \in H^{1/2}(U)$ such that $\rho^{-1/2} f \in L^2(U)$, where ρ denotes the distance to the boundary of U (see [21, Theorem 11.7]). It is a strict subspace of $H^{1/2}(U) = H_0^{1/2}(U)$, equipped with the Hilbertian norm

$$\left(\|f\|_{H^{1/2}(U)}^2 + \left\| \frac{f}{\sqrt{d(\cdot, \partial U)}} \right\|_{L^2(U)}^2 \right)^{1/2}.$$

Equivalently, $H_{00}^{1/2}(U)$ is the subspace of functions $f \in H^{1/2}(U)$ such that their extension $\tilde{f}$ by 0 outside U belongs to $H^{1/2}(\mathbb{R}^n)$, and the space $H_{00}^{1/2}(U)$ can be endowed with the equivalent norm $\|\tilde{f}\|_{H^{1/2}(\mathbb{R}^n)}$ (see [25, Chapter 33]); for instance this latter norm can be computed by Fourier transform.

A.2.3. Fractional Laplacian operator

For U bounded with a smooth boundary, the relation between $\|f\|_{H^s(U)}$ and $\|(-\Delta)^{\alpha/2} f\|_{L^2(\mathbb{R}^n)}^2$ is more intricate than in the case $U = \mathbb{R}^n$. First, for $0 \leqslant s < 1/2$, every $f \in H^s(U)$ can be extended by 0 outside U into a function of $H^s(\mathbb{R}^n)$, and hence for such values of s the space $H^s(U)$ can be as well defined using the equivalent norm

$$\|f\|_{H^s(U)} = \left(\|f\|_{L^2(U)}^2 + \left\|(-\Delta)^{s/2} f\right\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2},$$

where f is extended by 0 outside U. Note however that, although f has a compact support, the function $(-\Delta)^{s/2} f$ is not of compact support.

The same fact holds for the spaces $H_0^s(U)$, for $s \geqslant 0$, $s \notin \mathbb{N} + 1/2$, and for the space $H_{00}^{1/2}(U)$, since functions of these spaces can be extended by 0 to functions of $H^s(\mathbb{R}^n)$.

For other values of s (and actually, for every $s \in \mathbb{R}$), the existence of a continuous linear extension mapping P_U, previously mentioned, permits to endow $H^s(U)$, for instance, with the equivalent norm

$$\|f\|_{H^s(U)} = \left(\|f\|_{L^2(U)}^2 + \left\|(-\Delta)^{s/2} P_U f\right\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2}.$$

A.2.4. Relation with the domain of the fractional Dirichlet Laplacian operator

Let A denote the opposite of the Dirichlet Laplacian on $L^2(U)$, of domain $D(A) = H_0^1(U) \cap H^2(U)$. It must not be confused with the previous Laplacian operator. The operator A is positive, selfadjoint, and has a discrete spectrum. The domains of its real powers define a scale of Hilbert spaces $D(A^\alpha)$ (see [11]), which can be characterized as follows. Let $(e_n)_{n\in\mathbb{N}}$ denote an orthonormal basis of eigenvectors of A, and let $(\lambda_n)_{n\in\mathbb{N}}$ be the associated eigenvalues. Then,

$$D(A^\alpha) = \left\{ f \in L^2(U) \;\middle|\; \sum_{k\in\mathbb{N}} \lambda_k^{2\alpha} \langle f, e_k\rangle^2 < +\infty \right\},$$

for every $\alpha \in \mathbb{R}$. Using the interpolation theory of [21], one can establish that

$$\forall s \in [0,1) \quad D\big(A^s\big) = \begin{cases} H_0^{2s}(U) & \text{if } s \neq 1/4, \\ H_{00}^{1/2}(U) & \text{if } s = 1/4, \end{cases}$$

where the Lions–Magenes space $H_{00}^{1/2}(U)$ has been defined previously (note that, for $s = 1$, $D(A) = H_0^1(U) \cap H^2(U)$). In particular, for every $s \in [0,1]$,

$$\left\{ f \in L^2(U) \;\Big|\; \sum_{k \in \mathbb{N}} \lambda_k^s \langle f, e_k \rangle^2 < +\infty \right\} = \begin{cases} H_0^s(U) & \text{if } s \neq 1/2, \\ H_{00}^{1/2}(U) & \text{if } s = 1/2. \end{cases}$$

Note that $A^s f$ must be distinguished from $(-\Delta)^s f$, when both functions can be given a sense. For instance, for every $f \in H_0^s(U)$ with $s \in (0,1)$, $s \neq 1/2$ (and $f \in H_{00}^{1/2}(U)$ for $s = 1/2$), one has $f \in D(A^{s/2})$ and thus $A^{s/2} f \in L^2(U)$ by definition, whereas $(-\Delta)^s f \in L^2(\mathbb{R}^n)$ (where f is extended[8] by 0 outside U) is not even of compact support.

The space $D(A^{s/2})$ is a Hilbert space, when equipped with the graph norm $(\|f\|_{L^2(U)}^2 + \|A^{s/2} f\|_{L^2(U)}^2)^{1/2}$. It follows that $H_0^s(U)$, for $s \in (0,1)$, $s \neq 1/2$, can be equivalently defined with this graph norm, and similarly the space $H_{00}^{1/2}(U)$ can be endowed with the equivalent norm $(\|f\|_{L^2(U)}^2 + \|A^{1/4} f\|_{L^2(U)}^2)^{1/2}$.

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[8] Recall that functions of $H_0^s(U)$ can be extended by 0 to $H^s(\mathbb{R}^n)$ for every $s \geqslant 0$ such that $s \notin \mathbb{N} + 1/2$; for $s = 1/2$ for instance, functions of $H_{00}^{1/2}(U)$ can be extended by 0 to $H^{1/2}(\mathbb{R}^n)$.

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