Sub-Riemannian Brownian motion, functional inequalities on path space and horizontal Ricci curvature

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Outline

1 Sub-Riemannian structures
2 Ricci curvature bounds and gradient estimates
3 Ricci curvature and analysis on path space
4 Analysis on path space over sub-Riemannian manifolds
5 Ricci curvature bounds in sub-Riemannian geometry
I. Sub-Riemannian structures

- \((M, H, g_H)\) where
  - \(M\) smooth manifold, \(\dim M = n\)
  - \(H \subsetneq TM\) subbundle ("horizontal directions"), \(\text{rank } H = m\)
  - \(g_H\) fiberwise inner product on \(H\)
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Sub-Riemannian geometry

\(\hat{=}\) geometry intrinsically associated to \((M, H, g_H)\).
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- Let

\[
d_H(x, y) = \inf \left\{ \int_0^1 |\dot{\gamma}(t)|\, dt : \gamma(0) = x, \gamma(1) = y, \dot{\gamma}(t) \in H_{\gamma(t)} \forall t \right\}
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**Sub-Riemannian geometry**

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\]

- \(H\) bracket generating (i.e. \(\text{Lie}(H)(x) = T_x M\) for each \(x \in M\))

\[ \implies (M, d_H) \text{ metric space} \]
Canonical sub-Riemannian Laplacian?

\[ \Delta^H = \sum_{i=1}^{m} A_i^2 + Z \quad \text{(locally)} \]

\( A_1, \ldots, A_m \) local orthonormal frame of \( H \),
\( Z \) first order term (horizontal vector field)
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Some notation:

Consider

$$\#^H : T^*M \to H \subset TM, \quad \langle \#^H \alpha, v \rangle_{g_H} := \alpha(v),$$

for $\alpha \in T_x^*M$, $v \in H_x$, $x \in M$.

Note that $\ker \#^H = \text{Ann } H.$
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Some notation:
1. Consider

\[ \#^H : T^*M \to H \subset TM, \quad \langle \#^H \alpha, \nu \rangle_{g_H} := \alpha(\nu), \]

for \( \alpha \in T^*_x M, \ \nu \in H_x, \ x \in M. \)

Note that \( \ker \#^H = \text{Ann} \ H. \)
2. The map \( \#^H \) induces a (degenerate) co-metric \( g^*_H \) on \( T^*M \) via

\[ \langle \alpha, \beta \rangle_{g^*_H} = \langle \#^H \alpha, \#^H \beta \rangle_{g_H}. \]
Let $L$ be a second order partial differential operator on $M$. Its symbol $\sigma(L)$ is the symmetric, bilinear 2-tensor on $T^*M$ determined by the relation

$$\sigma(L)(df, dh) = \frac{1}{2}(L(fh) - fLh - hLf), \quad f, h \in C^\infty(M).$$
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A second order PDO $L$ (without constant term) is called sub-Laplacian with respect to $(M, H, g_H)$ if

\[
\sigma(L) = g_H^*.
\]

We write $L = \Delta^H$. 
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$$TM = H \oplus V, \quad g = g_H \oplus g_V \quad (\text{where } V := H^\perp).$$
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Let
\[ TM = H \oplus V, \quad g = g_H \oplus g_V \quad (\text{where } V := H^\perp). \]
Define
\[ \nabla^H f = \text{pr}_H \nabla f \equiv \#^H df \]
and let $\Delta^H$ be the generator of the Dirichlet form
\[ \mathcal{E}(f, h) := - \int_M \langle \nabla^H f, \nabla^H h \rangle_H d\text{vol}_g. \]
Then $\Delta^H := - (\nabla^H)^* \nabla^H = \text{trace}_H \nabla^2$ is a sub-Laplacian.
In the situation of the last example:

- **Canonical variation of the metric**

  \[ \varepsilon > 0 : \quad g_\varepsilon := g_H \oplus \frac{1}{\varepsilon} g_V \]

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In the limit only horizontal curves have finite length.

- **Observation**
  \[ \text{Ric}^g_\varepsilon (u, u) \xrightarrow{\varepsilon \downarrow 0} -\infty \quad \text{for any horizontal unit vector } u \]
Natural connections on a sub-Riemannian manifold \((M, H, g_H)\)

- Would like to have a connection \(\nabla\) on \(M\) which is horizontally compatible with \((H, g_H)\) in the sense that the horizontal subbundle \(H\) is preserved under parallel transport, as well as its metric \(g_H\)
Natural connections on a sub-Riemannian manifold $(M, H, g_H)$

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- Actually, a metric partial connection

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\nabla : \Gamma(H) \times \Gamma(H) \rightarrow \Gamma(H), \quad (A, B) \mapsto \nabla_A B,
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Natural connections on a sub-Riemannian manifold \((M, H, g_H)\)

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- In terms of the corresponding **horizontal Hessian**, \(\nabla^2 f \equiv \text{Hess } f \in \Gamma(H^* \otimes H^*)\),

\[
(\nabla^2 f)(A, B) = ABf - (\nabla_A B)f,
\]

the associated **sub-Laplacian** \(\Delta^H\) is given by

\[
\Delta^H f = \text{trace}_H \nabla^2 f, \quad f \in C^\infty(M).
\]
Note that horizontally compatible connections $\nabla$ will always have torsion $T$:

$$\nabla_A B - \nabla_B A - [A, B] = T(A, B), \quad A, B \in \Gamma(H).$$
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The map $(A, B) \mapsto T(A, B) \mod H$ does not depend on the choice of $\nabla$. 

A horizontally compatible connection $\nabla$ is uniquely determined by its torsion $T$. Let $V$ be a choice of complement to $H$. There exists a unique horizontally compatible partial connection $\nabla$ with $T(H, H) \subseteq V$. 
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Example  Let again \((M, g)\) and \(g_H = g|H\). Then \(TM = H \oplus_\perp V\) and
\[
g = g_H \oplus g_V
\]
Denote by \(\nabla^g\) the Levi-Civita connection on \(M, g\).

- **(Bott connection)** There is a canonical connection \(\nabla\) preserving the decomposition \(TM = H \oplus V\):
\[
\nabla_X Y = \begin{cases} 
\text{pr}_H(\nabla^g_X Y), & X, Y \in \Gamma(H), \\
\text{pr}_H([X, Y]), & X \in \Gamma(V), \ Y \in \Gamma(H), \\
\text{pr}_V([X, Y]), & X \in \Gamma(H), \ Y \in \Gamma(V), \\
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\text{pr}_V(\nabla^g_X Y), & X, Y \in \Gamma(V), 
\end{cases}
\]

- $\nabla g = 0$
- its torsion $T^\nabla(X, Y)$ is vertical for $X$ and $Y$ horizontal, and zero if either $X$ or $Y$ is vertical
Standing assumptions

Let $V$ be a choice of a complement to $H$ in $(M, H, g_H)$. Let $\text{pr}_H$ and $\text{pr}_V$ be the corresponding projections. Write $\nabla$ for the unique partial connection with $\mathcal{T}(H, H) \subseteq V$. 

Connections of this form satisfy the following properties:

1. Both $H$ and $V$ are parallel with respect to $\nabla$.
2. $\mathcal{T}(H, H) \subseteq V$.
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Conversely, any connection satisfying (i)-(iii) is of this form.
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- We shall extend

$$\nabla_X Y, \quad X, Y \in \Gamma(H),$$

to an affine connection on $M$ as follows:

$$\nabla_X Y = \begin{cases} 
\text{pr}_H[X, Y] & \text{if } X \in \Gamma(V), Y \in \Gamma(H) \\
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while $\nabla$ on $V$ can be an arbitrary partial connection on $V$ in the direction of $V$. 
Standing assumptions

Let $V$ be a choice of a complement to $H$ in $(M, H, g_H)$. Let $pr_H$ and $pr_V$ be the corresponding projections. Write $\nabla$ for the unique partial connection with $T(H, H) \subseteq V$.

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(Metric preserving complement $V$) For simplicity, assume that

$$(L_Z \text{pr}^*_H g_H)(X, X) = 0$$

for all $Z \in \Gamma(V)$ and $X \in \Gamma(H)$

where $L_Z$ denotes the Lie derivative with respect to $Z$. 

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Let \( \text{Ric}: TM \to TM \) be the Ricci tensor with respect to \( \nabla \):

\[
\text{Ric}(v) = \text{trace}_H R^\nabla(v, \times) \times
\]

The object of our interest is

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\text{Ric}^H \in \Gamma(H^* \otimes H), \quad \text{Ric}^H := \text{Ric}|_H \quad \text{(horizontal Ricci)}
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We have

$$\text{Ric}(v) = \text{pr}_H \text{Ric}^H \text{pr}_H v, \quad v \in TM,$$

where $\text{pr}_H: TM \to H$ is the projection with kernel $V$. 

Example

Let \((M, g)\) be a Riemannian manifold and \(g_H = g|_H\) such that \(TM = H \oplus V\), and

\[
g = g_H \oplus g_V \quad \text{and} \quad g_\varepsilon = g_H \oplus \frac{1}{\varepsilon} g_V, \quad \varepsilon > 0.
\]

Then

\[
\text{Ric}_{g_\varepsilon}(X, X) = \text{Ric}^H(X, X) + \frac{1}{2\varepsilon} \langle J^2 X, X \rangle_H, \quad X \in \Gamma(H),
\]

where for \(Z \in \Gamma(V)\), \(J_Z \in \Gamma(\text{End} TM)\) is defined by

\[
\langle J_Z X, Y \rangle_{g_H} = \langle Z, T^\nabla(X, Y) \rangle_{g_V},
\]

and, for \(Z_1, \ldots, Z_r\) any local vertical frame,

\[
J^2 := \sum_{i=1}^r J_{Z_i} J_{Z_i}.
\]
(Laplacian) For a compatible connection $\nabla$ as above let

$$\Delta^H = \text{trace}_H \nabla^2_{x,x}$$

be the subelliptic Laplacian (the trace of the Hessian $\nabla^2$ is taken over $H$ with respect to the inner product $g_H$)
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(Stochastic development) Let $X_0 = x$ then

$$dX_t = //_{0,t} \circ dB_t \quad \text{or} \quad dB_t = //^{-1}_{0,t} \circ dX_t$$

where $B_t$ is a (classical) Brownian motion in $H_x$ and

$$//_{0,t} := U_t \circ U_0^{-1} : H_x M \to H_{X_t} M$$

is stochastic parallel transport along of horizontal vectors along $X$ (by construction isometries with respect to $g_H$).

Here $U_t$ is the horizontal lift of $X_t$ to the orthonormal frame bundle $O(H)$ over $M$. 
Functional inequalities

Consider the semigroup generated by $\Delta^H$:

$$P_t f = e^{t\Delta^H} f$$

We have

$$P_t f(x) = \mathbb{E}[f(X_t^x) 1_{\{t < \zeta(x)\}}], \quad x \in M.$$
Functional inequalities

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- **Question**: How is $\text{Ric}^H$ related to functional inequalities for $P_t$?
II. Ricci curvature bounds and gradient estimates
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Let $(M, g)$ be a complete Riemannian manifold and

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- Let \((M, g)\) be a complete Riemannian manifold and
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- (Bakry-Émery Ricci tensor)
  \[
  \text{Ric}^Z = \text{Ric} - \nabla Z
  \]
  where \(\text{Ric}^Z(X, Y) := \text{Ric}(X, Y) - \langle \nabla_X Z, Y \rangle\)
Theorem (classical probabilistic representations)

Let $f \in \mathcal{B}_b(M)$ and $u(x, t) = P_t f(x)$ be the (minimal) solution to

$$\frac{\partial}{\partial t} u = Lu, \quad u|_{t=0} = f.$$
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- **(Semigroup formula)** Then \( P_t f(x) = \mathbb{E}[f(X^x_t) 1_{\{t < \zeta(x)\}}] \).
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- (Semigroup formula) Then \( P_t f(x) = \mathbb{E}[f(X^x_t) \mathbf{1}_{\{t < \zeta(x)\}}] \).

- (Derivative formula) If \( f \in C^1_b(M) \) and \( \text{Ric}^Z \) bounded below,
\[
(\nabla P_t f)(x) = \mathbb{E}\left[ Q_t \parallel_t^{-1} \nabla f(X^x_t) \right]
\]

where the random transformations \( Q_t \in \text{End}(T_x M) \) are defined as solution to the pathwise ODE
\[
dQ_t = -Q_t \text{Ric}^Z_{\parallel_t} \, dt, \quad Q_0 = \text{id}_{T_x M}.
\]

Here
\[
\text{Ric}^Z_{\parallel_t} := \parallel_t^{-1} \circ \text{Ric}^Z_{X_t} \circ \parallel_t \in \text{End}(T_x M)
\]
is the equivariant representation of \( \text{Ric}^Z \).
In particular, if

\[ \text{CD}(K, \infty) \quad \text{Ric}^Z(v, v) \geq K|v|^2, \quad v \in TM, \]

for some constant \( K \), then

\[ |Q_t| \leq e^{-Kt} \]

and

(gradient estimate) \[ |\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|, \quad f \in C^1_b(M). \]
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\]

Actually, for $K \in \mathbb{R}$ the following two conditions are equivalent:

- CD($K, \infty$) $\quad \text{Ric}(v, v) \geq K|v|^2, \quad v \in TM.$

- (gradient estimate) $\quad |\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|, \quad f \in C^1_b(M).$
Well-known and classical: Let $K$ be a real constant. The following conditions are equivalent:

- **(Bakry-Émery lower curvature bound)\)**

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- **(Poincaré inequality)** for $p \in (1, 2]$ and all $f \in C^\infty_c(M)$,
  \[ \frac{p}{4(p-1)} \left(P_t f^2 - (P_t f^2/p)^p \right) \leq \frac{1 - e^{-2Kt}}{2K} P_t|\nabla f|^2; \]
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- **(log-Sobolev inequality)** for all $f \in C^\infty_c(M)$,
  \[ P_t (f^2 \log f^2) - (P_t f^2) \log (P_t f^2) \leq \frac{2(1 - e^{-2Kt})}{K} P_t |\nabla f|^2. \]
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- **(Bakry-Émery lower curvature bound)**
  
  $$\text{CD}(K, \infty) \quad \text{Ric}^Z(X, X) \geq K|X|^2, \quad X \in TM;$$

- **(gradient estimate)** for $p \in [1, \infty[$ and all $f \in \mathcal{C}_c^\infty(M)$,
  
  $$|\nabla P_t f|^p \leq e^{-pKt} P_t |\nabla f|^p;$$

- **(Poincaré inequality)** for $p \in (1, 2]$ and all $f \in \mathcal{C}_c^\infty(M)$,
  
  $$\frac{p}{4(p-1)} (P_t f^2 - (P_t f^2/p)^p) \leq \frac{1 - e^{-2Kt}}{2K} P_t |\nabla f|^2;$$

- **(log-Sobolev inequality)** for all $f \in \mathcal{C}_c^\infty(M)$,
  
  $$P_t (f^2 \log f^2) - (P_t f^2 \log(P_t f^2)) \leq \frac{2(1 - e^{-2Kt})}{K} P_t |\nabla f|^2.$$

Many other equivalent statements, e.g., transportation-cost inequalities; convexity properties of the entropy; Wang’s dimension-free Harnack inequalities; Wang’s log-Harnack inequalities, ...
Comparison with the sub-Riemannian case

Example (Heisenberg group $\mathbb{H}^3$)

$X, Y, Z \in \Gamma(\mathbb{H}^3), \quad [X, Y] = Z, \quad [X, Z] = [Y, Z] = 0$

$\mathbb{H} = \text{span}(X, Y), \quad V = \mathbb{R} \cdot Z$

Let

$$\Delta^H := X^2 + Y^2 \quad \text{and} \quad P_t f = (e^{t\Delta^H})f$$
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Theorem (Hong-Quan Li, 2006)

$\exists C > 0, \quad |\nabla^H P_t f|_{g_H} \leq C P_t |\nabla^H f|_{g_H}, \quad \forall f \in C_c^\infty(\mathbb{H}^3)$,

where $\nabla^H f = \text{pr}_H \nabla f$.

The constant $C$ must be strictly larger than 1!
Boundedness of $\text{Ric}$

The problem of characterizing boundedness of $\text{Ric}$ in Riemannian geometry has been solved by A. Naber via analysis on path space:

$$|\text{Ric}| \leq K \quad (\text{i.e. } -K \leq \text{Ric} \leq K \text{ for some constant } K \geq 0)$$

$\iff$ certain functional inequalities on path space
III. Ricci curvature and analysis on path space

For fixed $T > 0$, let $W^T = C([0, T]; M)$ and

$$\mathcal{F} C_0^\infty, T = \left\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : 0 < t_1 < \ldots < t_n \leq T, f \in C_c^\infty(M^n) \right\}.$$ 

be the class of smooth cylindrical functions on $W^T$. 

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Denote

$$X_{[0,\tau]} = \{X_t : 0 \leq t \leq \tau\}.$$ 

For $F \in \mathcal{F} C_{0,T}^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, the intrinsic gradient is defined as

$$D_t^\parallel F(X_{[0,\tau]}) = \sum_{i=1}^n \mathbb{1}_{\{t < t_i\}} \, /_{t,t_i}^{-1} \nabla^i f(X_{t_1}, \ldots, X_{t_n}), \quad t \in [0, T],$$

where $\nabla^i$ denotes the gradient with respect to the $i$-th component.
Theorem [A. Naber (2015) and R. Haslhofer and A. Naber (2018)]

The following conditions are equivalent ($K \geq 0$):

| $\text{Ric}_Z| \leq K$; (Gradient inequality on path space) for $F \in F_{\infty 0}$, $\left| \nabla_x E[F(X_{x[0,T]})] \right| \leq E_x[|D//0F| + K \int_0^T e^{Kr}|D//rF|dr]$. (L^2 gradient inequality on path space) for $F \in F_{\infty 0}$, $\left| \nabla_x E[F(X_{x[0,T]})] \right|^2 \leq e^{KT}E_x[|D//0F|^2 + K \int_0^T e^{K(r-T)}|D//rF|^2dr]$. Important observation It is sufficient to check the estimates for very special $F \in F_{\infty 0}$. Namely:

(i) for $F(X_{x[0,T]}) = f(X_{x[t]})$, and

(ii) for 2-point cylindrical functions of the form $F(X_{x[0,T]}) = f(x) - \frac{1}{2} f(X_{x[t]})^2$.
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The following conditions are equivalent ($K \geq 0$):

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Namely:

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  \]
- ($L^2$ gradient inequality on path space) for $F \in \mathcal{F}C_0^\infty$,
  \[
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Important observation  It is sufficient to check the estimates for very special $F \in \mathcal{F} C_0^\infty$. Namely:

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(ii) for 2-point cylindrical functions of the form

\[ F(X_x^{[0,T]}) = f(x) - \frac{1}{2} f(X_t^x) \]
From this observation, equivalence of the following two items follows:

(i) \(|\text{Ric}^Z| \leq K\) for \(K \geq 0\);

(ii) for \(f \in C^\infty_c(M)\) and \(t > 0\),

\[
|\nabla P_t f|^2 \leq e^{2Kt} P_t |\nabla f|^2 \quad \text{and} \quad
\left| \nabla f - \frac{1}{2} \nabla P_t f \right|^2 \leq e^{Kt} \mathbb{E} \left[ \left| \nabla f - \frac{1}{2} \int_0^t \nabla f(X_t) \right|^2 \right] + \frac{1}{4} \left( e^{Kt} - 1 \right) |\nabla f|^2 (X_t).
\]
Path space characterization of pinched curvature

Let $F \in \mathcal{F} C_0^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$. Consider the gradients:

- **(intrinsic gradient)**

$$D_{t//} F(X^x_{[0,T]}) = \sum_{i=1}^{n} 1_{\{t < t_i\}} \left//_{t,t_i} \nabla f(X_{t_1}^x, \ldots, X_{t_n}^x)\right;$$

- **(damped gradient)**

- **(balanced gradient)**

$$D_t L F(X^x_{[0,T]}) = \sum_{i=1}^{n} 1_{\{t < t_i\}} Q_{t,t_i} \left//_{t,t_i} \nabla f(X_{t_1}^x, \ldots, X_{t_n}^x)\right;$$
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- **(damped gradient)**
  
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where $Q_{t,r}$ takes values in the linear automorphisms of $T_{X^x_t} M$ satisfying for fixed $t \geq 0$:

$$\frac{dQ_{t,r}}{dr} = -Q_{t,r} \text{Ric}^{Z}_{t,r}, \quad Q_{t,t} = \text{id}; \quad r \geq t$$
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  $$D_t^{\parallel} F(X^x_{[0,T]}) = \sum_{i=1}^n 1\{t<t_i\} \frac{-1}{t_{i-1}} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n});$$

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- (balanced gradient) For constants $k_1 \leq k_2$ let

  $$\bar{D}_t^{\parallel} F(X^x_{[0,T]}) = \sum_{i=1}^n 1\{t\leq t_i\} e^{-\frac{k_1+k_2}{2}(t_i-t)} \frac{-1}{t_{i-1}} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n}).$$
Theorem (Path space characterization of pinched curvature)

The following conditions are equivalent:

(i) \( k_1 \leq \text{Ric}^Z \leq k_2 \);

(ii) \((\text{Gradient estimate})\) for any \( F \in \mathcal{F} C_{0,T}^\infty \),

\[
\left| \nabla_x \mathbb{E} F(\mathbf{X}^x_{[0,T]}) \right| \leq \mathbb{E} |\bar{D}^F_0| + \frac{k_2-k_1}{2} \int_0^T e^{-k_1(s-t)} \mathbb{E} |\bar{D}^F_s| ds;
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Theorem (Path space characterization of pinched curvature)

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\]

(iii) (Log-Sobolev inequality) for any \( F \in \mathcal{F} C_{0,T}^\infty \) and \( t_1 < t_2 \) in \([0, T] \),

\[
\mathbb{E}\left[ \mathbb{E}\left[ F^2(X_{[0,T]}) | \mathcal{F}_{t_2} \right] \log \mathbb{E}\left[ F^2(X_{[0,T]}) | \mathcal{F}_{t_2} \right] \right] - \mathbb{E}\left[ \mathbb{E}\left[ F^2(X_{[0,T]}) | \mathcal{F}_{t_1} \right] \log \mathbb{E}\left[ F^2(X_{[0,T]}) | \mathcal{F}_{t_1} \right] \right] \\
\leq 2 \int_{t_1}^{t_2} \left( 1 + \frac{k_2-k_1}{2} \right) \int_t^T e^{-k_1 (s-t)} ds \right) \times \left( \mathbb{E}|\bar{D}_t^/ F|^2 + \frac{k_2-k_1}{2} \int_t^T e^{-k_1 (s-t)} \mathbb{E}|\bar{D}_s^/ F|^2 ds \right) dt.
\]
(iv) (Poincaré type inequality) for $F \in \mathcal{F} C_{0,T}^\infty$ and $t_1 < t_2$ in $[0, T]$,

$$
\mathbb{E}\left[\mathbb{E}[F(X_{[0,T]}|\mathcal{F}_{t_2})]^2\right] - \mathbb{E}\left[\mathbb{E}[F(X_{[0,T]}|\mathcal{F}_{t_1})]^2\right] \\
\leq \int_{t_1}^{t_2} \left(1 + \frac{k_2 - k_1}{2} \int_t^T e^{-k_1(s-t)} \, ds\right) \\
\times \left(\mathbb{E}|\bar{D}_t|F|^2 + \frac{k_2 - k_1}{2} \int_t^T e^{-k_1(s-t)} \mathbb{E}|\bar{D}_s|F|^2 \, ds\right) \, dt.
$$
Let $\mathcal{L}$ be the Ornstein-Uhlenbeck operator defined as generator associated to the Dirichlet form

$$
\mathcal{E}(F, F) = \mathbb{E} \left[ \int_0^T |D_t F|^2(X_{[0,T]}) \, dt \right].
$$
Let $L$ be the Ornstein-Uhlenbeck operator defined as generator associated to the Dirichlet form

$$\mathcal{E}(F, F) = \mathbb{E} \left[ \int_0^T |D_t^\perp F|^2 (X_{[0,T]}) dt \right].$$

The log-Sobolev inequality or Poincaré inequality on path space can be used to derive spectral gap-lower bounds for the operator $L$. 
Let $\mathcal{L}$ be the Ornstein-Uhlenbeck operator defined as generator associated to the Dirichlet form

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The log-Sobolev inequality or Poincaré inequality on path space can be used to derive spectral gap-lower bounds for the operator $\mathcal{L}$.

It is well-known that a log-Sobolev inequality

$$\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2 \, H(T, k_1, k_2) \int_0^T |D_t F|^2 (X_{[0,T]}) \, dt$$

or a Poincaré inequality

$$\mathbb{E}[(F - \mathbb{E}[F])^2] \leq H(T, k_1, k_2) \int_0^T |D_t F|^2 (X_{[0,T]}) \, dt$$

for some explicit bound $H(T, k_1, k_2)$, give the spectral gap lower bound $H(T, k_1, k_2)^{-1}$ for the operator $\mathcal{L}$. 
IV. Analysis on path space over sub-Riemannian manifolds

Let again $\nabla$ be a partial connection on $H$, extended as above to a compatible connection on $M$.

Weitzenböck formula

- Consider the corresponding rough sub-Laplacian

$$L(\nabla) := \text{trace}_H \nabla^2$$

(on functions and 1-forms).
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- Would like to have a Weitzenböck type commutation formula of the form:

$$dLf = (L - R)df, \quad L = L(\nabla),$$

where $R \in \Gamma(\text{End}(T^*M))$. 

Let $\hat{\nabla}$ be the adjoint connection to $\nabla$, i.e.

$$\hat{\nabla}_X Y = \nabla_X Y - T(X, Y).$$
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**Proposition** Let $L$ be a rough sub-Laplacian of a connection on $M$. There exists a vector bundle endomorphism

$$\mathcal{R} : T^* M \to T^* M$$

such that

$$(L - \mathcal{R}) df = dLf, \quad f \in C^\infty(M),$$

if and only if $L = L(\hat{\nabla})$ for some adjoint $\hat{\nabla}$ of a connection $\nabla$ that is compatible with $(H, g_H)$.
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In this case,

$$\mathcal{R} = \text{Ric}^\nabla$$

where for $(\alpha, v) \in T^* M \oplus TM$,

$$\text{Ric}^\nabla(\alpha)(v) = \text{trace}_H R^\nabla(\cdot, v)\alpha(\cdot)$$
**Proposition** *(Weitzenböck formula)*
Then, for all $f \in C^\infty(M)$,

$$
\left( L(\hat{\nabla}) - R \right) df = dL(\hat{\nabla})f = dL(\nabla)f = d\Delta^H f
$$
Derivative formula

Define $\hat{Q}_t = \hat{Q}_t(x) \in \text{End}(T_x M)$ by

$$\frac{d}{dt} \hat{Q}_t = -\hat{R}_{/t} \hat{Q}_t, \quad \hat{Q}_0 = \text{id}_{T_x M},$$

where $\hat{R} = \text{Ric}^\nabla$ and $\hat{R}_{/t} = \hat{R}_{/t}^{-1} \hat{R}_{/t}.$
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where $\mathbb{R} = \text{Ric}^\nabla$ and $\hat{\mathbb{R}}_t = \mathbb{R}^{-1} \hat{\mathbb{R}}_t$.

- (Derivative formula) For $P_t = e^{t \Delta_H}$ and $f \in C^\infty(M)$, we have

$$dP_t f(x) = \mathbb{E}[\hat{Q}_t^* \hat{\mathbb{R}}^{-1}_t df_{X_t(x)}]$$
Integration by parts on path space over a sub-Riemannian manifold

Let \((M, H, g_H)\) be a sub-Riemannian manifold equipped with a compatible connection \(\nabla\) and let

\[ L = \text{trace}_H \nabla^2 \]

be defined as the trace of the Hessian \(\nabla^2\) over \(H\) with respect to the inner product \(g_H\).
Integration by parts on path space over a sub-Riemannian manifold

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Assume that there is a decomposition \(TM = H \oplus V\) such that

\begin{itemize}
  \item both \(H\) and \(V\) are parallel with respect to \(\nabla\)
  \item \(T(H, H) \subseteq V\)
  \item \(T(H, V) = 0\).
\end{itemize}
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b. \(T(H, H) \subseteq V\)

c. \(T(H, V) = 0\).

No choice of a Riemannian metric \(g\) on \(M\) satisfying \(g|_H = g_H\) is required.

Assume again that the complement \(V\) metric preserving.
Let $X_t(x) \equiv X^x_t$ be the sub-Riemannian Brownian motion with generator $L$ such that $X_0(x) = x$ and

$$ dB^x_t = \frac{1}{t} \circ dX_t(x), \quad B_0 = 0 \in H_x $$

Recall that $B^x_t$ is a standard Brownian motion in $H_x$. 

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Recall that \( B^x_t \) is a standard Brownian motion in \( H_x \).

(Cameron-Martin space) Let

\[
 H = \left\{ h : [0, T] \rightarrow H_x \text{ abs. cont.} \left| \int_0^T |\dot{h}(t)|^2_{g_H} dt < \infty \right. \right\}
\]

which becomes a Hilbert space with inner product

\[
 \langle h_1, h_2 \rangle_H = \int_0^T \langle \dot{h}_1(t), \dot{h}_2(t) \rangle_{g_H} dt.
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\[
\langle h_1, h_2 \rangle_{\mathcal{H}} = \int_0^T \langle \dot{h}_1(t), \dot{h}_2(t) \rangle_{g_H} dt.
\]

As usual, we write \( \langle h, B^x \rangle_{\mathcal{H}} = \int_0^t \langle \dot{h}_s, dB^x_s \rangle_{g_H} \).
Derivatives on path space of sub-Riemannian manifolds

For fixed $T > 0$, let $W^T = C([0, T]; M)$ and

$$\mathcal{F} C^\infty_{0,T} = \left\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : 0 < t_1 < \ldots < t_n \leq T, f \in C^\infty_c(M^n) \right\}$$

be the class of smooth cylindrical functions on $W^T$. 
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\]

be the class of smooth cylindrical functions on \( W^T \).

Let the operator \( A_t : T_x M \to T_x M \) be given by

\[
A_t = \int_0^t T_{//t} (\circ dB_x^s, \cdot)
\]

(Note that \( A_t(H_x) \subseteq V_x \) and \( A_t(V_x) = 0 \))

For an adapted process \( h \) with paths in \( \mathbb{H} \) let

\[
S(h)_t = h_t + \int_0^t T_{//s} (\circ dB_x^s, h_s)
= h_t + \int_0^t dA_s h_s = \int_0^t (id + A_t + A_s) dh_s.
\]
(Derivative operator on path space) For $F \in \mathcal{F} C_{0,T}^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$ and $h \in \mathbb{H}$, let

$$D_h F(\gamma) = \sum_{i=1}^{n} \langle [-1]^{-1} d_i f(\gamma_{t_1}, \ldots, \gamma_{t_n}), S(h)_{t_i} \rangle$$
(Derivative operator on path space) For $F \in \mathcal{F} C_{0, T}^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$ and $h \in \mathbb{H}$, let

$$D_h F(\gamma) = \sum_{i=1}^{n} \langle \left/ \left/ \left/ \left/ t_i \right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\right\r
Define $D_t F \in H_x$ such that

$$D_h F = \int_0^t \langle D_t F, \dot{h}_t \rangle_{g_H} dt.$$  

It is straightforward to check that

$$D_t F := \sum_{i=1}^n 1_{\{t \leq t_i\}} \#^H (\text{id} + A_{t_i} - A_t)^* / t_i^{-1} df(\gamma_{t_1}, \ldots, \gamma_{t_n}).$$

The gradient $DF$ is then given by the relation

$$\langle DF, h \rangle_{H} = D_h F$$
Proposition (Integration by parts formula)

For $F \in \mathcal{F} C_{0,T}^\infty$ and any adapted process $h_t$ with paths in $\mathbb{H}$, we have

$$
\mathbb{E}[\langle DF, h \rangle_{\mathbb{H}}] = \mathbb{E}
\left[F \int_0^T \langle \dot{h}_t + \text{Ric}_{/t} h_t, dB_t \rangle_{g_{\mathbb{H}}} \right].
$$

In particular, for $f \in C^\infty(M)$,

$$
\mathbb{E}[\langle /_t^{-1} df_{X_t(x)}, S(h)_t \rangle] = \mathbb{E}
\left[f(X_t(x)) \int_0^t \langle \dot{h}_s + \text{Ric}_{/s} h_s, dB_s \rangle_{g_{\mathbb{H}}} \right].
$$
Damped gradients and Quasi-invariance

For $F \in \mathcal{F} C_{0,T}^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, define

$$
\tilde{D}_t F(\gamma) := \sum_{i=1}^{n} 1_{\{t \leq t_i\}} \#^H H^{-1}_t \hat{Q}_{t, t_i}^{-1} d_i f(\gamma_{t_1}, \ldots, \gamma_{t_n})
$$

and

$$
\tilde{D}_h F = \langle \tilde{D} F, h \rangle_H = \int_0^T \tilde{D}_t F \, dh_t
$$
Damped gradients and Quasi-invariance

For \( F \in \mathcal{F} C_{0,T}^\infty \) with \( F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n}) \), define

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\]

and

\[
\tilde{D}_h F = \langle \tilde{D} F, h \rangle_H = \int_0^T \tilde{D}_t F dh_t
\]

For adapted process \( h \) with paths in \( H \) one has

\[
\mathbb{E}_x[\langle \tilde{D} F, h \rangle_H] = \lim_{\varepsilon \to 0} \mathbb{E}\left[ \frac{F(X^\varepsilon_{[0,T]}) - F(X_{[0,T]})}{\varepsilon} \right]
\]

where

\[
dX^\varepsilon_t = //^{\varepsilon}_t dB_t + \varepsilon //^{\varepsilon}_t dh_t, \quad X^\varepsilon_0 = x
\]
Let $Q_t : T_x M \to T_x M$ be the solution of

$$Q_0 = \text{id}_{T_x M}, \quad dQ_t = -\text{Ric}_{//t} Q_t dt$$
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For any adapted process $h_t$ with paths in $\mathbb{H}$, we then have

$$\langle \tilde{D}F, h \rangle_{\mathbb{H}} = \langle DF, k \rangle_{\mathbb{H}}, \quad k_t = Q_t \int_0^t Q_s^{-1} dh_s$$

and hence

$$\mathbb{E}[\langle \tilde{D}F, h \rangle_{\mathbb{H}}] = \mathbb{E} \left[ F \int_0^T \langle h, B^x \rangle_{\mathbb{H}} \right]$$
V. Ricci curvature bounds in sub-Riemannian geometry

(Derivative formula on path space)

For $F \in \mathcal{F} \mathcal{C}_0^\infty$ and $t > 0$, we have

$$D_t \mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[\tilde{D}_t F|\mathcal{F}_t]$$
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(Derivative formula on path space)

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(Semigroup derivative formula)

$$dP_t f(v) = \mathbb{E} \left[ \left\langle t^{-1} df_{X_t(x)} , Q_t v + \int_0^t dA_r Q_r v \right\rangle \right], \quad v \in T_x M.$$
Theorem (Characterization of horizontal Ricci curvature)

Assume that $V$ is metric preserving. For a non-negative constant $K$ the following conditions are equivalent:

1. **(Bounded Ricci curvature)** the horizontal Ricci curvature $Ric^H = Ric|_H \in \text{End}(H)$ is bounded by $K$, i.e.

   $$-K \leq Ric^H \leq K$$

2. **(Gradient estimate)** for any $F \in \mathcal{F} C_0^\infty$,

   $$|D_0E_x[F]|_{g_H} \leq E_x\left[|D_0F|_{g_H} + K \int_0^T e^{Ks}|D_s F|_{g_H} \, ds\right]$$

3. **($L^2$ gradient estimate)** for any $F \in \mathcal{F} C_0^\infty$,

   $$|D_0E_x[F]|^2_{g_H} \leq e^{-KT}E_x\left[|D_0F|^2_{g_H} + K \int_0^T e^{Ks}|D_s F|^2_{g_H} \, ds\right]$$
Theorem (continuation)

(***Log-Sobolev inequality***) for any $F \in \mathcal{F} C_0^\infty$ and $t > 0$ in $[0, T]$,

$$
\mathbb{E}_x \left[ \mathbb{E}_x [F^2 | \mathcal{F}_t] \log \mathbb{E}_x [F^2 | \mathcal{F}_t] \right] - \mathbb{E}_x [F^2] \log \mathbb{E}_x [F^2] \\
\leq \int_0^t e^{K(T-r)} \left( \mathbb{E}_x |D_r F|_{g_H}^2 + \frac{K}{2} \int_r^T e^{K(s-r)} \mathbb{E}_x |D_s F|_{g_H}^2 \, ds \right) \, dr;
$$

(****Poincaré inequality***) for any $F \in \mathcal{F} C_0^\infty$ and $t > 0$ in $[0, T]$,

$$
\mathbb{E}_x \left[ \mathbb{E}_x [F | \mathcal{F}_t]^2 \right] - \mathbb{E}_x [F]^2 \\
\leq \int_0^t e^{K(T-r)} \left( \mathbb{E}_x |D_r F|_{g_H}^2 + \frac{K}{2} \int_r^T e^{K(s-r)} \mathbb{E}_x |D_s F|_{g_H}^2 \, ds \right) \, dr.
$$
For non-symmetric bounds, i.e. \( K_1 \leq \text{Ric}^H \leq K_2 \), one can give similar equivalent conditions redefining \( \bar{D}_t F \) by

\[
\bar{D}_t F = \sum_{i=1}^{n} 1\{t \leq t_i\} e^{-\frac{K_1+K_2}{2}(t_i-t)} \#^H (\text{id} + A_{t_i} - A_t)^* //_{t_i}^{-1} d_i F
\]

(Ornstein-Uhlenbeck operator)

For \( F, G \in \mathcal{F} C_0^\infty \) let

\[
\mathcal{E}(F, G) = \mathbb{E} \langle DF, DG \rangle_H = \mathbb{E} \left[ \int_0^T \langle D_t F, D_t G \rangle_{g_H} dt \right].
\]

Integration by parts formula implies the closability of the form.
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\[
\bar{D}_t F = \sum_{i=1}^{n} \mathbb{1}_{\{t \leq t_i\}} e^{-\frac{K_1 + K_2}{2} (t_i-t)} \#^H (\text{id} + A_t - A_t)^* \text{//} t_i^{-1} d_i F
\]

(Ornstein-Uhlenbeck operator)

For \( F, G \in \mathcal{F} C_0^\infty \) let

\[
\mathcal{E}(F, G) = \mathbb{E} \langle DF, DG \rangle_H = \mathbb{E} \left[ \int_0^T \langle D_t F, D_t G \rangle_{g_H} dt \right].
\]

Integration by parts formula implies the closability of the form.

Let \( \mathcal{L} \) be the generator of the the Dirichlet form

\[
\mathcal{E}(F, F) = \mathbb{E} \left[ \int_0^T |D_t F|^2_{g_H} dt \right].
\]

Let \( \text{gap}(\mathcal{L}) \) denote its spectral gap.
Theorem  Suppose there exists a constant $K \geq 0$ such that

$$|\text{Ric}^H| \leq K.$$ 

Then

(i) (Poincaré inequality) for any $F \in \text{dom}(\mathcal{E})$ with $\mathbb{E}[F] = 0,$

$$\mathbb{E}[F^2] \leq \frac{1}{2}(e^{KT} + 1)\mathcal{E}(F, F)$$

(ii) (Log-Sobolev inequality) for any $F \in \text{dom}(\mathcal{E})$ with $\mathbb{E}[F^2] = 1,$

$$\mathbb{E}[F^2 \log F^2] \leq (e^{KT} + 1)\mathcal{E}(F, F)$$

(iii) (Spectral gap estimate) the following estimate holds:

$$\text{gap}(\mathcal{L})^{-1} \leq \frac{1}{2}(e^{KT} + 1)$$
References


