Control of the 1D continuous version of the Cucker-Smale model*

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***Abstract*— The well-known Cucker-Smale model is a macroscopic system reproducing the alignment of velocities in a group of autonomous agents. Here, we focus on its mean-field limit, which we call *the continuous Cucker-Smale model*. It is a transport partial differential equation with nonlocal terms.**

For some choices of the parameters in the Cucker-Smale model (and the continuous one), alignment is not ensured for some initial configurations, therefore it is natural to study the enforcing of alignment via an external force.

We provide a control strategy enforcing alignment for every initial data and acting only on a small portion of the crowd at each time. This is an adapted version of the *sparse* control for a finite number of agent, that is the constraint of acting on a small number of agents at each time.

Keywords: control of transport PDEs, PDEs with nonlocal terms, Cucker-smale model, collective behavior.

I. INTRODUCTION

In recent years, the study of collective behavior of a crowd of autonomous agents has drawn a great interest from scientific communities, e.g. in civil engineering (for evacuation problems), robotics (coordination of robots), computer science and sociology (social networks), and biology (crowds of animals). In particular, it is well-known that some simple rules of interaction between agents can provide the formation of special patterns, like in formations of bird flocks, lines, etc... This phenomenon is often referred to as *self-organization*. Beside the problem of analysing the collective behavior of a "closed" system, it is now interesting to understand what changes of behavior can be induced by an external agent (e.g. a policy maker) to the crowd. For example, one can try to enforce the creation of patterns when they are not formed naturally, or break the formation of such patterns. This is the problem of **control of crowds**, that we address in this article in a specific case.

In our article, we focus on a well-known model for crowds dynamics, called the Cucker-Smale model, first proposed in [3]. Such model may, for instance, reproduce the behavior of a group of birds, in which each bird tries to align its velocity to the velocities of its neighbors. The dynamics of the i-th bird is given by

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{1}{N}\sum_{j=1}^N \phi(|x_j - x_i|)(v_j - v_i), \quad i = 1, \dots N, \end{cases} \tag{1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a nonincreasing positive function, accounting for the influence between two individuals, that depends on the distance only. Depending on the strength of the interaction ϕ for large distances, the crowd can either exhibit asymptotic alignment of the velocities of birds (flocking) or it disperses into separate clusters without aligning.

Then, it is interesting to understand in which case an external controller can enforce flocking even when it does not appear naturally. For the original Cucker-Smale problem, a solution has been given in [2]. In that case, the authors considered only *sparse* strategies, in the sense that the controller was allowed to act on all birds, but just one at each time. Under this assumption, they were able to enforce flocking for every initial configuration.

In this article, instead, we focus on the control of the mean-field limit of the Cucker-Smale model, which we call the continuous Cucker-Smale model. In [5], the authors have formally proved the existence of the mean-field limit of the Cucker-Smale model when the number of agents tends to infinity. The limit of the dynamics (1) is given by the following transport PDE

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [\xi[f]\, f] = 0, \tag{2}$$

where $f = f(t, x, v)$ is the density of crowd and $\xi[f](x, v) := \int \phi(|x - y|)(w - v)\, df(x, v)$ is the interaction kernel. As it is evident from the expression of $\xi[f]$, the velocity field $\xi[f]$ acting on the v variable depends globally on f; i.e. $\xi[f](x)$ is not uniquely determined by the value of $f(x)$, but instead on the value of f in the whole space $\mathbb{R}^{2d}$. Equations of this kind are called transport equations with nonlocal interactions. Results of existence, uniqueness and regularity of solutions for this kind of equations are quite recent (see [1]). We recall them in Section II.

To study the control of equation (2), we make the following assumptions. The control system is given by:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(\xi[f] + \chi_\omega u) f] = 0,$$

where ω is the set on which the control acts, and u is the control. We study the following **continuous sparsity constraint**: we assume to act on a small amount of the crowd, and with a finite strength of our control. Then, we will assume $\int_\omega df(x, v) \le c$ with a fixed quantity c and $\|u\|_{L^\infty} \le 1$.

Under this assumption, we prove controllability to flocking for each initial condition with compact support. We prove

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this result in the 1D case, while a generalization to any dimension of the space is given in [8].

II. Transport equation with nonlocal velocity

In this section, we recall known results about nonlocal transport equations:

$$\partial_t \mu + \nabla \cdot (v\left[\mu\right] \mu) = 0. \tag{3}$$

Such transport PDE is written in terms of measures $\mu \in \mathcal{P}(\mathbb{R}^d)$, where $\mathcal{P}(\mathbb{R}^d)$ is the space of probability measures on $\mathbb{R}^d$. We also deal with the space $\mathcal{P}_c(\mathbb{R}^d)$ of probability measures with compact support, as well as the space $\mathcal{P}_c^{ac}(\mathbb{R}^{2d})$ of probability measures with compact support and absolutely continuous with respect to the Lebesgue measure.

The fundamental tool we use to analyse such PDEs is the $1-$**Wasserstein distance**, defined below.

Definition 1: Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. The 1-Wasserstein distance is

$$W_1(\mu, \nu) := \sup \left\{ \int f \, d(\mu - \nu) \mid \|f\|_{Lip} \leq 1 \right\},$$

where $\|f\|_{Lip}$ is the Lipschitz constant of the function f. For a description of Wasserstein distances, see e.g [9].

We now recall a result ensuring existence and uniqueness for the solution of the Cauchy problem with dynamics (3).

Theorem 2: Let v satisfy the following hypotheses **(V)**:

(V)

The function $v\left[\mu\right] : \mathcal{P}(\mathbb{R}^d) \to C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfies

- $v\left[\mu\right]$ is uniformly Lipschitz and uniformly sublinear, i.e. there exist L, M not depending on μ, such that for all $\mu \in \mathcal{P}(\mathbb{R}^d), x, y \in \mathbb{R}^n$,

$$|v\left[\mu\right](x) - v\left[\mu\right](y)| \leq L|x-y|, |v\left[\mu\right](x)| \leq M(1+|x|).$$

- v is a Lipschitz function, i.e. there exists K such that

$$\|v\left[\mu\right] - v\left[\nu\right]\|_{C^0} \leq K W_1(\mu, \nu).$$

Then for every $\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)$ there exists a unique solution of the Cauchy problem:

$$\partial_t \mu + \nabla \cdot (v\mu) = 0, \qquad \mu_{|t=0} = \mu_0.$$

Moreover, if $\mu_0 \in \mathcal{P}_c^{ac}(\mathbb{R}^{2d})$, then the solution satisfies $\mu(t) \in \mathcal{P}_c^{ac}(\mathbb{R}^{2d})$ for all times t.

Proof: The proof under the stronger hypothesis $|v\left[\mu\right](x)| \leq M$ is given in [7]. The proof with constraint $|v\left[\mu\right](x)| \leq M(1+|x|)$ is a straight generalization. ■

These results can be generalized to transport PDEs in presence of sources, see [6].

III. The Cucker-Smale model

In this section, we recall the definition and properties of the Cucker-Smale model. We first recall the original model defined by Cucker and Smale in [3]. Consider N agents evolving in a space of dimension d. We denote with x_i, v_i the space-velocity coordinates of each agent, hence $i = 1, \ldots, N$. The generalized Cucker-Smale model is given by the following equations:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^N \phi(|x_j - x_i|)(v_j - v_i), \quad i = 1, \ldots N. \end{cases} \tag{4}$$

Here $\phi : \mathbb{R} \to \mathbb{R}$ is a nonincreasing positive function, that models the influence between two individuals, that depends on the distance only.

Such model has some interesting features. Dynamics of the barycenter is given by the following proposition.

Proposition 3: Let $x^* := \frac{1}{N}\sum_i x_i$ and $v^* := \frac{1}{N}\sum_i v_i$ be the space and velocity barycenter of the crowd. If the crowd evolves with dynamics (4), then it holds

$$\dot{v}^* = 0 \qquad \text{and} \qquad \dot{x}^* = v^*.$$

Proof: See the original proof in [3]. ■

The most interesting feature is the self-organization of the crowd. Indeed, if the influence of each agent on the others is sufficiently big (i.e. ϕ does not decrease too fast), then the crowd converges to the common mean velocity v^*. Such phenomenon is called "flocking", since it is a common phenomenon in birds' flocks. The formal definition is the following, see [3].

Definition 4: Let $((x_1(t), v_1(t)), \ldots, (x_N(t), v_N(t)))$ be a solution of (4). We say that it flocks if the two following conditions hold:

- it exists X_M such that $|x_i(t) - x^*(t)| \leq X_M$ for all $i = 1, \ldots, N$ and all times $t > 0$;
- it holds $\lim_{t\to\infty} |v_i(t) - v^*| = 0$ for all $i = 1, \ldots, N$.

Conditions ensuring flocking are well known, see e.g. [3], [4]. At the same time, if the initial configuration is "too dispersed" and the interaction among agents is "too weak", then the crowd does not converge to flocking. See an example in the original paper [3].

We now recall the mean-field limit to pass from the finite dimensional Cucker-Smale system (4) to a transport PDE when the number of agents N tends to infinity.

Theorem 5: Consider the following Cauchy problem

$$\begin{cases} \partial_t \mu + v \cdot \nabla_x \mu + \nabla_v \cdot \left[\xi\left[\mu\right] \mu\right] = 0, \\ \mu(0) = \mu_0 \end{cases} \tag{5}$$

where

$$\xi\left[\mu\right](x, v) := \int \phi(|x-y|)(w - v) \, d\mu(y, w). \tag{6}$$

Then there exists a unique solution of (5). Moreover, if $\mu_0 = \frac{1}{N}\sum_i \delta_{(x_i^0, v_i^0)}$, then $\mu(t) = \frac{1}{N}\sum_i \delta_{(x_i(t), v_i(t))}$, where $((x_1(t), v_1(t)), \ldots, (x_N(t), v_M(t)))$ is the solution of the finite-dimensional Cucker-Smale problem (4).

Proof: See [5]. Existence and uniqueness of the solution are consequences of Theorem 2, since the vector field $(v, \xi\left[\mu\right])$ satisfies **(V)**. ■

One can define the barycenter (x^*, v^*) in the measure setting too. We observe that the barycenter satisfies properties identical to the finite-dimensional setting.

Proposition 6: Define $x^* := \int x \, d\mu(x,v)$ and $v^* := \int v \, d\mu(x,v)$ the space and velocity barycenter of the crowd μ. If μ is a solution of (5), then x^*, v^* satisfy

$$\dot{v}^* = 0 \qquad \text{and} \qquad \dot{x}^* = v^*.$$

Proof: See e.g. [5, Prop. 3.1]. ■

We now give the definition of flocking for a measure.

Definition 7: Let $\mu(t)$ be a solution of (5). We say that it flocks if the two following conditions hold:

- it exists X_M such that $\mathrm{supp}(\mu(t)) \subseteq B(x^*(t), X_M) \times \mathbb{R}^d$ for all times $t > 0$;
- it holds $\int |v - v^*|^2 \, d\mu(t)(x,v) \to 0$ for $t \to \infty$.

We now recall a condition ensuring flocking of a given initial measure.

Corollary 8: Let μ_0 be a probability measure with compact support contained in $B_{X_0}(x_0) \times B_{V_0}(v_0)$, for a point (x_0, v_0) and some X_0, V_0. If X_0, V_0 satisfy

$$2V_0 \le \int_{2X_0}^{+\infty} \phi(2x) \, dx, \tag{7}$$

then the solution $\mu(t)$ converges to flocking.

Proof: This is an adapted version of [4, Thm. 3.2]. ■

A. Estimates on the vector field $\xi[\mu]$ for 1D systems

In this section, we study the behavior of the vector field $\xi[\mu]$ as a function of the support of μ. We study the 1-dimensional case, i.e. $d = 1$.

Consider the velocity variable v, and denote with e_2 the corresponding unit vector in the canonical base. For simplicity of notation, we assume that the velocity variable satisfies $v \in [0, W]$ for some W, eventually after a traslation in the velocity variables. Similarly, we assume $x \in [0, Y]$. This choice provides invariance of the positive space $(\mathbb{R}^+)^2$. We will use the quantity Y to denote the size of the support in the variable x and W for the size of the support in the variable v.

The result of this section is an estimate showing that, given (Y, W) the space-velocity size of the support, the vector field satisfies $\xi[\mu](x,v) \perp (v - v^*)e_2$ only for v that are "sufficiently inside" the support.

Proposition 9: Let μ be a probability density in $\mathbb{R}^2$, with velocity barycenter v^* and support in $B_Y(\bar{x}) \times [0, W]$ for some $\bar{x}$ and X, W. Let $(x,v) \in \mathbb{R}^2$ be such that $(\xi[f](x,v)) \perp (v - v^*)e_2$. Then it either holds $0 \le v - v^* \le r^+$ with

$$r^+ = \frac{\phi(0)}{\phi(0) + \phi(2X)}(W - v^*), \tag{8}$$

or $0 \ge v - v^* \ge -r^-$ with $r^- = \frac{\phi(0)}{\phi(0)+\phi(2X)}(v^*)$.

Proof: We give the proof of the first case only, in which $v - v^* \ge 0$. For the second case, it is sufficient to apply the change of variable $v \to W - v$.

Let (x,v) satisfy $\xi[\mu](x,v) \perp (v - v^*)e_2$, i.e.

$$\int_{\mathbb{R}^2} \phi(|x-y|)(w - v)(v - v^*) \, d\mu(y,w) = 0. \tag{9}$$

Divide the space $\mathbb{R}^{2d}$ in three subsets A, B, C as follows:

$$\begin{aligned} A &:= \left\{(y,w) \in \mathbb{R}^{2d} \mid w \ge v\right\}, \\ B &:= \left\{(y,w) \in \mathbb{R}^{2d} \mid v^* \le w < v\right\}, \\ C &:= \left\{(y,w) \in \mathbb{R}^{2d} \mid w < v^*\right\}. \end{aligned}$$

We first prove that $\mu(A) > 0$ by contradiction. Assume that $\mu(A) = 0$. Observe that (9) reads as

$$\begin{aligned} &\int_A \phi(|x-y|)(w - v)(v - v^*) \, d\mu(y,w) = \\ &\int_B \phi(|x-y|)(v - w)(v - v^*) \, d\mu(y,w) + \\ &+ \int_C \phi(|x-y|)(v - w)(v - v^*) \, d\mu(y,w). \end{aligned} \tag{10}$$

Since $\phi(|x-y|)(v-w)(v-v^*) > 0$ for all $(y,w) \in B \cup C$, then the right-hand side of (10) is strictly positive when $\mu(B) > 0$ or $\mu(C) > 0$. Hence, $\mu(A) = 0$ implies $\mu(B) = \mu(C) = 0$. This is in contradiction with the fact that $|\mu| = 1$.

We use again (10). Using that $\int_B \phi(|x-y|)(v-w)(v-v^*) \, d\mu(y,w) \ge 0$, that $(w - v) \le (W - v^*) - (v - v^*)$ in A and that $\phi(|x-y|) \le \phi(0)$, then (10) becomes

$$\begin{aligned} &\int_C \phi(|x-y|)(v - w)(v - v^*) \, d\mu(y,w) \le \\ &\phi(0)((W - v^*) - (v - v^*))(v - v^*)\mu(A). \end{aligned} \tag{11}$$

Since v^* is the velocity barycenter of μ, then

$$\int_{\mathbb{R}^2} (w - v^*) \cdot z \, d\mu(y,w) = 0,$$

for any vector z. Choosing the vector $z = v - v^*$ we get:

$$\begin{aligned} &\int_{A \cup B} (w - v^*)(v - v^*) \, d\mu(y,w) = \\ &= \int_C (v^* - w)(v - v^*) \, d\mu(y,w). \end{aligned}$$

For the left hand side, observe that $(w - v^*)(v - v^*) \ge (v - v^*)^2$ in A and that $\int_B (w - v^*)(v - v^*) \, d\mu(y,w) \ge 0$. For the right hand side, observe that $(v^* - w)(v - v^*) \le (v - w)(v - v^*)$ in C. This gives

$$\begin{aligned} \mu(A)(v - v^*)^2 &\le \int_A (w - v^*)(v - v^*) \, d\mu(y,w) \le \\ &\le \int_C (v^* - w)(v - v^*) \, d\mu(y,w) \le \\ &\le \int_C (v - w)(v - v^*) \, d\mu(y,w). \end{aligned} \tag{12}$$

Since for all $x, y \in B_X(x_0)$ it holds $\phi(|x-y|) \ge \phi(2X)$, then combine (11) and (12) to have

$$\begin{aligned} &\mu(A)(v - v^*)^2 \le \\ &\le \frac{\phi(0)}{\phi(2X)}((W - v^*) - (v - v^*))(v - v^*)\mu(A). \end{aligned}$$

Simplify this expression by $\mu(A) \cdot (v - v^*) > 0$ and find (8). ■

B. The continuous Cucker-Smale model with control

In this section we consider the following equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(\xi [f] + \chi_\omega u) f] = 0. \qquad (13)$$

Here the extra term $\chi_\omega u$ appears. It is a control term, i.e. a term chosen by an external agent (e.g. a policy maker) to achieve a certain goal. Our goal is to achieve flocking of the crowd for any initial data f_0, by choosing the set ω and the control u as function of the time, with some constraints specified in the following.

We only deal with probability measures $f \in \mathcal{P}_c^{ac}(\mathbb{R}^{2d})$, i.e. absolutely continuous with respect to the Lebesgue measure and with compact support. For this reason, we will limit our choice of the extra term $\chi_\omega u$ to ensure that for any $f(0) \in \mathcal{P}_c^{ac}(\mathbb{R}^{2d})$, the corresponding solution f of (13) satisfies $f(t) \in \mathcal{P}_c^{ac}(\mathbb{R}^{2d})$ for all $t \geq 0$.

Here we prove some properties of the solution of the controlled PDE (13). In particular, we will show some differences with respect to the uncontrolled PDE (5). All throughout this article, we consider the following assumptions on the control $\chi_\omega u$:

(U): For each time t, the function $\chi_{\omega(t)} u(t, x, v)$ is C^0 and piecewise-C^1 with respect to its arguments (x, v). The function $\chi_{\omega(t)} u(t, x, v)$ is measurable with respect to time and its support is uniformly bounded in time. The function u is uniformly bounded in time, i.e. there exists U such that $|u(t, x, v)| \leq U$ for all t, x, v.

First, we estimate the evolution of the barycenter $(x^*(t), v^*(t))$.

Proposition 10: Let $f(t)$ be a solution of (13) with $f(0) \in \mathcal{P}_c^{ac}(\mathbb{R}^{2d})$. Let $(x^*(t), v^*(t))$ be the barycenter of the solution of (13). Then it holds

$$\dot{x}^*(t) = v^*(t), \qquad \dot{v}^*(t) = \int_{\omega(t)} u(t, x, v) f(t, x, v)\, dx\, dv.$$

Proof: This is a straight consequence of (13). ■

We now estimate the evolution of the L^∞ norm of the solution of (13). This will be useful to check that a given control leaves $\|f\|_\infty$ finite for finite times, hence $f \in \mathcal{P}_c^{ac}(\mathbb{R}^{2d})$.

Proposition 11: Let f be a solution of (13) for a given control $\chi_\omega u$ satisfying **(U)**. Then it holds

$$\partial_t \|f\|_\infty \leq \|f\|_\infty \left(d\phi(0) + \|\nabla_v \cdot u\|_{L^\infty(\omega(t))} \right). \qquad (14)$$

Proof: Adapt the proof of [5, Pr. 3.1]. ■

IV. CONTROL OF (13) TO FLOCKING

In this section we consider the controlled PDE

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(\xi [f] + \chi_\omega u) f] = 0, \qquad (15)$$

with the following contraint on the control ω, u, that we call **continuous sparse constraint**:

- the external control acts on **a small amount of the crowd**: this is modeled by $\int_\omega df(x, v) \leq c < 1$;
- the action itself is bounded on the crowd in ω: this is modeled by $\|u\|_{L^\infty} \leq 1$.

Our goal is to achieve flocking for a crowd of agents evolving according to (15). More precisely, first we will force the crowd inside the flocking region in a finite time T with a suitable control; then, the uncontrolled evolution, given by (5), will provide convergence of the velocities to a Dirac delta in infinite time, together with a uniform bound in the space variable. The time T depends on the initial distribution f_0 of the crowd: The more "dispersed" is f_0, the larger will be the time T.

We present here the 1-dimensional case, i.e. $d = 1$. We first define the FUNDAMENTAL STEP $\mathcal{S}$ of our algorithm, and then show that a finite number of iterations of this fundamental step $\mathcal{S}$ provides convergence to flocking.

The fundamental step $\mathcal{S}$ of our algorithm has as input a function $f^0 = f(0)$ representing the initial data in (13), and provides as outputs a time T^0 and a function $f^1 = f(T^0)$, representing the time horizon in (13) and the corresponding solution at time T^0 for some specific choice of the control $\chi_\omega u$. We define it as follows.

As in Section III-A, we assume that the space variable satisfies $x \in [0, Y]$, and the velocity variable satisfies $v \in [0, W]$, eventually after translation.

Definition 12 (FUNDAMENTAL STEP $\mathcal{S}$): Take $f^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$ with a compact support contained in $[0, Y^0] \times [0, W^0]$. Consider the velocity barycenter $v^*(0) \in (0, W^0)$. Compute $\alpha^+(0) := \frac{\phi(0)}{\phi(0) + \phi(Y^0 + W^0)} (W^0 - v^*(0))$,

$$\begin{aligned}
\beta^+(0) &:= \frac{1}{3}(W^0 - \alpha^+(0) - v^*(0)), \\
\alpha^-(0) &:= \frac{\phi(0)}{\phi(0) + \phi(Y^0 + W^0)} v^*(0), \\
\beta^-(0) &:= \frac{1}{3}(v^*(0) - \alpha^-(0)), \\
\alpha^0 &:= \max\left\{\alpha^+(0), \alpha^-(0)\right\}, \\
\beta^0 &:= \max\left\{\beta^+(0), \beta^-(0)\right\},
\end{aligned}$$

$$\begin{aligned}
\gamma_a &:= \alpha^0 + \beta^0, \qquad \gamma_b := \alpha^0 + 2\beta^0, \\
\gamma_c &:= \alpha^0 + 3\beta^0, \qquad \gamma_d := \alpha^0 + 4\beta^0.
\end{aligned}$$

Divide the interval $[0, Y^0]$ in $n = \lceil \frac{2}{c} \rceil$ intervals $[0, x_1], [x_1, x_2], \ldots, [x_{n-1}, Y^0]$ as follows:

- x_1 is the minimum value such that $\Omega_1^0 := [0, x_1] \times [0, W^0]$ satisfies $f^0(\Omega_1^0) = \frac{c}{2}$; ...
- x_i is the minimum value such that $\Omega_i^0 := [x_{i-1}, x_i] \times [0, W^0]$ satisfies $f^0(\Omega_i^0) = \frac{c}{2}$; ...
- $x_n = Y^0$ and $\Omega_n^0 := [x_{n-1}, x_n] \times [0, W^0]$ satisfies $f^0(\Omega_n^0) \leq \frac{c}{2}$;

Choose the biggest $\varepsilon^0 > 0$ such that each set $[x_i - 3\varepsilon^0, x_{i+1} + 3\varepsilon^0] \times [0, W^0]$ contains at most a mass c of f^0, i.e. for all $i = 1 \ldots, n$ it holds

$$f^0([x_i - 3\varepsilon^0, x_{i+1} + 3\varepsilon^0] \times [0, W^0]) \leq c.$$

Define the control sets $\omega_i := \left[x_i - 2\varepsilon^0, x_{i+1} + 2\varepsilon^0\right] \times$

$$([v^*(0) - \gamma_d, v^*(0) - \gamma_a] \cup [v^*(0) + \gamma_a, v^*(0) + \gamma_d]).$$

Define the following family of functions ψ_i. First define the set $A_i := \big([-\infty, x_i - 2\varepsilon^0] \times \mathbb{R}\big) \cup \big([x_{i+1} + 2\varepsilon^0, +\infty] \times \mathbb{R}\big) \cup (\mathbb{R} \times ([-\infty, v^*(0) - \gamma_d] \cup$

$$\cup [v^*(0) - \gamma_a, v^*(0) + \gamma_a] \cup [v^*(0) + \gamma_d, +\infty))$$

and $\psi_i = 0$ on A_i.

Then define $B_i := [x_i - \varepsilon^0, x_{i+1} + \varepsilon^0] \times$

$$([v^*(0) - \gamma_c, v^*(0) - \gamma_b] \cup [v^*(0) + \gamma_b, v^*(0) + \gamma_c])$$

and $\psi_i = 1$ on B_i.

Then define $\tilde{\psi}_i(x, v) := \min \Big\{ \frac{|x - x_i - 2\varepsilon^0|}{\varepsilon^0}, \frac{|x - (x_{i+1} + 2\varepsilon^0)|}{\varepsilon^0}, \frac{|v - (v^*(0) - \gamma_a)|}{\beta^0}, \frac{|v - (v^*(0) + \gamma_a)|}{\beta^0}, \frac{|v - (v^*(0) - \gamma_d)|}{\beta^0}, \frac{|v - (v^*(0) + \gamma_d)|}{\beta^0} \Big\}$ and define $\psi_i = \tilde{\psi}_i$ on $\mathbb{R}^2 \setminus (A_i \cup B_i)$.

Define the controls $u_i(x, v) := \psi_i(x, v) \frac{v - v^*(0)}{|v - v^*(0)|}$.

Choose $T^0 := \min \left\{ \frac{\varepsilon}{W^0}, \frac{\beta^0}{2c}, 1 \right\}$.

Define the strategy on the time interval $[0, T^0]$:

- On $[0, \frac{T^0}{n})$ apply the control $\chi_{\omega_1} u_1$; ...
- On $[\frac{(i-1)T^0}{n}, \frac{iT^0}{n})$ apply the control $\chi_{\omega_i} u_i$; ...
- On $[\frac{(n-1)T^0}{n}, T^0]$ apply the control $\chi_{\omega_n} u_n$;

We now show that such fundamental step $\mathcal{S}$ is well defined, and it provides decreasing of the velocity support.

Proposition 13: Take $f^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$, with a compact support contained in $[0, Y^0] \times [0, W^0]$, then it holds:

1) the fundamental step $\mathcal{S}$ is well-defined;
2) applying $\mathcal{S}$ to f^0, we get $f^1 := f(T^0) \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$;
3) there exists a constant $a > 0$ such that f^1 has support contained in $[0, Y^1] \times [a, a + W^1]$ with $Y^1 \leq Y^0 + W^0$;
4) each set $\mathbb{R} \times [v^*(0) - \gamma_a - k^-, v^*(0) + \gamma_a + k^+]$ with $k^-, k^+ \geq 0$ is invariant;
5) either $[a, a + W^1] \subset [0, W^0 - \frac{T^0}{n}]$ or $[a, a + W^1] \subset [\frac{T^0}{n}, W^0]$, that implies $W^1 \leq W^0 - \frac{T^0}{n}$;
6) $\varepsilon^0 \geq \frac{c}{12\|f^0\|_\infty W^0}$;
7) the control satisfies **(U)** and the constraints $\int_{\omega(t)} f(t) \leq c$ and $\|u(t)\|_\infty \leq 1$.

Proof: **Property 1.** We denote with $\mathcal{S}.1, \mathcal{S}.2, \ldots$ the substeps of the fundamental step. We prove that each substep is well-defined. For $\mathcal{S}.1$, computations are explicit.

For $\mathcal{S}.2$, each x_i is uniquely determined, since $f^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$. For the same reason, in $\mathcal{S}.3$ for each interval $[x_{i-1}, x_i]$ there exists a maximum ε_i, and ε^0 is a minimum on a finite set. For $\mathcal{S}.4, \mathcal{S}.5, \mathcal{S}.6, \mathcal{S}.7$ there is nothing to prove.

Property 2. Since ψ_i is a C^0 and piecewise C^∞ function, then both the vector field $\nu_0 := (v, \xi[f])$ and each vector field $\nu_i := (v, \xi[f] + \chi_{\omega_i} u_i)$ ensure existence and uniqueness of the solution of (13). Moreover, by Theorem 2, $f^0 = f(0) \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$ implies $f\left(\frac{T}{n}\right) \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$, and iteratively we have $f^1 = f(T) \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$.

Property 3. The set $\mathbb{R} \times [0, W^0]$ is invariant with respect to the controlled dynamics, since the vector field $\xi(f)$ always points inwards on the boundary of the domain, and each u_i points inwards too on the boundary. As a consequence, the support of $f(t)$ is contained in $[0, Y^0 + W^0] \times [0, W^0]$ on the interval $[0, T^0]$ since $T^0 \leq 1$.

Property 4. We first study the case $k^- = k^+ = 0$. Consider the set $\mathbb{R} \times [v^*(0) - \alpha^0 - \beta^0, v^*(0) + \alpha^0 + \beta^0]$: it is invariant with respect to the dynamics. Indeed, first observe that the control does not act on such set, due to the definition of ω_i. Then, observe that $\dot{v}^* = \int_\omega u$ as in Proposition 10, and $\left|\int_\omega u\right| \leq c$, then $T^0 \leq \frac{\beta^0}{2c}$ implies $|v^*(t) - v^*(0)| \leq \frac{\beta(0)}{2} < \beta(t)$. Hence, one can prove

$$\xi[f(t)](x, v^*(0) + \gamma_a) \cdot (v^*(0) + \gamma_a - v^*(0)) < 0;$$
$$\xi[f(t)](x, v^*(0) - \gamma_a) \cdot (v^*(0) - \gamma_a - v^*(0)) < 0,$$

for all $t \in [0, T^0]$, by using Proposition 9. The same proof gives invariance of each set $\mathbb{R} \times [v^*(0) - \gamma_a - k^-, v^*(0) + \gamma_a + k^+]$ with $k^-, k^+ \geq 0$.

Property 5. Assume that $\beta^0 = \beta^+(0)$, the case $\beta^0 = \beta^-(0)$ being equivalent. Consider the evolution of each set $\Omega_i(t)$ via the vector field $(v, \xi[f(t)] + \chi_\omega u)$. For each time $t \in [0, i\frac{T^0}{n})$, the set $\mathbb{R} \times [0, W^0]$ is invariant, then $\Omega_i^0 \subset \mathbb{R} \times [0, W^0]$ implies $\Omega_i\left(i\frac{T^0}{n}\right) \subset \mathbb{R} \times [0, W^0]$.

For the times $t \in \left[i\frac{T^0}{n}, (i+1)\frac{T^0}{n}\right)$, denote with $0 \leq a(t) \leq b(t) \leq W^0$ the boundaries for which $\Omega_i(t) \subset \mathbb{R} \times [a(t), b(t)]$, and observe that $b(t) \geq W^0 - \beta^0$ implies that the control acts with value -1, hence $\dot{b} < -1$. Either this holds on the whole interval $\left[i\frac{T^0}{n}, (i+1)\frac{T^0}{n}\right)$ and then $b\left((i+1)\frac{T^0}{n}\right) \leq W^0 - \frac{T^0}{n}$, or it holds $b(t) < W^0 - \beta^0$ for some t. In both cases, we have $\Omega_i\left((i+1)\frac{T^0}{n}\right) \subset \mathbb{R} \times \left[0, W^0 - \frac{T^0}{n}\right]$. For each time $t \in \left[(i+1)\frac{T^0}{n}, T^0\right]$, invariance of $\mathbb{R} \times \left[0, W^0 - \frac{T^0}{n}\right]$ gives $\Omega_i(T^0) \subset \mathbb{R} \times \left[0, W^0 - \frac{T^0}{n}\right]$. Since the estimate holds for all $\Omega_i(t)$, then the support of f^1 is contained in $\mathbb{R} \times \left[0, W^0 - \frac{T^0}{n}\right]$.

Property 6. The mass contained in $[x_i, x_{i+1}] \times [0, W^0]$ is $\frac{c}{2}$, then the mass contained in $[x_i - l, x_{i+1} + l] \times [0, W^0]$ is less or equal to $\frac{c}{2} + 2\|f^0\|_\infty l W^0$. Since we want to have a mass c, we need to have $l \geq \frac{c}{4\|f^0\|_\infty W^0}$. Choose $3\varepsilon^0 = l$.

Property 7. The regularity of $\chi_\omega u$ is evident, since u is piecewise constant with respect to time and it is C^0 and piecewise-C^∞ with respect to (x, v). The constraint $\|u(t)\|_\infty \leq 1$ is evident too, due to the definition of ψ_i. For the constraint $\int_{\omega(t)} f(t) \leq c$, we prove the stronger condition $\int_{\omega_i} f(t) \leq c$ for all i. Observe that, since $\dot{x} = v \leq W^0$ for all $t \in [0, T^0]$, then the mass can travel on the x coordinate of at most a distance $T^0 W^0 \leq \varepsilon^0$. Then $\int_{\omega_i} f(t)\, dv \leq \int_{x_i - 3\varepsilon^0}^{x_{i+1} + 3\varepsilon^0} dx \int dv f^0 = c$. ■

We now repeat the FUNDAMENTAL STEP $\mathcal{S}$ until reaching a prescribed size η of the velocity support. We will then choose η to satisfy condition (7), that ensures flocking.

Definition 14 (COMPLETE ALGORITHM $\mathbb{S}$): Take $f^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$ with compact support in $[0, Y^0] \times [0, W^0]$ and fix $\eta > 0$. Apply the following strategy:

WHILE $W^i > \eta$ COMPUTE $f^{i+1} = \mathcal{S}(f^i)$.

We now prove that the algorithm given above terminates, together with other useful properties.

Proposition 15: Take $f^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$ with a compact support contained in $[0, Y^0] \times [0, W^0]$ and fix $\eta > 0$. Then we have the follwoing properties:

1) the COMPLETE ALGORITHM $\mathbb{S}$ terminates, i.e. there exists f^k computed via the algorithm with support in $[0, Y^k] \times [a^k, a^k + W^k]$ such that $W^k \le \eta$;
2) it holds $f^k = f(\sum_{j=0}^k T^j)$ with $\sum_{j=0}^k T^j \le W^0 \lceil \frac{2}{c} \rceil$;
3) it holds $Y^k \le Y^0 + (W^0)^2 \lceil \frac{2}{c} \rceil$;
4) the control satisfies **(U)** and the constraints $\int_{\omega(t)} f(t) \le c$ and $\|u(t)\|_\infty \le 1$.

Proof: **Property 1.** Consider $\beta^i, \varepsilon^i, T^i$ given by the FUNDAMENTAL STEP $\mathcal{S}$ applied to f^i, together with the size W^i of the support in the velocity variable. Proposition 13.5 can be read as $W^{i+1} \le W^i - \frac{T^i}{n}$. Since $W^i \ge 0$ for all i, then we have $\sum_{j=1}^i T^j \le nW^0$ for all i. Define $\bar{T} := \sum_{j=1}^\infty T^j$, that satisfies $\bar{T} \le nW^0$. As a consequence, the dynamics given by the FUNDAMENTAL STEP $\mathcal{S}$ is contained in $[0, Y^0 + n(W^0)^2] \times [0, W^0]$ for all times $t \in [0, \bar{T})$, where the right extreme is open since we have not proved yet the convergence of the COMPLETE ALGORITHM. In other terms $Y^i \le Y^0 + n(W^0)^2$ for all i.

Since the sequence W^i is bounded by $0 \le W^i \le W^0$ and it is decreasing due to $W^{i+1} \le W^i - \frac{T^i}{n}$ with $T^i > 0$, then it admits a limit. We want to prove that such limit is 0, by contradiction. Assume that there exists $\bar{W} > 0$ such that $W^i \ge \bar{W}$ for all i. Then, either $W^i - (v^*)^i \ge \frac{W^i}{2} \ge \frac{\bar{W}}{2}$ or $(v^*)^i \ge \frac{W^i}{2} \ge \frac{\bar{W}}{2}$. In both cases we have
$\beta^i \ge \frac{1}{3} \frac{\phi(Y^i + W^i)}{\phi(0) + \phi(Y^i + W^i)} \frac{\bar{W}}{2} \ge \frac{\phi(Y^0 + n(W^0)^2 + W_0)}{\phi(0) + \phi(\bar{W})} \frac{\bar{W}}{2}$,
where we have used that $0 \le Y^i \le (Y^0 + nW^0)^2$, that $\bar{W} \le W^i \le W^0$ and that ϕ is decreasing. Since the estimate does not depend on i, then there exists $\bar{\beta} := \frac{\phi(Y^0 + n(W^0)^2 + W^0)}{\phi(0) + \phi(\bar{W})} \frac{\bar{W}}{2} > 0$ such that $\beta^i \ge \bar{\beta}$ for all i.

Consider now the function $\|f(t)\|_\infty$ on the interval $[0, \bar{T})$. A direct computation gives

$$\begin{aligned} \|\nabla_v \cdot u_k^i\|_{L^\infty(\omega_k^i)} &\le \|\partial_v \psi_k^i(x, v)\|_{L^\infty(\omega_k^i)} + \\ &+ \|\psi_k^i(x, v)\|_{L^\infty(\omega_k^i)} \le (1/\bar{\beta}^i) + 1 \end{aligned}$$

for all $t \in [0, \bar{T})$. Then, applying (14), we have $\|f(t)\|_{L^\infty} \le \bar{F}$ with $\bar{F} := e^{(\phi(0) + 1/\bar{\beta} + 1)\bar{T}} \|f^0\|_{L^\infty} < +\infty$. Then observe that, for each f^i it holds $\|f^i\|_\infty \le \bar{F}$, that implies $\varepsilon^i \ge \bar{\varepsilon}$ with $\bar{\varepsilon} := \frac{c}{2\bar{F}W^0} > 0$ due to Proposition 13.6. Since we have proved $\beta^i \ge \bar{\beta}, \varepsilon^i \ge \bar{\varepsilon}$ for all i and we know $W^i \le W^0$ for all i, then $T^i = \min\left\{\frac{\varepsilon^i}{W^i}, \frac{\beta^i}{2c}, 1\right\} \ge \min\left\{\frac{\bar{\varepsilon}}{W^0}, \frac{\bar{\beta}}{2c}, 1\right\}$ does not converge to 0. Then, $\sum_{i=1}^\infty T^i = +\infty$. This contradicts $\sum_{i=1}^\infty T^i \le nW^0$.

Since $W^i \to 0$,then there exists I such that $W^I < \eta$ and the algorithm terminates.

Property 2 and 3. We already proved that $\sum_{j=1}^i T^j \le nW^0$ and $Y^i \le Y^0 + n(W^0)^2$ for all i. By choosing $i = I$, and recalling $n = \lceil \frac{c}{2} \rceil$, we have the results.

Property 4. Apply Proposition 13.7 for each step. ∎

We now use this algorithm to prove controllability to flocking of any initial configuration $f^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$.

Theorem 16 (Flocking in 1D): Let $f^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^2)$ with a compact support contained in $[0, Y^0] \times [0, W^0]$. Then, for a given $c > 0$, the algorithm $\mathbb{S}$ with

$$\eta := \frac{1}{2} \int_{2(Y^0 + \lceil \frac{c}{2} \rceil (W^0)^2)}^\infty \phi(2x)\, dx$$

drives f^0 to flocking in time $T \le W^0 \lceil \frac{2}{c} \rceil$ with controls satisfying both **(U)** and the constraints $\int_{\omega(t)} f(t) \le c$ and $\|u(t)\|_\infty \le 1$.

Proof: Apply $\mathbb{S}$ with the given η to find f^k, that satisfies $W^k \le \eta$. Since Proposition 15.3 gives $Y^k \le Y^0 + \lceil \frac{c}{2} \rceil (W^0)^2$, then we have

$$2W^k \le 2\eta = \int_{2(Y^0 + \lceil \frac{c}{2} \rceil (W^0)^2)}^\infty \phi(2x)\, dx \le \int_{2Y^k}^\infty \phi(2x)\, dx.$$

Then, Corollary 8 ensures flocking. The estimate on T is given by Proposition 15.2 and conditions on the control by Proposition 15.4. ∎

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