

VALUE FUNCTION FOR REGIONAL CONTROL PROBLEMS VIA DYNAMIC PROGRAMMING AND PONTRYAGIN MAXIMUM PRINCIPLE

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ABSTRACT. In this paper we consider regional deterministic finite-dimensional optimal control problems, where the dynamics and the cost functional depend on the region of the state space where one is and have discontinuities at their interface.

Under the assumption that optimal trajectories have a locally finite number of switchings (i.e., no Zeno phenomenon), we use the duplication technique to show that the value function of the regional optimal control problem is the minimum over all possible structures of trajectories of value functions associated with classical optimal control problems settled over fixed structures, each of them being the restriction to some submanifold of the value function of a classical optimal control problem in higher dimension.

The lifting duplication technique is thus seen as a kind of desingularization of the value function of the regional optimal control problem.

In turn, we establish sensitivity relations for regional optimal control problems and we prove that the regularity of the value function of such problems is the same (i.e., is not more degenerate) than the one of the higher-dimensional classical optimal control problem that lifts the problem.

1. Introduction. We consider regional optimal control problems in finite dimension, the word “regional” meaning that the dynamics and the cost functional may depend on the region of the state space and therefore present discontinuities at the interface between these different regions. Our objective is to provide a description of these trajectories by exploiting the usual Pontryagin maximum principle and the Dynamic Programming approach (recall that the value function is the viscosity solution of the corresponding Hamilton-Jacobi equation). We establish a relationship between these two approaches, which is new in the context of regional control problems.

There is a wide existing literature on regional optimal control problems, which have been studied with different approaches and within various related contexts: *stratified optimal control problems* in [9, 11, 22], *optimal multiprocesses* in [17, 18];

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they also enter into the wider class of *hybrid optimal control* (see [10, 30, 33]). Necessary optimality conditions have been developed in [20, 21, 35] in the form of an appropriate Pontryagin maximum principle. For regional optimal control problems, the main feature is the jump of the adjoint vector coming from the Pontryagin maximum principle at the interface between two regions (see [21]). An alternative approach is the Bellman one, developed in [7, 8, 29] in terms of an appropriate Hamilton-Jacobi equation studied whose solutions are studied in the viscosity sense (see also [24, 26, 28] for transmission conditions at the interface).

In this paper we exploit both the Dynamic Programming approach and Pontryagin maximum principle in order to describe the optimal trajectories of regional control problems. Although the techniques are not new we believe that the approach is interesting and helpful.

We are going to use in an instrumental way the *lifting duplication technique*, nicely used in [19] in order to prove that the hybrid version of the Pontryagin maximum principle can be derived from the classical version (i.e., for classical, non-hybrid problems) under the assumption that optimal trajectories are regular enough. More precisely, we assume throughout that optimal trajectories have a *locally finite number of switchings*, or, in other words, we assume that wild oscillation phenomena (known as *Fuller*, *Robbins* or *Zeno* phenomena in the existing literature, see [13] for a survey) do not occur; or at least, if they happen then we deliberately ignore the corresponding wildly oscillating optimal trajectories and we restrict our search of optimal trajectories to those that have a regular enough structure, i.e., a locally finite number of switchings. Under this assumption, the duplication technique developed in [19] can be carried out and shows that the regional optimal control problem can be lifted to a higher-dimensional optimal control problem that is “classical”, i.e., non-regional. As we are going to see, this construction has a number of nice applications.

In order to point out the main ideas, we consider the following simplified framework with only two different regions. Let $N \in \mathbb{N}^*$. We assume that

$$\begin{aligned}\mathbb{R}^N &= \Omega_1 \cup \Omega_2 \cup \mathcal{H}, & \Omega_1, \Omega_2 \text{ open}, & \quad \Omega_1 \cap \Omega_2 = \emptyset \\ \mathcal{H} &= \partial\Omega_1 = \partial\Omega_2 \text{ is a } C^1 \text{ submanifold of } \mathbb{R}^N\end{aligned}$$

and we consider a regional optimal control problem, that is, a nonlinear optimal control problem in \mathbb{R}^N stratified according to the above partition:

$$\begin{aligned}\dot{\mathbb{X}}(t) &= f(\mathbb{X}(t), a(t)) \\ \mathbb{X}(t^0) &= x^0, \quad \mathbb{X}(t^f) = x^f \\ \inf \int_{t^0}^{t^f} \ell(\mathbb{X}(t), a(t)) dt\end{aligned}\tag{1}$$

where the dynamics f and the running cost ℓ are defined as follows. If $x \in \Omega_i$ for $i = 1$ or 2 then

$$f(x, a) = f_i(x, a), \quad \ell(x, a) = l_i(x, a),$$

where $f_i : \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}^N$ and $l_i : \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}$ are C^1 mappings. If $x \in \mathcal{H}$ then

$$f(x, a) = f_{\mathcal{H}}(x, a), \quad \ell(x, a) = l_{\mathcal{H}}(x, a)$$

where $f_{\mathcal{H}} : \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}^N$ and $l_{\mathcal{H}} : \mathbb{R}^N \times \mathbb{R}^m \rightarrow \mathbb{R}$ are C^1 mappings. The set \mathcal{H} is called the *interface* between the two open regions Ω_1 and Ω_2 (see Figures 1 and 2).

The class of controls that we consider also depends on the region. As long as $\mathbb{X}(\cdot)$ is in the region Ω_i , we assume that a is a L^∞ function of time with values in A_i , where A_i is a measurable subset of \mathbb{R}^m . Accordingly, as long as $\mathbb{X}(\cdot)$ is in \mathcal{H} , we assume that a is a L^∞ function of time with values in $A_{\mathcal{H}}$, where $A_{\mathcal{H}}$ is a measurable subset of \mathbb{R}^m .

The terminal times t^0 and t^f and the terminal points x^0 and x^f may be fixed or free according to the problem under consideration. For instance, if we fix x^0, t^0, x^f, t^f , we define the value function

$$S(x^0, t^0, x^f, t^f)$$

of the regional optimal control problem (1) as being the infimum of the cost functional over all possible admissible trajectories steering the control system from (x^0, t^0) to (x^f, t^f) .

Our objective is to show that the value function S of the regional optimal control problem (1) can be recovered from the study of a *classical* (i.e., non-hybrid) optimal control problem settled in higher dimension, under the assumption of finiteness of switchings. To this aim, we list all possible structures of optimal trajectories of (1). We recall that, for regional optimal control problems, existence of an optimal control and Cauchy uniqueness results are derived using Filippov-like arguments, allowing one to tackle the discontinuities of the dynamics and of the cost functional (see, e.g., [9, 11, 22]).

In what follows, we assume that the regional optimal control problem under consideration admits at least one optimal solution. We consider such an optimal trajectory $\mathbb{X}(\cdot)$ associated with a control $a(\cdot)$ on $[t^0, t^f]$. Assuming that $x^0 \in \Omega_1$ and that $x^f \in \Omega_2$, we consider various structures as explained next.

The simplest case is when the trajectory $\mathbb{X}(\cdot)$ consists of two arcs, denoted by $([t^0, t^1], X_1(\cdot))$ and $([t^1, t^f], X_2(\cdot))$, lying respectively in Ω_1 for the first part and in Ω_2 for the second part of the trajectory, with $X_1(t^1) = X_2(t^1) \in \mathcal{H}$ (the interface between the two regions). Such optimal trajectories are studied in [21] under the assumption of a *transversal crossing* and an explicit jump condition is derived in that reference for the adjoint vector obtained by the Pontryagin maximum principle. This is the simplest possible trajectory structure. We denote it by 1-2 (see Figure 1). It has only one switching.

The second structure is when the trajectory $\mathbb{X}(\cdot)$ consists of three arcs, denoted by $([t^0, t^1], X_1(\cdot))$, $([t^1, t^2], X_{\mathcal{H}}(\cdot))$ and $([t^2, t^f], X_2(\cdot))$, lying respectively in the region Ω_1 for the first arc, along the interface \mathcal{H} for the second (middle) arc and in the region Ω_2 for the third arc. Such a structure is denoted by 1- \mathcal{H} -2 (see Figure 2). The trajectory has two switchings.

Accordingly, we consider all possible structures 1-2- \mathcal{H} -1, 1- \mathcal{H} -1-2, 1-2- \mathcal{H} -2, etc, made of a finite number of successive arcs. Restricting ourselves to any such fixed structure, we can define a specific optimal control problem consisting of finding an optimal trajectory steering the system from the initial point x^0 to the desired target point x^f and minimizing the cost functional over all admissible trajectories having exactly such a structure. Denoting by $S_{12}, S_{1\mathcal{H}2}$, etc, the corresponding value functions, we have then

$$S = \inf\{S_{12}, S_{1\mathcal{H}2}, \dots\}$$

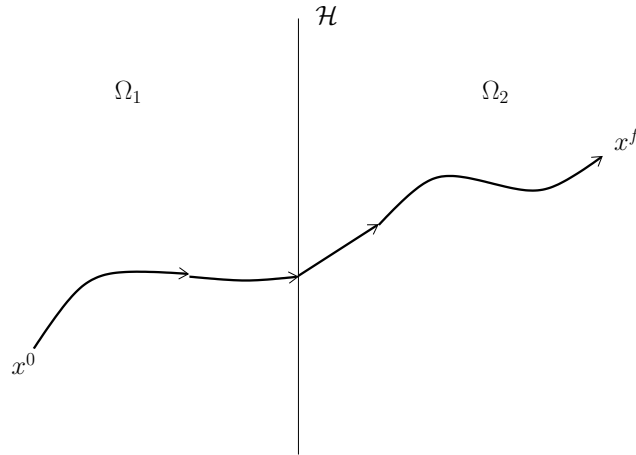
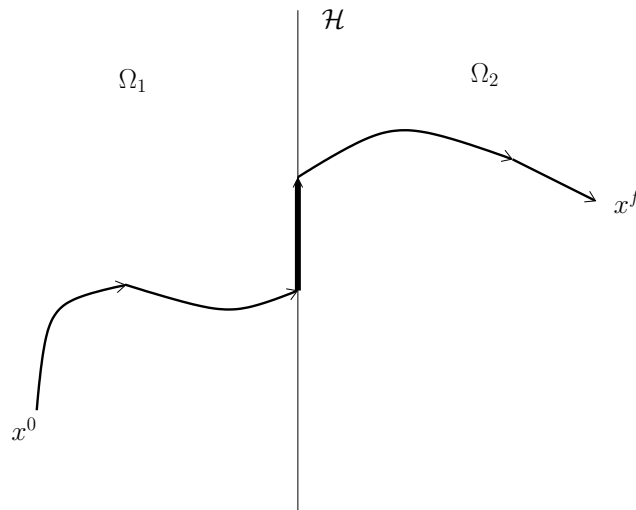


FIGURE 1. Structure 1-2.

FIGURE 2. Structure 1- \mathcal{H} -2.

provided all optimal trajectories of the initial regional optimal control problem have a locally finite number of switchings – and thus, the infimum above runs over a finite number of possibilities.

Using the duplication argument of [19], we will show that each of the above value functions (restricted to some fixed structure) can be written as the projection / restriction of the value function of a *classical* optimal control problem in higher dimension (say p , which is equal to the double of the number of switchings of the corresponding structure), the projection being considered along some coordinates, and the restriction being done to some submanifolds of the higher dimensional space \mathbb{R}^p . The word “duplication” reflects the fact that each arc of the trajectory gives two components of the dynamics of the problem in higher dimension.

Thanks to this technique, we characterize the value function S as a viscosity solution of an Hamilton-Jacobi equation and we apply the classical Pontryagin maximum principle. We thus provide an explicit relationship between the gradient of the value function S of the regional optimal control problem evaluated along the optimal trajectory and the adjoint vector. This sensitivity relation extends to the framework of regional optimal control problems the relation known in the classical framework. This fact allows us to derive conditions at the interface: continuity of the Hamiltonian and jump condition for the adjoint vector.

In Section 2 we provide the details of the lifting duplication procedure for the structures 1-2 and 1- \mathcal{H} -2. The construction is similar for other structures and consists of designing a duplicated problem of dimension two times the number of arcs of the structure.

The value function S of the regional optimal control problem is then the infimum of “classical” value functions associated with all possible structures, provided that optimal trajectories have a locally finite number of switchings. The latter assumption is required to apply the duplication technique.

However in general it may happen that the structure of switchings have a complex structure, even fractal, and thus the set of switching points may be countably or even uncountably infinite. In the context of hybrid optimal control problems, the *Zeno phenomenon* is a well known chattering phenomenon, meaning that the control switches an infinite number of times over a compact interval of times. It is analyzed for instance in [25, 39], and necessary and/or sufficient conditions for the occurrence of the Zeno phenomenon are provided in [3, 23]. However, we are not aware of any existing result providing sufficient conditions for hybrid optimal control problems under which the number of switchings of optimal trajectories is locally finite or even only countable. Anyway, although the Zeno phenomenon may occur in general, restricting the search of optimal strategies to trajectories having only a locally finite number of switchings is a reasonable assumption in practice in particular in view of numerical implementation (as done for instance in [13, 38, 40, 41]).

Under the local finiteness assumption on the number of switchings, it follows from our analysis that the regularity of the value function S of the regional optimal control problem is the same (i.e., is not more degenerate) than the one of the higher-dimensional classical optimal control problem lifting the initial problem. More precisely, we will prove that each value function $S_{12}, S_{1\mathcal{H}2}, \dots$, for each fixed structure, is the restriction to a submanifold of the value function of a classical optimal control problem in higher dimension.

Our main result, Theorem 2.2, gives a precise representation of the value function and of the corresponding sensitivity relations, in relation with the adjoint vector coming from the Pontryagin maximum principle. In particular, if for instance all classical value functions above are Lipschitz then the value function S of the regional optimal control problem is Lipschitz as well. This regularity result is new in the framework of hybrid or regional optimal control problems.

The paper is organized as follows.

In Section 2 we define the regional optimal control problem and we state the complete set of assumptions that we consider throughout. We analyze in detail the structures 1-2 and 1- \mathcal{H} -2 (the other cases being similar) by giving an explicit construction of the duplicated problem. As a result, we obtain the abovementioned representation of the value function of the regional optimal control problem and the consequences for its regularity.

In Section 3 we study in detail an example of a regional optimal control problem, having a structure 1- \mathcal{H} -2, modelling for instance the motion of a pedestrian walking in Ω_1 and Ω_2 and having the possibility of taking a tramway along \mathcal{H} at any point of this interface \mathcal{H} .

Section 4 gathers the proofs of all results stated in Section 2.

2. Value function for regional optimal control problems.

2.1. Problem and main assumptions. We assume that:

(H \mathcal{H}) $\mathbb{R}^N = \Omega_1 \cup \Omega_2 \cup \mathcal{H}$ with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\mathcal{H} = \partial\Omega_1 = \partial\Omega_2$ being a C^1 submanifold.

More precisely, there exists a function $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ of class C^1 such that

$$\mathcal{H} = \{x \in \mathbb{R}^N \mid \Psi(x) = 0\}$$

with $\nabla\Psi \neq 0$ on \mathcal{H} .

We consider the problem of minimizing the cost of trajectories going from (x^0, t^0) to (x^f, t^f) . These trajectories follow the dynamics f_i or $f_{\mathcal{H}}$ when they are respectively in Ω_i ($i = 1, 2$) or in \mathcal{H} with respective costs l_i on Ω_i and $l_{\mathcal{H}}$ on \mathcal{H} .

The tangent bundle of \mathcal{H} is $T\mathcal{H} = \bigcup_{z \in \mathcal{H}} (\{z\} \times T_z\mathcal{H})$ where $T_z\mathcal{H}$ is the tangent space to \mathcal{H} at z (which is isomorphic to \mathbb{R}^{N-1}). For $\phi \in C^1(\mathcal{H})$ and $x \in \mathcal{H}$, we denote by $\nabla_{\mathcal{H}}\phi(x)$ the gradient of ϕ at x , which belongs to $T_x\mathcal{H}$. The scalar product in $T_z\mathcal{H}$ is denoted by $\langle u, v \rangle_{\mathcal{H}}$. This definition makes sense if the two vectors u and v belong to $T_z\mathcal{H}$ and without ambiguity we will use the same notation when one of the vectors u or v is in \mathbb{R}^N . The notation $\langle u, v \rangle$ refers to the usual Euclidean scalar product in \mathbb{R}^N .

We make the following assumptions:

(Hg) Let \mathcal{M} be a submanifold of \mathbb{R}^N and A be a measurable subset of \mathbb{R}^m . Let $g : \mathcal{M} \times A \rightarrow \mathbb{R}^N$ be a continuous bounded function, with Lipschitz continuous derivatives with respect to the first variable. More precisely, we assume that there exists $M > 0$ such that for any $x \in \mathcal{M}$ and $\alpha \in A$,

$$|g(x, \alpha)| \leq M$$

and there exist $L, L^1 > 0$ such that for any $z, z' \in \mathcal{M}$ and $\alpha \in A$,

$$|g(z, \alpha) - g(z', \alpha)| \leq L |z - z'|,$$

$$\left| \frac{\partial}{\partial z_j} g(z, \alpha) - \frac{\partial}{\partial z_j} g(z', \alpha) \right| \leq L^1 |z - z'|, \quad j = 1, \dots, N.$$

(Hfl $_i$) Let $A_i, i = 1, 2$, be measurable subsets of \mathbb{R}^m . We assume that $f_i : \Omega_i \times A_i \rightarrow \mathbb{R}^N$ and $l_i : \Omega_i \times A_i \rightarrow \mathbb{R}$, ($i = 1, 2$) satisfy Assumption (Hg) for a suitable choice of positive constants M, L and L^1 .

(Hfl $_{\mathcal{H}}$) Let $A_{\mathcal{H}}$ be a measurable subset of \mathbb{R}^m . We assume that $(x, f_{\mathcal{H}}(x, a_{\mathcal{H}})) : \mathcal{H} \times A_{\mathcal{H}} \rightarrow T\mathcal{H}$ and $l_{\mathcal{H}} : \mathcal{H} \times A_{\mathcal{H}} \rightarrow \mathbb{R}$ satisfy Assumption (Hg) for a suitable choice of positive constants M, L and L^1 .

We consider optimal trajectories that are made of successive arcs staying in one of the regions Ω_1 or Ω_2 or along the interface \mathcal{H} and touching the boundary of Ω_1 or of Ω_2 only at initial or final time.

The problem in the region Ω_i (for $i = 1$ or 2). The trajectories $X_i : \mathbb{R}^+ \rightarrow \mathbb{R}^N$ are solutions of

$$\dot{X}_i(t) = f_i(X_i(t), \alpha_i(t)), \quad X_i(t) \in \Omega_i \quad \forall t \in (t^0, t^f) \tag{2}$$

$$X_i(t^0) = x^0, \quad X_i(t^f) = x^f \quad \text{with } x^0 \neq x^f \in \bar{\Omega}_i. \tag{3}$$

The value function $S_i : \bar{\Omega}_i \times \mathbb{R}^+ \times \bar{\Omega}_i \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is

$$S_i(x^0, t^0; x^f, t^f) = \inf \left\{ \int_{t^0}^{t^f} l_i(X_i(t), \alpha_i(t)) dt : X_i \text{ is solution of (2) - (3)} \right\}.$$

We define the Hamiltonian $\widetilde{H}_1 : \Omega_i \times \mathbb{R}^N \times \mathbb{R} \times A_i \rightarrow \mathbb{R}$ of the corresponding optimal control problem by

$$\widetilde{H}_1(X_i, Q_i, p^0, \alpha_i) = \langle Q_i, f_i(X_i, \alpha_i) \rangle + p^0 l_i(X_i, \alpha_i)$$

and the maximized Hamiltonian $H_1 : \Omega_i \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$H_1(X_i, Q_i, p^0) = \sup_{\alpha_i \in A_i} \widetilde{H}_1(X_i, Q_i, p^0, \alpha_i)$$

(see [1, 27, 36] for the statement and use of the classical Pontryagin maximum principle).

The problem along the interface \mathcal{H} . The trajectories $X_{\mathcal{H}} : \mathbb{R}^+ \rightarrow \mathcal{H}$ are solutions of

$$\dot{X}_{\mathcal{H}}(t) = f_{\mathcal{H}}(X_{\mathcal{H}}(t), a_{\mathcal{H}}(t)), \quad X_{\mathcal{H}}(t) \in \mathcal{H} \quad \forall t \in (t^0, t^f) \tag{4}$$

$$X_{\mathcal{H}}(t^0) = x^0, \quad X_{\mathcal{H}}(t^f) = x^f \quad \text{with } x^0 \neq x^f \in \mathcal{H}. \tag{5}$$

The value function $S_{\mathcal{H}} : \mathcal{H} \times \mathbb{R}^+ \times \mathcal{H} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is

$$S_{\mathcal{H}}(x^0, t^0; x^f, t^f) = \inf \left\{ \int_{t^0}^{t^f} l_{\mathcal{H}}(X_{\mathcal{H}}(t), a_{\mathcal{H}}(t)) dt : X_{\mathcal{H}} \text{ is solution of (4) - (5)} \right\}.$$

We define the Hamiltonian $\widetilde{H}_{\mathcal{H}} : T\mathcal{H} \times \mathbb{R} \times A_{\mathcal{H}} \rightarrow \mathbb{R}$ of the corresponding optimal control problem by

$$\widetilde{H}_{\mathcal{H}}(X_{\mathcal{H}}, Q_{\mathcal{H}}, p^0, a_{\mathcal{H}}) = \langle Q_{\mathcal{H}}, f_{\mathcal{H}}(X_{\mathcal{H}}, a_{\mathcal{H}}) \rangle_{\mathcal{H}} + p^0 l_{\mathcal{H}}(X_{\mathcal{H}}, a_{\mathcal{H}})$$

and the maximized Hamiltonian $H_{\mathcal{H}} : T\mathcal{H} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$H_{\mathcal{H}}(X_{\mathcal{H}}, Q_{\mathcal{H}}, p^0) = \sup_{a_{\mathcal{H}} \in A_{\mathcal{H}}} \widetilde{H}_{\mathcal{H}}(X_{\mathcal{H}}, Q_{\mathcal{H}}, p^0, a_{\mathcal{H}}).$$

2.2. Analysis of the structure 1-2. We describe here the simplest possible structure: trajectories consisting of two arcs living successively in Ω_1, Ω_2 and crossing the interface \mathcal{H} at some given time (see Figure 1). This case has already been studied in the literature. As explained in [17], the jump condition (10) hereafter is a rather straightforward generalization of the problem solved by Snell's Law. Besides, the Pontryagin maximum principle is also well established in this case. We recall it hereafter in detail because it is interesting to compare this result with the one obtained for more general structures (see Theorem 2.2 and Remark 3).

We make the following *transversal crossing* assumption:

- (H 1-2) There exist a time $t_c \in (t^0, t^f)$ and an optimal trajectory that starts from Ω_1 , stays in Ω_1 in the interval $[t^0, t_c]$, does not arrive tangentially at time t_c on \mathcal{H} and stays in Ω_2 on the interval $(t_c, t^f]$.

Such trajectories are described as follows: for each initial and final data $(x^0, t^0, x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times (0, +\infty)$, the trajectory is given by the vector $\mathbb{X}(t) = (X_1(t), X_2(t)) : \mathbb{R}^+ \rightarrow \mathbb{R}^N \times \mathbb{R}^N$ Lipschitz solution of the system

$$\begin{cases} \dot{X}_1(t) = f_1(X_1(t), \alpha_1(t)) & t \in (t^0, t_c) \\ \dot{X}_2(t) = f_2(X_2(t), \alpha_2(t)) & t \in (t_c, t^f) \end{cases} \quad (6)$$

completed with the mixed conditions

$$X_1(t_0) = x^0, \quad X_1(t_c) = X_2(t_c), \quad X_2(t^f) = x^f, \quad (7)$$

the non tangential conditions

$$\langle \nabla \Psi(X_1(t_c^-)), f_1(X_1(t_c^-), \alpha_1(t_c^-)) \rangle \neq 0 \quad \langle \nabla \Psi(X_2(t_c^+)), f_2(X_2(t_c^+), \alpha_2(t_c^+)) \rangle \neq 0 \quad (8)$$

and the state constraints

$$X_1(t) \in \Omega_1 \quad \forall t \in (t^0, t_c), \quad X_2(t) \in \Omega_2 \quad \forall t \in (t_c, t^f). \quad (9)$$

The cost of such a trajectory is

$$C(x^0, t^0; x^f, t^f; \mathbb{X}) = \int_{t^0}^{t_c} l_1(X_1(t), \alpha_1(t)) dt + \int_{t_c}^{t^f} l_2(X_2(t), \alpha_2(t)) dt.$$

Hence the value function $S_{1,2} : \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is given by

$$S_{1,2}(x^0, t^0; x^f, t^f) = \inf \left\{ C(x^0, t^0; x^f, t^f; \mathbb{X}) : \mathbb{X} \text{ is solution of (6)-(7)-(8)-(9), } t^f > t_c > t^0 \right\}.$$

Under the assumptions $(H\mathcal{H})$, (Hfl_i) , $(Hfl_{\mathcal{H}})$, the results of [20, 21] can be applied and we infer that, for any $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+$, we have

$$S_{1,2}(x^0, t^0; x^f, t^f) = \min \left\{ S_1(x^0, t^0; x_c, t_c) + S_2(x_c, t_c; x^f, t^f) : t^0 < t_c < t^f, x_c \in \mathcal{H} \right\}$$

where we recall that S_i is the value function of the problem restricted to the region Ω_i . Moreover, if $\mathbb{X}(\cdot)$ is an optimal trajectory for the value function $S_{1,2}(x^0, t^0; x^f, t^f)$ and $P(\cdot)$ is the corresponding adjoint vector given by the Pontryagin maximum principle, then we have the continuity condition

$$H_1(X_1(t_c^-), P(t_c^-), p^0) = H_2(X_2(t_c^+), P(t_c^+), p^0)$$

and the *jump condition* on the adjoint vectors

$$P_2(t_c^+) - P_1(t_c^-) = \frac{\langle P_1(t_c^-), f_1(t_c^-) - f_2(t_c^+) \rangle + p^0(l_1(t_c^-) - l_2(t_c^+))}{\langle \nabla \Psi(X_2(t_c^+)), f_2(t_c^+) \rangle} \nabla \Psi(X_1(t_c^-)) \quad (10)$$

where, above, the short notation $f_i(t_c^\pm)$ stands for $f_i(X_i(t_c^\pm), t_c, \alpha_i(t_c^\pm))$ and $l_i(t_c^\pm)$ stands for $l_i(x_i(t_c^\pm), t_c, \alpha_i(t_c^\pm))$ ($i = 1, 2$).

2.3. Analysis of the structure 1- \mathcal{H} -2. In this section we analyze the structure with three arcs drawn on Figure 2. Precisely, given $(x^0, t^0, x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+$ with $x^0 \neq x^f$ we make the following assumption:

(H 1 \mathcal{H} 2) There exist $t^0 < t_1 < t_2 < t^f$ and an optimal trajectory that starts from Ω_1 , stays in Ω_1 in the interval $[t^0, t_1)$, stays on \mathcal{H} on a time interval $[t_1, t_2]$ and stays in Ω_2 in the interval $(t_2, t^f]$.

Such trajectories are described as follows: the trajectory associated with any initial and final data $(x^0, t^0, x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}_*^+$ will be given by the vector $\mathbb{X}(\cdot) = (X_1(\cdot), X_{\mathcal{H}}(\cdot), X_2(\cdot)) : \mathbb{R}^+ \rightarrow \mathbb{R}^N \times \mathcal{H} \times \mathbb{R}^N$ which is a Lipschitz solution of the system

$$\begin{cases} \dot{X}_1(t) = f_1(X_1(t), \alpha_1(t)) & t \in (t^0, t_1) \\ \dot{X}_{\mathcal{H}}(t) = f_{\mathcal{H}}(X_{\mathcal{H}}(t), a_{\mathcal{H}}(t)) & t \in (t_1, t_2) \\ \dot{X}_2(t) = f_2(X_2(t), \alpha_2(t)) & t \in (t_2, t^f) \end{cases} \quad (11)$$

with mixed conditions

$$\begin{aligned} X_1(t_0) = x^0, \quad X_1(t_1) \in \mathcal{H}, \quad X_1(t_1) = X_{\mathcal{H}}(t_1), \\ X_{\mathcal{H}}(t_2) \in \mathcal{H}, \quad X_2(t_2) = X_{\mathcal{H}}(t_2), \quad X_2(t^f) = x^f \end{aligned} \quad (12)$$

and which satisfies the state constraints

$$X_1(t) \in \Omega_1 \quad \forall t \in (t^0, t_1), \quad X_{\mathcal{H}} \in \mathcal{H} \quad \forall t \in (t_1, t_2), \quad X_2(t) \in \Omega_2 \quad \forall t \in (t_2, t^f). \quad (13)$$

The cost of such a trajectory is

$$\begin{aligned} C(x^0, t^0; x^f, t^f; \mathbb{X}) = \int_{t^0}^{t_1} l_1(X_1(t), \alpha_1(t)) dt + \int_{t_1}^{t_2} l_{\mathcal{H}}(X_{\mathcal{H}}(t), a_{\mathcal{H}}(t)) dt \\ + \int_{t_2}^{t^f} l_2(X_2(t), \alpha_2(t)) dt. \end{aligned}$$

Our aim is to characterize the value function $S_{1,\mathcal{H},2} : \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$

$$S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \inf \left\{ C(x^0, t^0; x^f, t^f; \mathbb{X}) : \mathbb{X} \text{ is solution of (11)-(12)-(13), } t^f > t_2 > t_1 > t^0 \right\}. \quad (14)$$

Remark 1. This definition does not include the cases where $x^0 \in \mathcal{H}$ and/or $x^f \in \mathcal{H}$. However, it can be modified in order to involve only vectors X_1 , X_2 or $X_{\mathcal{H}}$. Moreover, note that if both $x^0, x^f \in \mathcal{H}$ then $S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = S_{\mathcal{H}}(x^0, t^0; x^f, t^f)$.

Herafter, we use the following notations.

Notations. Let $u = u(x^0, t^0; x^f, t^f) : (\Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}_*^+) \rightarrow \mathbb{R}$ be a generic function. We denote by $\nabla_{x^0} u$, $\nabla_{x^f} u$ the gradients with respect to the first and the second state variable respectively, so $\nabla_{x^0} u$ and $\nabla_{x^f} u$ take values in \mathbb{R}^N . We denote by u_{t^0} and u_{t^f} the partial derivatives with respect to the first and the second time variable respectively, so u_{t^0} and u_{t^f} take values in \mathbb{R} .

If $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \mathcal{H} \times (0, +\infty)$ we define $\nabla_{x^f}^{\mathcal{H}} u$ such that $(x^f, \nabla_{x^f}^{\mathcal{H}} u) \in T\mathcal{H}$.

If $(x^0, t^0; x^f, t^f) \in \mathcal{H} \times \mathbb{R}^+ \times \Omega_2 \times (0, +\infty)$ we define $\nabla_{x^0}^{\mathcal{H}} u$ such that $(x^0, \nabla_{x^0}^{\mathcal{H}} u) \in T\mathcal{H}$.

Definition of the duplicated problem. The main ingredient of our analysis is the construction of the duplicated problem (following [19]), the advantage being that the latter will be a classical (non-regional) problem in higher dimension. The idea is to change the time variable to let the possible optimal trajectories evolve “at the same time” on the three arcs: the one on Ω_1 , the one on \mathcal{H} and the one on Ω_2 . In this duplicated optimal control problem we will not need to impose the mixed conditions (12) and the state constraints (13). Therefore we will be able to characterize the value function by a Hamilton-Jacobi equation, apply the usual Pontryagin maximum principle and exploit the classical link (usual sensitivity relations) between them.

We set $V = (A_1 \times [0, T]) \times (A_{\mathcal{H}} \times [0, T]) \times (A_2 \times [0, T])$, for $T > 0$ large enough. For fixed $T_0, T_1 \in \mathbb{R}^+$ the *admissible controls* are

$$\mathbb{V}(\tau) = (v_1(\tau), w_1(\tau), v_{\mathcal{H}}(\tau), w_{\mathcal{H}}(\tau), v_2(\tau), w_2(\tau)) \in L^\infty([T_0, T_1]; V).$$

The *admissible trajectories* are Lipschitz continuous vector functions

$$\mathbb{Z}(\cdot) = (Y_1(\cdot), \rho_1(\cdot), Y_{\mathcal{H}}(\cdot), \rho_{\mathcal{H}}(\cdot), Y_2(\cdot), \rho_2(\cdot)) : (T_0, T_1) \rightarrow \Omega_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \Omega_2 \times \mathbb{R}_*^+$$

solutions of the so-called *duplicated system*

$$\begin{cases} Y_1'(\tau) = f_1(Y_1(\tau), v_1(\tau))w_1(\tau) & \tau \in (T_0, T_1) \\ \rho_1'(\tau) = w_1(\tau) & \tau \in (T_0, T_1) \\ Y_{\mathcal{H}}'(\tau) = f_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau))w_{\mathcal{H}}(\tau) & \tau \in (T_0, T_1) \\ \rho_{\mathcal{H}}'(\tau) = w_{\mathcal{H}}(\tau) & \tau \in (T_0, T_1) \\ Y_2'(\tau) = f_2(Y_2(\tau), v_2(\tau))w_2(\tau) & \tau \in (T_0, T_1) \\ \rho_2'(\tau) = w_2(\tau) & \tau \in (T_0, T_1) \end{cases} \quad (15)$$

with initial and final conditions

$$\mathbb{Z}(T_0) = Z_0, \quad \mathbb{Z}(T_1) = Z_1. \quad (16)$$

Note that to take into account the mixed conditions on the original problem, we will allow initial and final states Z_0, Z_1 in $\bar{\Omega}_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \bar{\Omega}_2 \times \mathbb{R}_*^+$. More precisely, given $(Z_0, T_0), (Z_1, T_1) \in (\bar{\Omega}_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \bar{\Omega}_2 \times \mathbb{R}_*^+) \times \mathbb{R}^+$ we consider the subset of admissible trajectories

$$\begin{aligned} \mathcal{Z}_{(Z_0, T_0), (Z_1, T_1)} = & \left\{ \mathbb{Z} \in Lip((T_0, T_1); \Omega_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \Omega_2 \times \mathbb{R}_*^+) : \right. \\ & \text{there exists an admissible control } \mathbb{V} \in L^\infty([T_0, T_1]; V) \\ & \left. \text{such that } \mathbb{Z} \text{ is a solution of (15)-(16)} \right\}. \end{aligned}$$

For each admissible trajectory \mathbb{Z} we consider the cost functional

$$\begin{aligned} \mathcal{C}(\mathbb{Z}) = \int_{T_0}^{T_1} & \left(l_1(Y_1(\tau), v_1(\tau))w_1(\tau) + l_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau))w_{\mathcal{H}}(\tau) \right. \\ & \left. + l_2(Y_2(\tau), v_2(\tau))w_2(\tau) \right) d\tau \end{aligned}$$

and hence the value function $\Sigma : (\bar{\Omega}_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \bar{\Omega}_2 \times \mathbb{R}_*^+ \times \mathbb{R}^+)^2 \rightarrow \mathbb{R}$ is defined by

$$\Sigma(Z_0, T_0; Z_1, T_1) = \inf \left\{ \mathcal{C}(\mathbb{Z}) : \mathbb{Z} \in \mathcal{Z}_{(Z_0, T_0), (Z_1, T_1)} \right\}. \quad (17)$$

Link between the regional optimal control problem and the duplicated problem. To establish the link between the original and the duplicated problem, given $(x^0, t^0; x^f, t^f) \in \Omega_1 \times (0, +\infty) \times \Omega_2 \times (0, +\infty)$, we define the submanifold of $\mathbb{R}^{6(N+1)}$

$$\begin{aligned} \mathcal{M}(x^0, t^0; x^f, t^f) &= \left\{ (Z_0, Z_1) \in \mathbb{R}^{6(N+1)} : Z_0 = (x^0, t^0, x_1, t_1, x_2, t_2), \right. \\ &\quad \left. Z_1 = (x_1, t_1, x_2, t_2, x^f, t^f) \text{ with } x_1 \in \mathcal{H}, x_2 \in \mathcal{H}, \text{ and } t^f > t_2 > t_1 > t^0 \right\}. \end{aligned}$$

The following result says that the original value function is the minimum of the value functions $\Sigma(Z_0, T_0; Z_1, T_1)$ restricted to the submanifold $\mathcal{M}(x^0, t^0; x^f, t^f)$.

Proposition 1. *Under the assumptions $(H\mathcal{H})$, (Hf_i) and $(Hf_{\mathcal{H}})$, given $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+$, we have*

$$\begin{aligned} &S_{1, \mathcal{H}, 2}(x^0, t^0; x^f, t^f) \\ &= \min \left\{ \Sigma(Z_0, T_0; Z_1, T_1) : (Z_0, Z_1) \in \mathcal{M}(x^0, t^0; x^f, t^f), 0 \leq T_0 < T_1 \right\}. \end{aligned} \quad (18)$$

Proposition 1 is proved in Section 4.1.

Application of the usual Pontryagin maximum principle to the duplicated problem. Let us introduce several further notations.

In order to write the partial derivatives of Σ at points $\mathbb{Z} \in \Omega_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \Omega_2 \times \mathbb{R}_*^+$ we enumerate the space variables as follows:

$$(\mathbb{Z}_0, \mathbb{Z}_1) = ((1, 2, 3, 4, 5, 6), (7, 8, 9, 10, 11, 12)).$$

Therefore $\partial_i \Sigma$ takes values in \mathbb{R} for $i = 2, 4, 6, 8, 10, 12$; $\nabla_i \Sigma$ takes values in \mathbb{R}^N for $i = 1, 5, 7, 11$ and $\nabla_{\mathcal{H}, i} \Sigma$ in $T\mathcal{H}$ for $i = 3, 9$. We set

$$\begin{aligned} \nabla_{\mathbb{Z}_0} &= (\nabla_1 \Sigma, \partial_2 \Sigma, \nabla_{\mathcal{H}, 3} \Sigma, \partial_4 \Sigma, \nabla_5 \Sigma, \partial_6 \Sigma), \\ \Sigma_{t_0}(\mathbb{Z}_0, t_0, \mathbb{Z}_1, t_1) &= -\frac{\partial}{\partial t_0} \Sigma(\mathbb{Z}_0, t_0, \mathbb{Z}_1, t_1), \end{aligned}$$

$$\begin{aligned} \nabla_{\mathbb{Z}_1} \Sigma &= (\nabla_7 \Sigma, \partial_8 \Sigma, \nabla_{\mathcal{H}, 9} \Sigma, \partial_{10} \Sigma, \nabla_{11} \Sigma, \partial_{12} \Sigma), \\ \Sigma_{t_1}(\mathbb{Z}_0, t_0, \mathbb{Z}_1, t_1) &= -\frac{\partial}{\partial t_1} \Sigma(\mathbb{Z}_0, t_0, \mathbb{Z}_1, t_1). \end{aligned}$$

Moreover, we respectively denote by $D_{\mathbb{Z}_0}^+ \Sigma$ and $D_{\mathbb{Z}_0}^- \Sigma$ (or $D_{\mathbb{Z}_1}^+ \Sigma$ and $D_{\mathbb{Z}_1}^- \Sigma$) the classical super- and sub-differential in the space variables 1, 3, 5, 7, 9.

Given $\mathbb{V} = (v_1, w_1, v_{\mathcal{H}}, w_{\mathcal{H}}, v_2, w_2) \in V$, $\mathbb{Z} = (Y_1, \rho_1, Y_{\mathcal{H}}, \rho_{\mathcal{H}}, Y_2, \rho_2) \in \mathbb{R}^N \times \mathbb{R} \times T\mathcal{H} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ and $\mathbb{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) \in \mathbb{R}^N \times \mathbb{R} \times T_{Y_{\mathcal{H}}} \mathcal{H} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, we define the Hamiltonian

$$\begin{aligned} \widetilde{\mathcal{H}}(\mathbb{Z}, \mathbb{Q}, p^0, \mathbb{V}) &= \langle Q_1, f_1(Y_1, v_1)w_1 \rangle + Q_2 w_1 + p^0 l_1(Y_1, v_1)w_1 \\ &\quad + \langle Q_3, f_{\mathcal{H}}(Y_{\mathcal{H}}, v_{\mathcal{H}})w_{\mathcal{H}} \rangle_{\mathcal{H}} + Q_4 w_{\mathcal{H}} + p^0 l_{\mathcal{H}}(Y_{\mathcal{H}}, v_{\mathcal{H}})w_{\mathcal{H}} \\ &\quad + \langle Q_5, f_2(Y_2, v_2)w_2 \rangle + Q_6 w_2 + p^0 l_2(Y_2, v_2)w_2 \end{aligned}$$

and we set

$$\mathcal{H}(\mathbb{Z}, \mathbb{Q}, p^0) = \sup_{\mathbb{V} \in V} \widetilde{\mathcal{H}}(\mathbb{Z}, \mathbb{Q}, p^0, \mathbb{V}).$$

The application of the usual Pontryagin maximum principle (see [27]) to the duplicated optimal control problem leads to the following lemma.

Lemma 2.1. *Under the assumptions $(H\mathcal{H})$, (Hfl_i) and $(Hfl_{\mathcal{H}})$, let $(Z_0, T_0), (Z_1, T_1) \in (\Omega_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \Omega_2 \times \mathbb{R}_*^+) \times \mathbb{R}^+$ and let $\mathbb{Z}(\cdot) \in \mathcal{Z}_{(Z_0, T_0), (Z_1, T_1)}$ be an optimal trajectory for the value function $\Sigma(Z_0, T_0; Z_1, T_1)$ defined in (17). Assume that $\mathbb{V}(\cdot)$ is the corresponding optimal control.*

There exist $p^0 \leq 0$ and a piecewise absolutely continuous mapping

$$\mathbb{P}_{\mathbb{Z}}(\cdot) = (P_{Y_1}(\cdot), P_{\rho_1}(\cdot), P_{Y_{\mathcal{H}}}(\cdot), P_{\rho_{\mathcal{H}}}(\cdot), P_{Y_2}(\cdot), P_{\rho_2}(\cdot))$$

from \mathbb{R}^+ to $\mathbb{R}^N \times \mathbb{R} \times T_{Y_{\mathcal{H}}}\mathcal{H} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ (adjoint vector) with $(\mathbb{P}_{\mathbb{Z}}(\cdot), p^0) \neq (0, 0)$, such that the extremal lift $(\mathbb{Z}(\cdot), \mathbb{P}_{\mathbb{Z}}(\cdot), p^0, \mathbb{V}(\cdot))$ is solution of

$$\mathbb{Z}'(\tau) = \frac{\partial \widetilde{\mathcal{H}}}{\partial P}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p^0, \mathbb{V}(\tau)), \quad \mathbb{P}'_{\mathbb{Z}}(\tau) = -\frac{\partial \widetilde{\mathcal{H}}}{\partial Z}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p^0, \mathbb{V}(\tau))$$

for almost every $\tau \in (T_0, T_1)$. Moreover, the maximization condition

$$\widetilde{\mathcal{H}}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p^0, \mathbb{V}(\tau)) = \max_{\mathbb{V} \in \mathbb{V}} \widetilde{\mathcal{H}}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p^0, \mathbb{V}) \quad (= \mathcal{H}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p^0)) \quad (19)$$

holds for almost every $\tau \in (T_0, T_1)$.

If $Z_0 = (x^0, t^0, x_1, t_1, x_2, t_2)$, $Z_1 = (x_1, t_1, x_2, t_2, x^f, t^f) \in \mathcal{M}(x^0, t^0; x^f, t^f)$ then the following transversality condition holds: there exist $\nu_1, \nu_2 \in \mathbb{R}$ such that

$$P_{\rho_{\mathcal{H}}}(T_0) = P_{\rho_1}(T_1) \quad (20)$$

$$P_{\rho_2}(T_0) = P_{\rho_{\mathcal{H}}}(T_1) \quad (21)$$

$$P_{Y_{\mathcal{H}}}(T_0) = P_{Y_1}(T_1) + \nu_1 \nabla \Psi(x_1) \quad (22)$$

$$P_{Y_2}(T_0) = P_{Y_{\mathcal{H}}}(T_1) + \nu_2 \nabla \Psi(x_2). \quad (23)$$

We provide a proof of Lemma 2.1 in Section 4.2.

Sensitivity relations. In order to establish the link between the adjoint vector and the gradient of the value function Σ , we assume the uniqueness of the extremal lift:

(Hu) We assume that the optimal trajectory $\mathbb{Z}(\cdot)$ in Lemma 2.1 admits a unique extremal lift $(\mathbb{Z}(\cdot), \mathbb{P}_{\mathbb{Z}}(\cdot), p^0, \mathbb{V}(\cdot))$ which is moreover normal, i.e., $p^0 = -1$.

The assumption of uniqueness of the solution of the optimal control problem and of uniqueness of its extremal lift (which is then moreover normal) is closely related to the differentiability properties of the value function. We refer to [4, 16] for precise results on differentiability properties of the value function and to [12, 31, 32, 34] for results on the size of the set where the value function is differentiable. For instance for control-affine systems the singular set of the value function has Hausdorff $(N-1)$ -measure zero, whenever there is no optimal singular trajectory (see [32]), and is a stratified submanifold of \mathbb{R}^N of positive codimension in an analytic context (see [37]). These results essentially say that, if the dynamics and cost function are C^1 , then the value function is of class C^1 at “generic” points. Moreover, note that the property of having a unique extremal lift, that is moreover normal, is generic in the sense of the Whitney topology for control-affine systems (see [14, 15] for precise statements).

We have the following result.

Proposition 2. *Assume $(H\mathcal{H})$, (Hfl_i) and $(Hfl_{\mathcal{H}})$. Let $(Z_0, T_0), (Z_1, T_1) \in (\overline{\Omega}_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \overline{\Omega}_2 \times \mathbb{R}_*^+) \times \mathbb{R}^+$ and let $\mathbb{Z}(\cdot) \in \mathcal{Z}_{(Z_0, T_0), (Z_1, T_1)}$ be an optimal trajectory for the value function $\Sigma(Z_0, T_0; Z_1, T_1)$ defined in (17). Let $\mathbb{P}_{\mathbb{Z}}$ be the corresponding absolutely continuous adjoint vector given by Theorem 2.1. Then:*

(i) For any time τ in the closed interval $[T_0, T_1]$ we have

$$D_{\bar{Z}_0}^- \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) \subseteq -\mathbb{P}_{\mathbb{Z}}(\tau) \subseteq D_{\bar{Z}_0}^+ \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) \tag{24}$$

in the sense that either $D_{\bar{Z}_0}^- \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1)$ is empty or the function $\tau \mapsto \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1)$ is differentiable and then $D_{\bar{Z}_0}^- \Sigma = D_{\bar{Z}_0}^+ \Sigma$ at this point.

Moreover, under the assumption (Hu) the function $\tau \mapsto \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1)$ is differentiable for every time in $[T_0, T_1]$, thus

$$\nabla_{Z_0} \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) = -\mathbb{P}_{\mathbb{Z}}(\tau) \quad \forall \tau \in [T_0, T_1]. \tag{25}$$

(ii) For any time τ in the closed interval $[T_0, T_1]$ we have

$$D_{\bar{Z}_1}^- \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) \subseteq \mathbb{P}_{\mathbb{Z}}(\tau) \subseteq D_{\bar{Z}_1}^+ \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) \tag{26}$$

in the sense that either $D_{\bar{Z}_1}^- \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau)$ is empty or the function $\tau \mapsto \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau)$ is differentiable and then $D_{\bar{Z}_1}^- \Sigma = D_{\bar{Z}_1}^+ \Sigma$ at this point.

Moreover, under the assumption (Hu) the function $\tau \mapsto \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau)$ is differentiable for every time in $[T_0, T_1]$, thus

$$\nabla_{Z_1} \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) = \mathbb{P}_{\mathbb{Z}}(\tau) \quad \forall \tau \in [T_0, T_1]. \tag{27}$$

Proposition 2 is proved in Section 2.

Remark 2. It is useful to write equalities (25) and (27) as a single equality. We have indeed

$$-\nabla_{Z_0} \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) = \mathbb{P}_{\mathbb{Z}}(\tau) = \nabla_{Z_1} \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) \quad \forall \tau \in [T_0, T_1] \tag{28}$$

that is, more precisely,

$$-\nabla_1 \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) = P_{Y_1}(\tau) = \nabla_7 \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) \quad \forall \tau \in [T_0, T_1] \tag{29}$$

$$-\partial_2 \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) = P_{\rho_1}(\tau) = \partial_8 \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) \quad \forall \tau \in [T_0, T_1] \tag{30}$$

$$-\nabla_3^{\mathcal{H}} \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) = P_{Y_{\mathcal{H}}}(\tau) = \nabla_9^{\mathcal{H}} \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) \quad \forall \tau \in [T_0, T_1] \tag{31}$$

$$-\partial_4 \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) = P_{\rho_{\mathcal{H}}}(\tau) = \partial_{10} \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) \quad \forall \tau \in [T_0, T_1] \tag{32}$$

$$-\nabla_5 \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) = P_{Y_2}(\tau) = \nabla_{11} \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) \quad \forall \tau \in [T_0, T_1] \tag{33}$$

$$-\partial_6 \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) = P_{\rho_2}(\tau) = \partial_{12} \Sigma(Z_0, T_0; \mathbb{Z}(\tau), \tau) \quad \forall \tau \in [T_0, T_1]. \tag{34}$$

Note that at times T_0 and T_1 the gradients are naturally defined as the limits of the gradients in the open interval (T_0, T_1) .

Application to the regional optimal control problem: main result. We now establish a result that is analogous to the one obtained for the structure 1-2. We first remark that for this structure one cannot directly define a global adjoint vector, therefore its role will be played by the limit of the gradient of the value function (vectors $Q_1, Q_2, Q_{\mathcal{H}}$ below). The main result is the following.

Theorem 2.2. Under the assumptions (HH), (Hfl_i), (Hfl_ℋ) and (Hu), for any $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}_*^+$ we have

$$\begin{aligned} S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) \\ = \min \{ S_1(x^0, t^0; x_1, t_1) + S_{\mathcal{H}}(x_1, t_1; x_2, t_2) + S_2(x_2, t_2; x^f, t^f) : \\ t_0 < t_1 < t_2 < t^f, x_1, x_2 \in \mathcal{H} \}. \end{aligned}$$

Let $\mathbb{X}(\cdot)$ be an optimal trajectory for the value function $S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f)$ defined by (14) and let

$$Q_1(t_1^-) = - \lim_{t \rightarrow t_1^-} \nabla_{x^0} S_{1,\mathcal{H},2}(X_1(t), t; x^f, t^f),$$

$$Q_2(t_2^+) = \lim_{t \rightarrow t_2^+} \nabla_{x^f} S_{1,\mathcal{H},2}(x^0, t^0; X_2(t), t),$$

$$Q_{\mathcal{H}}(t_1^+) = - \lim_{t \rightarrow t_1^+} \nabla_{x^0}^{\mathcal{H}} S_{1,\mathcal{H},2}(X_{\mathcal{H}}(t), t; x^f, t^f),$$

$$Q_{\mathcal{H}}(t_2^-) = \lim_{t \rightarrow t_2^-} \nabla_{x^f}^{\mathcal{H}} S_{1,\mathcal{H},2}(x^0, t^0; X_{\mathcal{H}}(t), t).$$

We have the continuity conditions

$$H_1(X_1(t_1^-), Q_1(t_1^-), p^0) = H_{\mathcal{H}}(X_{\mathcal{H}}(t_1^+), Q_{\mathcal{H}}(t_1^+), p^0), \quad (35)$$

$$H_{\mathcal{H}}(X_{\mathcal{H}}(t_2^-), Q_{\mathcal{H}}(t_2^-), p^0) = H_2(X_2(t_2^+), Q_2(t_2^+), p^0). \quad (36)$$

Moreover, there exist $\nu_1, \nu_2 \in \mathbb{R}$ such that

$$\begin{aligned} Q_{\mathcal{H}}(t_1^+) &= Q_1(t_1^-) + \nu_1 \nabla \Psi(X_1(t_1^-)), \\ Q_2(t_2^+) &= Q_{\mathcal{H}}(t_2^-) + \nu_2 \nabla \Psi(X_2(t_2^+)). \end{aligned} \quad (37)$$

Furthermore, if $\langle \nabla \Psi(X_1(t_1^-)), f_1(t_1^-) \rangle \neq 0$ and $\langle \nabla \Psi(X_2(t_2^+)), f_2(t_2^+) \rangle \neq 0$ then

$$\nu_1 = \frac{\langle Q_{\mathcal{H}}(t_1^+), f_1(t_1^-) \rangle - \langle Q_1(t_1^-), f_{\mathcal{H}}(t_1^+) \rangle_{\mathcal{H}} + p^0 (l_1(t_1^-) - l_{\mathcal{H}}(t_1^+))}{\langle \nabla \Psi(X_1(t_1^-)), f_1(t_1^-) \rangle}, \quad (38)$$

$$\nu_2 = \frac{\langle Q_{\mathcal{H}}(t_2^-), f_{\mathcal{H}}(t_2^-) \rangle_{\mathcal{H}} - \langle Q_2(t_2^+), f_2(t_2^+) \rangle + p^0 (l_{\mathcal{H}}(t_2^-) - l_2(t_2^+))}{\langle \nabla \Psi(X_2(t_2^+)), f_2(t_2^+) \rangle} \quad (39)$$

where we used the short notations

$$f_i(t_i^\pm) = f_i(X_i(t_i^\pm), \alpha_i(t_i^\pm)), \quad l_i(t_i^\pm) = l_i(X_i(t_i^\pm), \alpha_i(t_i^\pm))$$

with $i \in \{1, 2, \mathcal{H}\}$.

Theorem 2.2 is proved in Section 4.4.

Remark 3. Note the similarity between the jump conditions (37)-(39) and the ones in the transversal case (10): the difference is due to the fact that \mathcal{H} is of codimension 1.

2.4. More general structures. Proceeding as in Section 2.3, the analogue of Proposition 1 is obtained for any other structure 1-2- \mathcal{H} -1, 1- \mathcal{H} -1-2, 1-2- \mathcal{H} -2, etc, in a similar way. For each given such structure, the duplication technique permits to lift the corresponding regional control problem to a classical (i.e., non-regional) optimal control problem in higher dimension, and then the value function of the regional optimal control problem is written as the minimum of the value function of the high-dimensional classical optimal control problem over a submanifold, this submanifold representing the junction conditions of the regional problem (continuity conditions on the state and jump conditions on the adjoint vector).

For example, consider optimal trajectories with the structure 2- \mathcal{H} -2-1, i.e., trajectories starting in Ω_2 , staying in Ω_2 along the time interval $[t^0, t_1]$, then lying in \mathcal{H} on $[t_1, t_2]$, then going back to Ω_2 on $(t_2, t_3]$ and finally staying in Ω_1 in the time interval $(t_3, t^f]$. Then, the duplicated problem has four arcs and is settled in dimension 8. The whole approach developed previously can be applied as well and we obtain the corresponding analogues of Proposition 1 and then of Theorem 2.2.

In such a way, all possible structures can be described as consisting of a finite succession of arcs, and they are analyzed thanks to the duplication technique. If the structure has N arcs then the duplicated problem is settled in dimension $2N$.

As already said, from a practical point of view it is reasonable to restrict the search of optimal trajectories over all possible trajectories having only a finite number of switchings. This is always what is done in practice because, numerically and in real-life implementation, the Zeno phenomenon is not desirable. Under such an assumption, our approach developed above shows that the value function of the regional optimal control problem can be written as

$$U = \inf\{S_{1,2}, S_{1,\mathcal{H},2}, S_{1,2,\mathcal{H},2}, \dots\}$$

where each of the value functions S_* is itself the minimum of the value function of a classical optimal control problem (in dimension that is the double of the number of switchings of the corresponding structure) over terminal points running in some submanifold. An interesting consequence is that:

The regularity of the value function U of the regional optimal control problem is the same (i.e., not more degenerate) than the one of the higher-dimensional classical optimal control problem that lifts the problem.

The lifting duplication technique may thus be seen as a kind of *desingularization*, showing that the value function of the regional optimal control problem is the minimum over all possible structures of value functions associated with classical optimal control problems settled over fixed structures, each of them being the restriction to some submanifold of the value function of a classical optimal control problem in higher dimension.

In particular, if for instance all value functions above are Lipschitz then the value function of the regional optimal control problem is Lipschitz as well. Note that Lipschitz regularity is ensured if there is no abnormal minimizer (see [38]), and this sufficient condition is generic in some sense (see [14, 15]).

Such a regularity result is new in the context of regional optimal control problems.

Remark 4. In this paper, for the sake of simplicity we have analyzed regional problems in \mathbb{R}^N . Since all arguments are local, the same procedure can be applied to regional problems settled on a smooth manifold, which is stratified as $M = M^0 \cup M^1 \cup \dots \cup M^N$ (disjoint union) where M^j is a j -dimensional embedded submanifold of M .

Remark 5. Our results can also be straightforwardly extended to time-dependent dynamics and running costs, and to regions $\Omega_i(t)$ depending on time, always assuming at least a C^1 dependence.

2.5. What happens in case of Zeno phenomenon? In case the *Zeno phenomenon* occurs, optimal trajectories oscillate for instance between two regions Ω_1 and Ω_2 an infinite number of times over a compact time interval.

If the number of switchings is countably infinite, then the above procedure can, at least formally, be carried out, but then the duplicated (lifted) problem is settled in infinite (countable) dimension. In order to settle it rigorously, much more functional analysis work would be required. Anyway, formally the value function is then written as an infimum of countably many value functions of classical optimal control problems, but even if the latter are regular enough (for instance, Lipschitz), taking the infimum may break this regularity and create some degeneracy.

If the number of switchings is uncountably infinite, the situation may even go worst. The duplication technique cannot be performed, at least in the form we have done it, and we do not know if there would exist a somewhat related approach to capture any information. The situation is widely open there. We are not aware of any example of a regional (or, more generally, hybrid) optimal control problem for which the set of switching points of the optimal trajectory would have a fractal structure. Notice the related result stated in [2], according to which, for smooth bracket generating single-input control-affine systems with bounded scalar controls, the set of switching points of the optimal bang-bang controls cannot be a Cantor set.

3. Example. As an example we consider here a very simple regional optimal control problem where it is easy to see that a trajectory of the form 1- \mathcal{H} -2 is the best possible choice. The idea is to model situations where it is optimal to move along the interface \mathcal{H} as long as possible. One can think, for example, of a pedestrian walking in Ω_1 and Ω_2 with the possibility of taking a tramway along \mathcal{H} at any point of this interface \mathcal{H} .

More generally, this example models any problem where moving along a direction is much faster and/or cheaper than along others.

In \mathbb{R}^2 we set

$$\Omega_1 = \{(x, y) : y < 0\}, \quad \Omega_2 = \{(x, y) : y > 0\}, \quad \mathcal{H} = \{(x, y) : y = 0\}.$$

We choose the dynamics

$$f_1(X_1, \alpha_1) = \begin{pmatrix} \cos(\alpha_1) \\ \sin(\alpha_1) \end{pmatrix}, \quad f_{\mathcal{H}}(X_{\mathcal{H}}, \alpha_{\mathcal{H}}) = 10, \quad f_2(X_2, \alpha_2) = \begin{pmatrix} \cos(\alpha_2) \\ \sin(\alpha_2) \end{pmatrix}$$

where the controls α_i take values on $[-\pi, \pi]$. We consider the minimal time problem, therefore our aim is to compute the value function

$$U(x^0, 0; x^f) = \inf \left\{ t^f : \dot{\mathbb{X}}(t) = f(\mathbb{X}(t), a(t)) \quad \text{with} \quad \mathbb{X}(t^0) = x^0, \quad \mathbb{X}(t^f) = x^f \right\},$$

where the dynamics f coincide with $f_1, f_2, f_{\mathcal{H}}$ respectively in $\Omega_1, \Omega_2, \mathcal{H}$.

We analyze the case where we start from a point (x_0, y_0) in Ω_1 and we aim to reach a point (x_1, y_1) in Ω_2 with $x_1 > x_0$. In Ω_1 , the dynamics f_1 allow to move with constant velocity equal to one in any direction, therefore it is clear that the best choice is to go “towards \mathcal{H} but also in the direction of x_1 ”. Indeed, if we compare on Figure 3 below the dotted trajectory and the black one, they spend the same time in Ω_1 , but on \mathcal{H} the dotted one is not the minimal time. Therefore the black one is a better choice.

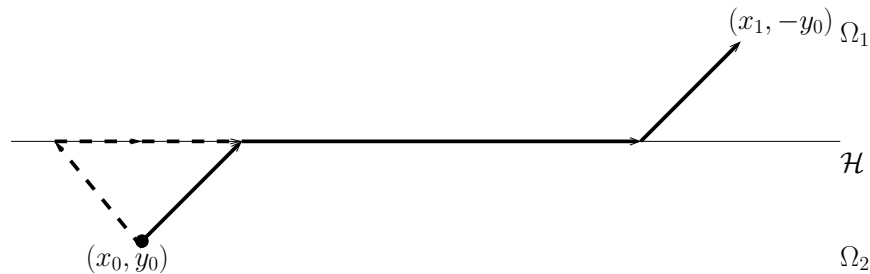


FIGURE 3. Going “to the left” is not optimal.

For this reason, and since the problem is symmetric, it is not restrictive to assume that $y_1 = -y_0$ and that trajectories with the structure 1- \mathcal{H} -2 are like the ones described on Figure 4 with $0 \leq a \leq \frac{x_1 - x_0}{2}$.

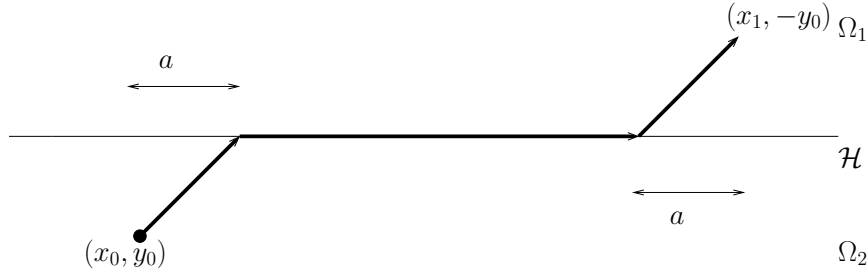


FIGURE 4. The trajectory 1- \mathcal{H} -2 .

For each trajectory steering (x_0, y_0) to $(x_1, -y_0)$ a simple computation gives the cost (as a function of the parameter a)

$$C(a) = 2\sqrt{y_0^2 + a^2} + \frac{x_1 - x_0}{10} - \frac{a}{5}.$$

Therefore, the value function is

$$U(x^0, 0; x^f) = \min_{0 \leq a \leq \frac{x_1 - x_0}{2}} \left(2\sqrt{y_0^2 + a^2} + \frac{x_1 - x_0}{10} - \frac{a}{5} \right)$$

and we obtain that:

- if $\frac{x_1 - x_0}{2} > \frac{|y_0|}{3\sqrt{11}}$ then the optimal trajectory has the structure 1- \mathcal{H} -2 with

$$a = \frac{|y_0|}{3\sqrt{11}}$$

and the optimal final time is

$$t_f = \frac{19}{3\sqrt{11}} - \frac{x_1 - x_0}{10}.$$

- if $\frac{x_1 - x_0}{2} \leq \frac{|y_0|}{3\sqrt{11}}$ then the optimal trajectory has the structure 1-2 with

$$a = \frac{x_1 - x_0}{2}$$

and the optimal final time is

$$t_f = 2\sqrt{y_0^2 + \frac{(x_1 - x_0)^2}{4}}$$

(see Figure 5).

We finally remark that, although this example is very simple, it is paradigmatic and illustrates many possible situations where one has two regions of the space (with specific dynamics) separated by an interface along which the dynamics are quicker than in the two regions. In this sense, the above example can easily be adapted and complexified to represent some more realistic situations.

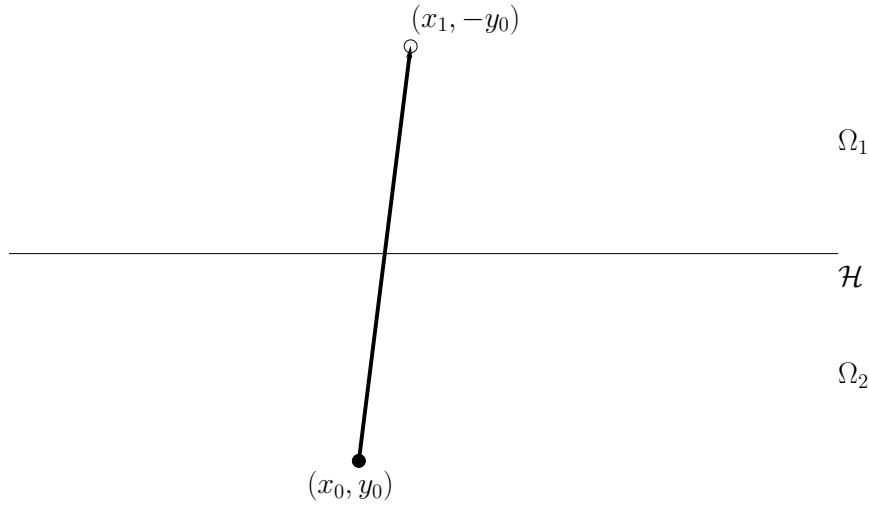


FIGURE 5. The trajectory 1-2.

4. Proofs.

4.1. **Proof of Proposition 1.** Fix $(x^0, t^0; x^f, t^f) \in \bar{\Omega}_1 \times (0, +\infty) \times \bar{\Omega}_2 \times (0, +\infty)$, with $x^0 \neq x^f$ and t_1, t_2 such that $t^f > t_2 > t_1 > t^0$. Let \mathbb{X} be the corresponding trajectory solution of (11)-(12)-(13). We construct three increasing C^1 diffeomorphisms:

$$\rho_1 : [T_0, T_1] \rightarrow [t^0, t_1], \quad \rho_{\mathcal{H}} : [T_0, T_1] \rightarrow [t_1, t_2], \quad \rho_2 : [T_0, T_1] \rightarrow [t_2, t^f]$$

with $0 \leq T_0 < T_1$ arbitrarily chosen. We solve then the state equation (15) with controls

$$w_1(\tau) = \rho'_1(\tau), \quad w_2(\tau) = \rho'_2(\tau), \quad w_{\mathcal{H}}(\tau) = \rho'_{\mathcal{H}}(\tau),$$

$$v_1(\tau) = \alpha_1(\rho_1(\tau)) = \alpha_1(t), \quad v_{\mathcal{H}}(\tau) = a_{\mathcal{H}}(\rho_{\mathcal{H}}(\tau)) = a_{\mathcal{H}}(t),$$

$$v_2(\tau) = \alpha_2(\rho_2(\tau)) = \alpha_2(t)$$

and initial and final data

$$\bar{Z}_0 = (x^0, t^0, X_1(t_1), t_1, X_2(t_2), t_2), \quad \bar{Z}_1 = (X_1(t_1), t_1, X_2(t_2), t_2, x^f, t^f).$$

Therefore $(\bar{Z}_0, \bar{Z}_1) \in \mathcal{M}(x^0, t^0; x^f, t^f)$ and the duplicated trajectory is such that

$$Y_1(\tau) = X_1(\rho_1(\tau)) = X_1(t), \quad Y_{\mathcal{H}}(\tau) = X_{\mathcal{H}}(\rho_{\mathcal{H}}(\tau)) = X_{\mathcal{H}}(t),$$

$$Y_2(\tau) = X_2(\rho_2(\tau)) = X_2(t)$$

for any $t \in (t^0, t^f)$, $\tau \in (T_0, T_1)$. Moreover, by the above change of time variable we have

$$\int_{t^0}^{t_1} l_1(X_1(t), \alpha_1(t)) dt + \int_{t_1}^{t_2} l_{\mathcal{H}}(X_{\mathcal{H}}(t), a(t)) dt + \int_{t_2}^{t^f} l_2(X_2(t), \alpha_2(t)) dt$$

$$= \int_{T_0}^{T_1} \left(l_1(Y_1(\tau), v_1(\tau))w_1(\tau) + l_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau))w_{\mathcal{H}}(\tau) + l_2(Y_2(\tau), v_2(\tau))w_2(\tau) \right) d\tau.$$

Hence $C(x^0, t^0; x^f, t^f; \mathbb{X}) = \mathcal{C}(\mathbb{Z})$. Conversely, since the time change of variable ρ is invertible given $(Z_0, Z_1) \in \mathcal{M}(x^0, t^0; x^f, t^f)$ and a corresponding admissible trajectory \mathbb{Z} we can construct a trajectory \mathbb{X} such that $\mathcal{C}(\mathbb{Z}) = C(x^0, t^0; x^f, t^f; \mathbb{X})$ and the proof is completed.

4.2. Proof of Lemma 2.1. The result follows by applying the usual Pontryagin maximum principle (see [27]). If $Z_0, Z_1 \in \mathcal{M}(x^0, t^0; x^f, t^f)$, the classical transversality condition holds (see [1, Theorem 12.15] or [36]):

$$(-\mathbb{P}_{\mathbb{Z}}(T_0), \mathbb{P}_{\mathbb{Z}}(T_1)) \perp T_{(\mathbb{Z}(T_0), \mathbb{Z}(T_1))} \mathcal{M}(x^0, t^0; x^f, t^f).$$

Now, if $\mathbb{Z}(T_0) = Z_0 = (x^0, t^0, x_1, t_1, x_2, t_2)$ and $\mathbb{Z}(T_1) = Z_1 = (x_1, t_1, x_2, t_2, x^f, t^f)$ the above relation gives:

- $t_1 = Z_0^4 = Z_1^2$ implies $P_{\rho_{\mathcal{H}}}(T_0) = P_{\rho_1}(T_1)$;
- $t_2 = Z_0^5 = Z_1^4$ implies $P_{\rho_2}(T_0) = P_{\rho_{\mathcal{H}}}(T_1)$;
- $x_1 = Z_0^3 = Z_1^1$ and $x_1 \in \mathcal{H}$ imply $P_{Y_{\mathcal{H}}}(T_0) = P_{Y_1}(T_1) + \nu_1 \nabla \Psi(x_1)$;
- $x_2 = Z_0^6 = Z_1^3$ and $x_2 \in \mathcal{H}$ imply $P_{Y_2}(T_0) = P_{Y_{\mathcal{H}}}(T_1) + \nu_2 \nabla \Psi(x_2)$.

The result follows.

4.3. Proof of Proposition 2. To apply the classical theory of viscosity solutions for Hamilton-Jacobi equations (see, e.g., [5, 6]), we define two different value functions by considering separately the case when we fix the initial data (Z_0, T_0) and we consider a function the final data (Z_1, T_1) or conversely. Precisely, to prove (i) we fix (T_1, Z_1) and for any $T_0 \in \mathbb{R}_*^+$, $Z_0 \in \bar{\Omega}_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \bar{\Omega}_2 \times \mathbb{R}_*^+$ we define $\Sigma^0(Z_0, T_0) = \Sigma(Z_0, T_0, Z_1, T_1)$.

Similarly to prove (ii), (T_0, Z_0) is given and for any $T_1 \in \mathbb{R}_*^+$, $Z_1 \in \bar{\Omega}_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \bar{\Omega}_2 \times \mathbb{R}_*^+$ we set $\Sigma^1(Z_1, T_1) = \Sigma(Z_0, T_0, Z_1, T_1)$.

In order to write the partial derivatives of Σ^0 and Σ^1 we consider a generic function $u(\mathbb{Z}, t) : (\Omega_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \Omega_2 \times \mathbb{R}_*^+) \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and we will enumerate the variables as follows $\mathbb{Z} = (1, 2, 3, 4, 5, 6)$. Therefore $\partial_i u(\mathbb{Z})$ takes values in \mathbb{R} for $i = 2, 4, 6$, $\nabla_i \Sigma(\mathbb{Z})$ takes values in \mathbb{R}^N for $i = 1, 5$ and $\nabla_{\mathcal{H}, i} \Sigma$ in $T\mathcal{H}$ for $i = 3$. We will set

$$\nabla u = (\nabla_1 u, \partial_2 u, \nabla_{\mathcal{H}, 3} u, \partial_4 u, \nabla_5 u, \partial_6 u) \quad u_t(\mathbb{Z}, t) = -\frac{\partial u}{\partial t}(\mathbb{Z}, t).$$

Moreover, we respectively denote by D^+u and D^-u the classical super- and sub-differential in the space variables 1, 3, 5. We have

$$\nabla_{Z_0} \Sigma(Z_0, T_0; Z_1, T_1) = \nabla \Sigma^0(Z_0, T_0) \quad \text{and} \quad \nabla_{Z_1} \Sigma(Z_0, T_0; Z_1, T_1) = \nabla \Sigma^1(Z_1, T_1).$$

By applying the standard theory of viscosity solution (see [5, Propositions 3.1 and 3.5], see also [6]) we know that $\Sigma^0(Z_0, T_0)$ is a bounded, Lipschitz continuous viscosity solution of

$$-\frac{\partial u}{\partial t}(\mathbb{Z}, t) + \mathcal{H}(\mathbb{Z}, -\nabla u, -1) = 0 \quad \text{in } (\Omega_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \Omega_2 \times \mathbb{R}_*^+) \times (T_0, T_1),$$

and $\Sigma^1(Z_1, T_1)$ is a bounded, Lipschitz continuous viscosity solution of

$$\frac{\partial u}{\partial t}(\mathbb{Z}, t) + \mathcal{H}(\mathbb{Z}, \nabla u, -1) = 0 \quad \text{in } (\Omega_1 \times \mathbb{R}_*^+ \times \mathcal{H} \times \mathbb{R}_*^+ \times \Omega_2 \times \mathbb{R}_*^+) \times (T_0, T_1).$$

Therefore, we can apply [5, Corollary 3.45] to obtain (24) and (26). Now, if assumption (Hu) holds, one can prove that the two functions $\tau \mapsto \Sigma^0(\mathbb{Z}(\tau), \tau)$ and $\tau \mapsto \Sigma^1(\mathbb{Z}(\tau), \tau)$ are differentiable (see [12, Theorem 7.4.16] or [4, 16]) thus (25) and (27) follow.

4.4. **Proof of Theorem 2.2.** Fix $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times (0, +\infty)$. To obtain the first result we rewrite the equality (18) of Proposition 1 as

$$S_{1,\mathcal{H},2}(\lambda^0, \lambda^f) = \inf \left\{ \Sigma((\lambda^0, \chi); (\chi, \lambda^f)) : \chi = (x_1, t_1, x_2, t_2), \right. \\ \left. \Psi(x_1) = 0, \Psi(x_2) = 0, \quad t^f > t_2 > t_1 > t^0 \right\} \quad (40)$$

where we set $\lambda = (\lambda^0, \lambda^f) = (x^0, t^0; x^f, t^f)$ and $\chi = (x_1, t_1, x_2, t_2)$. Thus, by the construction of the duplicated value function Σ we have

$$S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) \\ = \min \left\{ S_1(x^0, t^0; \cdot, x_1, t_1) + S_{\mathcal{H}}(x_1, t_1; x_2, t_2) + S_2(x_2, t_2; x^f, t^f) : \right. \\ \left. t_0 < t_1 < t_2 < t^f, \quad x_1, x_2 \in \mathcal{H} \right\}.$$

Thanks to (18) in Proposition 1 we can consider now $(Z_0, Z_1) \in \mathcal{M}(x^0, t^0; x^f, t^f)$ such that $S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \Sigma(Z_0, T_0; Z_1, T_1)$ for an optimal trajectory $\mathbb{Z}(\cdot) \in \mathcal{Z}_{(Z_0, T_0), (Z_1, T_1)}$ (note that we have $S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \Sigma(Z_0, T_0; Z_1, T_1) = C(\mathbb{Z})$). Let $\mathbb{P}_{\mathbb{Z}}(\cdot)$ be the adjoint vector given by Theorem 2.1, the maximality condition (19) implies that

$$\langle P_{Y_1}(\tau), f_1(Y_1(\tau), v_1(\tau)) \rangle + P_{\rho_1}(\tau) + p^0 l_1(Y_1(\tau), v_1(\tau)) = 0 \quad (41)$$

$$\langle P_{Y_{\mathcal{H}}}(\tau), f_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau)) \rangle_{\mathcal{H}} + P_{\rho_{\mathcal{H}}}(\tau) + p^0 l_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau)) = 0 \quad (42)$$

$$\langle P_{Y_2}(\tau), f_2(Y_2(\tau), v_2(\tau)) \rangle + P_{\rho_2}(\tau) + p^0 l_2(Y_2(\tau), v_2(\tau)) = 0 \quad (43)$$

for almost every $\tau \in (T_0, T_1)$. Moreover, by the transversality condition in Theorem 2.1, there exist $\nu_1, \nu_2 \in \mathbb{R}$ such that

$$P_{\rho_{\mathcal{H}}}(T_0) = P_{\rho_1}(T_1) \quad (44)$$

$$P_{\rho_2}(T_0) = P_{\rho_{\mathcal{H}}}(T_1) \quad (45)$$

$$P_{Y_{\mathcal{H}}}(T_0) = P_{Y_1}(T_1) + \nu_1 \nabla \Psi(x_1) \quad (46)$$

$$P_{Y_2}(T_0) = P_{Y_{\mathcal{H}}}(T_1) + \nu_2 \nabla \Psi(x_2). \quad (47)$$

Our aim is now to interpret these equalities on the original problem. By definition of the duplicated problem, we construct an optimal trajectory $\mathbb{X}(\cdot)$ for $S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f)$, such that

$$Y_1(\tau) = X_1(\rho_1(\tau)) = X_1(t) \quad \forall \tau \in (T_0, T_1) \quad \forall t \in (t^0, t_1) \\ Y_{\mathcal{H}}(\tau) = X_{\mathcal{H}}(\rho_{\mathcal{H}}(\tau)) = X_{\mathcal{H}}(t) \quad \forall \tau \in (T_0, T_1) \quad \forall t \in (t_1, t_2) \\ Y_2(\tau) = X_2(\rho_2(\tau)) = X_2(t) \quad \forall \tau \in (T_0, T_1) \quad \forall t \in (t_2, t^f)$$

Indeed, we recall that by construction $C(\mathbb{X}) = C(\mathbb{Z}) = S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \Sigma(Z_0, T_0; Z_1, T_1)$. We set now

$$P_1(t) = P_{Y_1}(\tau) = P_{Y_1}(\rho_1(t)) \quad \forall \tau \in (T_0, T_1) \quad \forall t \in (t^0, t_1), \\ P_{\mathcal{H}}(t) = P_{Y_{\mathcal{H}}}(\tau) = P_{Y_{\mathcal{H}}}(\rho_{\mathcal{H}}(t)) \quad \forall \tau \in (T_0, T_1) \quad \forall t \in (t_1, t_2), \\ P_2(t) = P_{\rho_2}(\tau) = P_{\rho_2}(\rho_2(\tau)) \quad \forall \tau \in (T_0, T_1) \quad \forall t \in (t_2, t^f).$$

Therefore, by definition of the Hamiltonians \widetilde{H}_1 , $\widetilde{H}_{\mathcal{H}}$ and \widetilde{H}_2 , the equalities (41)-(43) give

$$\widetilde{H}_1(X_1(t), P_1(t), p^0, \alpha_1(t)) = -P_{\rho_1}(\tau), \\ \widetilde{H}_{\mathcal{H}}(X_{\mathcal{H}}(t), P_{\mathcal{H}}(t), p^0, a_{\mathcal{H}}(t)) = -P_{\rho_{\mathcal{H}}}(\tau),$$

$$\widetilde{H}_2(X_2(t), P_2(t), p^0, \alpha_2(t)) = -P_{\rho_2}(\tau),$$

for almost all $t \in (t^0, t^f)$ and $\tau \in (T_0, T_1)$.

To obtain the continuity conditions on the Hamiltonians we consider the above equalities at times t_1, t_2 . By construction of the time change of variable and the continuity of the adjoint vector we have

$$\begin{aligned} \widetilde{H}_1(X_1(t_1^-), P_1(t_1^-), p^0, \alpha_1(t_1^-)) &= \lim_{t \rightarrow t_1^-} \widetilde{H}_1(X_1(t), P_1(t), p^0, \alpha_1(t)) \\ &= \lim_{\tau \rightarrow T_1} (-P_{\rho_1}(\tau)) = -P_{\rho_1}(T_1) \end{aligned}$$

$$\begin{aligned} \widetilde{H}_{\mathcal{H}}(X_{\mathcal{H}}(t_1^+), P_{\mathcal{H}}(t_1^+), p^0, a_{\mathcal{H}}(t_1^+)) &= \lim_{t \rightarrow t_1^+} \widetilde{H}_{\mathcal{H}}(X_{\mathcal{H}}(t), P_{\mathcal{H}}(t), p^0, a_{\mathcal{H}}(t)) \\ &= \lim_{\tau \rightarrow T_0} (-P_{\rho_{\mathcal{H}}}(\tau)) = -P_{\rho_{\mathcal{H}}}(T_0) \end{aligned}$$

$$\begin{aligned} \widetilde{H}_{\mathcal{H}}(X_{\mathcal{H}}(t_2^-), P_{\mathcal{H}}(t_2^-), p^0, a_{\mathcal{H}}(t_2^-)) &= \lim_{t \rightarrow t_2^-} \widetilde{H}_{\mathcal{H}}(X_{\mathcal{H}}(t), P_{\mathcal{H}}(t), p^0, a_{\mathcal{H}}(t)) \\ &= \lim_{\tau \rightarrow T_1} (-P_{\rho_{\mathcal{H}}}(\tau)) = -P_{\rho_{\mathcal{H}}}(T_1) \end{aligned}$$

$$\begin{aligned} \widetilde{H}_2(X_2(t_2^+), P_2(t_2^+), p^0, \alpha_2(t_2^+)) &= \lim_{t \rightarrow t_2^+} \widetilde{H}_2(X_2(t), P_2(t), p^0, \alpha_2(t)) \\ &= \lim_{\tau \rightarrow T_0} (-P_{\rho_2}(\tau)) = -P_{\rho_2}(T_0). \end{aligned}$$

Since by (44), (45) we have $P_{\rho_{\mathcal{H}}}(T_0) = P_{\rho_1}(T_1)$ and $P_{\rho_{\mathcal{H}}}(T_1) = P_{\rho_2}(T_0)$, the above equalities give

$$\widetilde{H}_1(X_1(t_1^-), P_1(t_1^-), p^0, \alpha_1(t_1^-)) = \widetilde{H}_{\mathcal{H}}(X_{\mathcal{H}}(t_1^+), P_{\mathcal{H}}(t_1^+), p^0, a_{\mathcal{H}}(t_1^+)),$$

$$\widetilde{H}_{\mathcal{H}}(X_{\mathcal{H}}(t_2^-), P_{\mathcal{H}}(t_2^-), p^0, a_{\mathcal{H}}(t_2^-)) = \widetilde{H}_2(X_2(t_2^+), P_2(t_2^+), p^0, \alpha_2(t_2^+)),$$

therefore, by the optimality of the trajectory, we can conclude that

$$H_1(X_1(t_1^-), P_1(t_1^-), p^0) = H_{\mathcal{H}}(X_{\mathcal{H}}(t_1^+), P_{\mathcal{H}}(t_1^+), p^0), \tag{48}$$

$$H_{\mathcal{H}}(X_{\mathcal{H}}(t_2^-), P_{\mathcal{H}}(t_2^-), p^0) = H_2(X_2(t_2^+), P_2(t_2^+), p^0). \tag{49}$$

To obtain the jump conditions on the adjoint vector we exploit the transversality conditions on the duplicated problem ((22) and (23) in Theorem 2.1). By applying the usual change of variable in (46) and (47) we have

$$P_{\mathcal{H}}(t_1^+) = P_1(t_1^-) + \nu_1 \nabla \Psi(X_1(t_1^-)) \quad \text{and} \quad P_2(t_2^+) = P_{\mathcal{H}}(t_2^-) + \nu_2 \nabla \Psi(X_2(t_2^+)). \tag{50}$$

Note now that by definition of $\widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_{\mathcal{H}}$ the continuity conditions (48)-(49) read

$$\langle P_1(t_1^-), f_1(t_1^-) \rangle + p^0 l_1(t_1^-) = \langle P_{\mathcal{H}}(t_1^+), f_{\mathcal{H}}(t_1^+) \rangle_{\mathcal{H}} + p^0 l_{\mathcal{H}}(t_1^+) \tag{51}$$

$$\langle P_{\mathcal{H}}(t_2^-), f_{\mathcal{H}}(t_2^-) \rangle_{\mathcal{H}} + p^0 l_{\mathcal{H}}(t_2^-) = \langle P_2(t_2^+), f_2(t_2^+) \rangle + p^0 l_2(t_2^+) \tag{52}$$

where we used the short notations

$$f_i(t_i^\pm) = f_i(X_i(t_i^\pm), \alpha_i(t_i^\pm)), \quad l_i(t_i^\pm) = l_i(X_i(t_i^\pm), \alpha_i(t_i^\pm))$$

with $i \in \{1, 2, \mathcal{H}\}$. By using twice $P_{\mathcal{H}}(t_1^+) = P_1(t_1^-) + \nu_1 \nabla \Psi(X_1(t_1^-))$ and by recalling that by construction $\langle \nabla \Psi(X_1(t_1^-)), f_{\mathcal{H}}(t_1^+) \rangle_{\mathcal{H}} = 0$ the equality (51) becomes

$$\begin{aligned} &\nu_1 \langle \nabla \Psi(X_1(t_1^-)), f_1(t_1^-) \rangle \\ &= \langle P_{\mathcal{H}}(t_1^+), f_1(t_1^-) \rangle - \langle P_1(t_1^-), f_{\mathcal{H}}(t_1^+) \rangle_{\mathcal{H}} + p^0 (l_1(t_1^-) - l_{\mathcal{H}}(t_1^+)) \end{aligned}$$

thus

$$\nu_1 = \frac{\langle P_{\mathcal{H}}(t_1^+), f_1(t_1^-) \rangle - \langle P_1(t_1^-), f_{\mathcal{H}}(t_1^+) \rangle_{\mathcal{H}} + p^0 (l_1(t_1^-) - l_{\mathcal{H}}(t_1^+))}{\langle \nabla \Psi(X_1(t_1^-)), f_1(t_1^-) \rangle},$$

since by assumption $\langle \nabla \Psi(X_1(t_1^-)), f_1(t_1^-) \rangle \neq 0$. Similarly, if we replace $P_2(t_2^+) = P_{\mathcal{H}}(t_2^-) + \nu_2 \nabla \Psi(X_2(t_2^+))$ in (52) we obtain

$$\langle P_{\mathcal{H}}(t_2^-), f_{\mathcal{H}}(t_2^-) \rangle_{\mathcal{H}} + p^0 l_{\mathcal{H}}(t_2^-) = \langle P_{\mathcal{H}}(t_2^-) + \nu_2 \nabla \Psi(X_2(t_2^+)), f_2(t_2^+) \rangle + p^0 l_2(t_2^+).$$

Thus

$$\nu_2 = \frac{\langle P_{\mathcal{H}}(t_2^-), f_{\mathcal{H}}(t_2^-) \rangle_{\mathcal{H}} - \langle P_{\mathcal{H}}(t_2^-), f_2(t_2^+) \rangle + p^0 (l_{\mathcal{H}}(t_2^-) - l_2(t_2^+))}{\langle \nabla \Psi(X_2(t_2^+)), f_2(t_2^+) \rangle}$$

thanks to the assumption $\langle \nabla \Psi(X_2(t_2^+)), f_2(t_2^+) \rangle \neq 0$.

In order to conclude the proof we need, roughly speaking, to replace $P_1, P_2, P_{\mathcal{H}}$ by Q_1, Q_2 and $Q_{\mathcal{H}}$. To this aim we compute the relation between $P_1, P_2, P_{\mathcal{H}}$ and the derivatives of $S_{1,\mathcal{H},2}$. This is done in Lemma 4.1 hereafter.

Lemma 4.1. *Under the assumptions $(H\mathcal{H}), (Hf_i), (Hf_{\mathcal{H}})$ and (Hu) , given any $(x^0, t^0; x^f, t^f) \in \Omega_1 \times (0, +\infty) \times \Omega_2 \times (0, +\infty)$, if $\chi = (x_1, t_1, x_2, t_2) = \chi((x^0, t^0; x^f, t^f))$ is a minimizer of (40), then*

$$\frac{\partial}{\partial t^0} S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \partial_2 \Sigma((x^0, t^0, \chi), (\chi, x^f, t^f)) \tag{53}$$

$$\begin{aligned} \frac{\partial}{\partial t^f} S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) &= \partial_{12} \Sigma((x^0, t^0, \chi), (\chi, x^f, t^f)) \\ \nabla_{x^0} S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) &= \nabla_1 \Sigma((x^0, t^0, \chi), (\chi, x^f, t^f)) \\ \nabla_{x^f} S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) &= \nabla_{11} \Sigma((x^0, t^0, \chi), (\chi, x^f, t^f)). \end{aligned} \tag{54}$$

Moreover, if $(x^0, t^0; x^f, t^f) \in \Omega_1 \times (0, +\infty) \times \mathcal{H} \times (0, +\infty)$ then

$$\nabla_{x^f}^{\mathcal{H}} S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \nabla_9^{\mathcal{H}} \Sigma((x^0, t^0, \chi), (\chi, x^f, t^f))$$

and if $(x^0, t^0; x^f, t^f) \in \mathcal{H} \times (0, +\infty) \times \Omega_2 \times (0, +\infty)$ then

$$\nabla_{x^0}^{\mathcal{H}} S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \nabla_3^{\mathcal{H}} \Sigma((x^0, t^0, \chi), (\chi, x^f, t^f)).$$

Before proving this lemma, let us conclude the proof. By (54) in Lemma 4.1 we have

$$\nabla_{x^0} S_{1,\mathcal{H},2}(X_1(t), t; x^f, t^f) = \nabla_1 \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) \quad \forall \tau \in (T_0, T_1) \quad \forall t \in (t^0, t_1),$$

therefore, by the continuity of the adjoint vector and (29), we have

$$\begin{aligned} P_1(t_1^-) &= \lim_{t \rightarrow t_1^-} P_1(t) = \lim_{\tau \rightarrow T_1} P_{Y_1}(\tau) \\ &= \lim_{\tau \rightarrow T_1} -\nabla_1 \Sigma(\mathbb{Z}(\tau), \tau; Z_1, T_1) = \lim_{t \rightarrow t_1^-} -\nabla_{x^0} S_{1,\mathcal{H},2}(X_1(t), t; x^f, t^f), \end{aligned}$$

that is, $P_1(t_1^-) = Q_1(t_1^-)$.

In a similar way, by Lemma 4.1 below, equalities (31)-(33) and the continuity of the adjoint vector, we obtain $P_2(t_2^+) = Q_2(t_2^+)$, $P_{\mathcal{H}}(t_2^-) = Q_{\mathcal{H}}(t_2^-)$. This concludes the proof of Theorem 2.2.

Proof of Lemma 4.1. Given any $\lambda = (x^0, t^0; x^f, t^f)$, let

$$\chi = (x_1, t_1, x_2, t_2) = (x_1(\lambda), t_1(\lambda), x_2(\lambda), t_2(\lambda))$$

be a minimizer of (40). We can then write

$$\begin{aligned} S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) \\ = \Sigma\left((x^0, t^0, x_1(\lambda), t_1(\lambda), x_2(\lambda), t_2(\lambda)); (x_1(\lambda), t_1(\lambda), x_2(\lambda), t_2(\lambda), x^f, t^f)\right). \end{aligned}$$

We first remark that putting together (20)-(23) in Theorem 2.1 and (28) in Remark 2 we have

$$\begin{aligned} (\partial_4\Sigma + \partial_8\Sigma)((\lambda^0, \chi(\lambda)); (\chi(\lambda), \lambda^f)) &= 0 \\ (\partial_6\Sigma + \partial_{10}\Sigma)((\lambda^0, \chi(\lambda)); (\chi(\lambda), \lambda^f)) &= 0 \\ (\nabla_3\Sigma + \nabla_7\Sigma)((\lambda^0, \chi(\lambda)); (\chi(\lambda), \lambda^f)) &= \nu_1 \nabla\Psi(x_1) \\ (\nabla_5\Sigma + \nabla_9\Sigma)((\lambda^0, \chi(\lambda)); (\chi(\lambda), \lambda^f)) &= \nu_2 \nabla\Psi(x_2) \\ \Psi(x_1(\lambda)) &= 0 \\ \Psi(x_2(\lambda)) &= 0. \end{aligned} \tag{55}$$

We will only provide the details of the proof of (53) and (54), the other proofs being similar. Setting $\bar{\chi} = ((\lambda^0, \chi(\lambda)), (\chi(\lambda), \lambda^f))$, by simple computations we get

$$\begin{aligned} \frac{\partial S}{\partial t^0}(\lambda) &= \partial_2\Sigma(\bar{\chi}) + \langle \nabla_3\Sigma(\bar{\chi}), \frac{\partial x_1}{\partial t^0}(\lambda) \rangle + \partial_4\Sigma(\bar{\chi}) \frac{\partial t_1}{\partial t^0}(\lambda) + \langle \nabla_5\Sigma(\bar{\chi}), \frac{\partial x_2}{\partial t^0}(\lambda) \rangle \\ &\quad + \partial_6\Sigma(\bar{\chi}) \frac{\partial t_2}{\partial t^0}(\lambda) + \langle \nabla_7\Sigma(\bar{\chi}), \frac{\partial x_1}{\partial t^0}(\lambda) \rangle + \partial_8\Sigma(\bar{\chi}) \frac{\partial t_1}{\partial t^0}(\lambda) \\ &\quad + \langle \nabla_9\Sigma(\bar{\chi}), \frac{\partial x_2}{\partial t^0}(\lambda) \rangle + \partial_{10}\Sigma(\bar{\chi}) \frac{\partial t_2}{\partial t^0}(\lambda). \end{aligned}$$

Therefore, thanks to (55), we have

$$\frac{\partial S}{\partial t^0}(\lambda) = \partial_2\Sigma(\bar{\chi}) + \langle \mu_1 \nabla\Psi(x_1(\lambda)), \frac{\partial x_1}{\partial t^0}(\lambda) \rangle + \langle \mu_2 \nabla\Psi(x_2(\lambda)), \frac{\partial x_2}{\partial t^0}(\lambda) \rangle.$$

Moreover, since differentiating conditions $\Psi(x_1(\lambda)) = 0$, $\Psi(x_2(\lambda)) = 0$ in (55) we obtain

$$\langle \nabla\Psi(x_1(\lambda)), \frac{\partial x_1}{\partial t^0}(\lambda) \rangle = 0 \quad \text{and} \quad \langle \nabla\Psi(x_2(\lambda)), \frac{\partial x_2}{\partial t^0}(\lambda) \rangle = 0$$

and we conclude that $\frac{\partial S}{\partial t^0}(\lambda) = \partial_2\Sigma(\bar{\chi})$.

Similarly,

$$\begin{aligned} \frac{\partial S}{\partial x^0}(\lambda) &= \nabla_1\Sigma(\bar{\chi}) + \langle \nabla_3\Sigma(\bar{\chi}), \frac{\partial x_1}{\partial x^0}(\lambda) \rangle + \partial_4\Sigma(\bar{\chi}) \frac{\partial t_1}{\partial x^0}(\lambda) + \langle \nabla_5\Sigma(\bar{\chi}), \frac{\partial x_2}{\partial x^0}(\lambda) \rangle \\ &\quad + \partial_6\Sigma(\bar{\chi}) \frac{\partial t_2}{\partial x^0}(\lambda) + \langle \nabla_7\Sigma(\bar{\chi}), \frac{\partial x_1}{\partial x^0}(\lambda) \rangle + \partial_8\Sigma(\bar{\chi}) \frac{\partial t_1}{\partial x^0}(\lambda) \\ &\quad + \langle \nabla_9\Sigma(\bar{\chi}), \frac{\partial x_2}{\partial x^0}(\lambda) \rangle + \partial_{10}\Sigma(\bar{\chi}) \frac{\partial t_2}{\partial x^0}(\lambda). \end{aligned}$$

Thanks to (55), this gives

$$\frac{\partial S}{\partial x^0}(\lambda) = \nabla_1\Sigma(\bar{\chi}) + \langle \nu_1 \nabla\Psi(x_1(\lambda)), \frac{\partial x_1}{\partial x^0}(\lambda) \rangle + \langle \nu_2 \nabla\Psi(x_2(\lambda)), \frac{\partial x_2}{\partial x^0}(\lambda) \rangle$$

by differentiating conditions $\Psi(x_1(\lambda)) = 0$, $\Psi(x_2(\lambda)) = 0$ in (55) we have

$$\langle \nabla\Psi(x_1(\lambda)), \frac{\partial x_1}{\partial x^0}(\lambda) \rangle = 0 \quad \text{and} \quad \langle \nabla\Psi(x_2(\lambda)), \frac{\partial x_2}{\partial x^0}(\lambda) \rangle = 0$$

and hence $\frac{\partial S}{\partial x^0}(\lambda) = \nabla_1 \Sigma(\bar{\chi})$. □

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