Sub-Laplacian Comparison Theorems on H-Type Foliations

Gianmarco Molino

SRGI, Université Sorbonne

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Riemannian Geometry

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- Metric, analytic, and even topological properties can be determined from a knowledge of curvature
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- Riemannian manifolds allow for many notions of curvature
- Metric, analytic, and even topological properties can be determined from a knowledge of curvature
- How do these ideas fit in a subRiemannian setting?
Let $(\mathcal{M}, g)$ be a Riemannian manifold of dimension $m$ and suppose there exists $\kappa \in \mathbb{R}$ such that $Ric \geq (n - 1)\kappa g$. 

**Theorem (Bonnet-Meyers)**

If $\kappa > 0$ then $\text{diam}(\mathcal{M}) \leq \pi \sqrt{\kappa}$.

The fundamental group of $\mathcal{M}$ must be finite.
Consequences of Laplacian Comparisons

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**Theorem (Bishop-Gromov)**

Let $\overline{M}_m^m$ be the Riemannian manifold of dimension $m$ and constant sectional curvature $\kappa$. Denote by $B_M(p, r)$ the Riemannian ball of radius $r$ around $p \in M$. Then

$$\phi(r) = \frac{B_M(p, r)}{B_{\overline{M}_m^m}(p_\kappa, r)}$$

is nonincreasing on $(0, \infty)$. 
Connections

Denoting the space of vector fields \( \Gamma(TM) \), an operator

\[
\nabla : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)
\]

is called a connection.
Connections

Denoting the space of vector fields $\Gamma(T\mathbb{M})$, an operator

$$\nabla : \Gamma(T\mathbb{M}) \times \Gamma(T\mathbb{M}) \to \Gamma(T\mathbb{M})$$

such that

$$\nabla_{fX + Y} Z = f\nabla_X Z + \nabla_Y Z$$
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2. $\nabla_X(fY) = df(X)Y + f\nabla_XY$

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Levi-Civita Connection

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**Theorem**

*Let* \((\mathbb{M}, g)\) *be a Riemannian manifold. There exists a unique connection* \(\nabla\) *on* \(\mathbb{M}\) *such that*

1. \(\nabla_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\)
2. \(\nabla_X Y - \nabla_Y X = [X, Y]\)
Curvature

- Riemannian Curvature:

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z \]
Curvature

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- **Sectional Curvature:** (for orthonormal \( X, Y \)):
  \[ K(X, Y) = g(R(X, Y)Y, X) \]
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- Ricci Curvature:
  \[ Ric(X, Y) = Tr(Z \mapsto g(R(Z, X)Y, Z)) \]
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- **Scalar Curvature:**
  \[ s(X) = Tr(Y \mapsto Ric(X, Y)) \]
Some Definitions

We set some notation. For a Riemannian manifold \((\mathbb{M}, g)\) and a point \(p \in \mathbb{M}\), we define the distance function

\[
d_p : \mathbb{M} \to \mathbb{R}, \quad d_p(q) = d(p, q)
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Let \(\gamma : [0, L] \to M\) be a minimizing geodesic. Then we define the curvatures

\[
K^+(t) = \sup \{ K(X_{\gamma(t)}, Y_{\gamma(t)}): \gamma'(t) \in \text{Span}(X_{\gamma(t)}, Y_{\gamma(t)}) \}
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K^-(t) = \inf \{ K(X_{\gamma(t)}, Y_{\gamma(t)}): \gamma'(t) \in \text{Span}(X_{\gamma(t)}, Y_{\gamma(t)}) \}
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We define

\[
\text{Hess} f(X, Y) = \nabla^2 f(X, Y) = g(\nabla_X \nabla f, Y)
\]

\[
\Delta f = \text{Tr}(\text{Hess} f)
\]
Theorem (Hessian Comparison)

Let $(\mathbb{M}_i, g_i), i \in \{1, 2\}$ be Riemannian manifolds, $\gamma_i : [0, L] \to \mathbb{M}_i$ be minimizing geodesics such that

$$K_{\mathbb{M}_2}^+(t) \leq K_{\mathbb{M}_1}^-(t)$$
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Let \(X_i \in \Gamma(T\mathbb{M}_i)\) be such that for all \(t \in [0, L]\)

- \(\|X_1(\gamma_1(t))\|_{g_1} = \|X_2(\gamma_2(t))\|_{g_2}\)
- \(g_1(X_1(\gamma_1(t)), \gamma_1'(t)) = g_2(X_2(\gamma_2(t)), \gamma_2'(t))\)
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- \(\|X_1(\gamma_1(t))\|_{g_1} = \|X_2(\gamma_2(t))\|_{g_2}\)
- \(g_1(X_1(\gamma_1(t)), \gamma'_1(t)) = g_2(X_2(\gamma_2(t)), \gamma'_2(t))\)

then denoting \(p_i = \gamma_i(0), q_i = \gamma_i(t),\)

\[
\text{Hess } d_{p_1}(X_1(q_1), X_1(q_1)) \leq \text{Hess } d_{p_2}(X_2(q_2), X_2(q_2))
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- Define index $I(X, X) = \int_0^r \langle \nabla_{\dot{\gamma}} X, \nabla_{\dot{\gamma}} X \rangle - R(X, \dot{\gamma}, \dot{\gamma}, X) \, dt$
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- Define index $I(X, X) = \int_0^r \langle \nabla_{\dot{\gamma}} X, \nabla_{\dot{\gamma}} X \rangle - R(X, \dot{\gamma}, \dot{\gamma}, X) \, dt$
- $K_1^+ \leq K_2^- \implies I(X_1, X_1) \leq I(X_2, X_2)$
- Theorem follows from $\nabla^2 d = \alpha I(X, X)$
Rauch Comparison Theorem

Corollary (Rauch Comparison)

Take the same assumptions as in the previous theorem. Then

$$\Delta_1 d_{p_1}(q_2) \leq \Delta_2 d_{p_2}(q_2)$$
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This presents a way to compare the behaviors of distance functions, but we still need to something to compare them to.
Model Spaces

Denote by $\mathbb{M}^m_\kappa$ the Riemannian manifold of constant sectional curvature $\kappa$ and dimension $m$. 
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$$\mathbb{M}_\kappa^m = \begin{cases} 
S^m(\kappa) & \kappa > 0 \\
\mathbb{R}^m & \kappa = 0 \\
H^m(\kappa) & \kappa < 0 
\end{cases}$$
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S^m(κ) & κ > 0 \\
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\mathbb{H}^m(κ) & κ < 0 
\end{cases}
$$

We refer to these as Model Spaces. We are able to compute $Δd_p$ explicitly on these spaces, and use this as a basis for comparison.
Laplacian Comparison

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$$\text{Ric} \geq (n - 1)\kappa g$$

Let $p, q \in \mathcal{M}$ and denote $r = d(p, q)$. 

Then

$$\Delta_d p(q) \leq \begin{cases} (n - 1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \kappa > 0 \\ \frac{n - 1}{\kappa}r & \kappa = 0 \\ (n - 1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \kappa < 0 \end{cases}$$
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\end{cases}$$
Comparison Function

On the model spaces, the Jacobi fields can be computed explicitly using the Jacobi equation

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V + R(V, \dot{\gamma}, \dot{\gamma}) = 0$$
Comparison Function

On the model spaces, the Jacobi fields can be computed explicitly using the Jacobi equation

\[ \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V + R(V, \dot{\gamma}, \ddot{\gamma}) = 0 \]

then the upper bound on \( \Delta d_p(q) \) is given by solving an ode.
Basic Definitions

Let $\mathbb{M}$ be a smooth manifold. We say that $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if

- $\mathcal{H}$ is a constant rank, bracket generating subbundle of $T\mathbb{M}$,
- and $g_{\mathcal{H}}$ is a fiberwise inner product on $\mathcal{H}$.

A main goal of sub-Riemannian geometry is to determine adequate notions of curvature that are able to support generalizations of the comparison theorems found in the Riemannian theory.
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Some History

- Li-Zelenko 2011, Lee-Li 2013, Agrachev-Lee 2015, Lee-Li-Zelenko 2016: Comparison theorems on Sasakian manifolds
- Rizzi-Silveira 2015, 2017, Barilari-Rizzi 2016: Comparison theorems in 3 Sasakian case
- Baudoin-Bonnefont-Garofalo 2014, Baudoin-Grong-Kuwada-Thalmaier 2017: Eulerian approach to comparison theorems on Sasakian and 3 Sasakian manifolds
Let $\mathcal{M}$ be a smooth manifold. We say that $(\mathcal{M}, \mathcal{H}, g)$ is a sub-Riemannian manifold with metric preserving complement or sRmc-manifold if

- $(\mathcal{M}, g)$ is a Riemannian manifold,
- the metric orthogonally splits as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$,
- and $(\mathcal{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold.

We denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ by $g$. 
Motivating Example: Hopf Fibration

Consider $S^{2n+1}$ foliated as

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$
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Define the vertical distribution as tangent to the leaves, $\mathbb{S}^1$. 
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Define the vertical distribution as tangent to the leaves, $S^1$,

Then setting $\mathcal{H}$ to be orthogonal to $\mathcal{V}$ will make $(S^{2n+1}, \mathcal{H}, g)$ a sRmc-manifold.
Gromov-Hausdorff Convergence

For a sRmc-manifold \((\mathcal{M}, \mathcal{H}, g)\) we define the canonical variation of the metric

\[
g_{\varepsilon} = g_{\mathcal{H}} + \frac{1}{\varepsilon}g_{\mathcal{N}}
\]
Gromov-Hausdorff Convergence

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which in the Gromov-Hausdorff sense

\[ (M, \mathcal{H}, g_{\varepsilon}) \xrightarrow{\varepsilon \to 0^+} (M, \mathcal{H}, g_{\mathcal{H}}) \]
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The idea is to consider the convergence of Riemannian structures to the sub-Riemannian one.
Hladky-Bott Connection

Theorem (Hladky ‘12 [5])

There exists a unique metric connection $\nabla$ on $(\mathbb{M}, \mathcal{H}, g)$ such that
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1. $\mathcal{H}$ and $\mathcal{V}$ are $\nabla$-parallel,
2. The torsion $T$ of $\nabla$ satisfies
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
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## Hladky-Bott Connection

**Theorem (Hladky ‘12 [5])**

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2. The torsion \( T \) of \( \nabla \) satisfies
   - \( T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V} \),
   - \( T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H} \)
3. For every \( X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V}) \),
   - \( \langle T(X, Z), Y \rangle_{\mathcal{H}} = \langle T(Y, Z), X \rangle_{\mathcal{H}} \)
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This is called the Hladky-Bott connection.
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This is called the **Hladky-Bott** connection.
Hladky-Bott Connection

We can explicitly write $\nabla$ in terms of the Levi-Civita connection $\nabla^g$ as

$$\nabla_X Y = \begin{cases} 
\pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\
\pi_{\mathcal{H}} [X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\
\pi_{\mathcal{V}} [X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\
\pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V})
\end{cases}$$
Hladky-Bott Connection

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\pi_V [X, Y] + A_X Y & Y \in \Gamma(V), X \in \Gamma(H) \\
\pi_V \nabla^g_X Y & X, Y \in \Gamma(V) 
\end{cases}$$

where the tensor $A$ is defined by

$$\langle A_X Y, Z \rangle = \frac{1}{2} (\langle \mathcal{L}_X g \rangle(Y_H, Z_H) + \langle \mathcal{L}_X g \rangle(Y_V, Z_V))$$
Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:
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- **Bundle-like metric**: A foliation is said to have a bundle-like metric if the metric locally splits orthogonally. This is equivalent to

\[ \mathcal{L}_V g(\mathcal{H}, \mathcal{H}) = 0 \]
Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:

- **Bundle-like metric:** A foliation is said to have a bundle-like metric if the metric locally splits orthogonally. This is equivalent to
  \[ \mathcal{L}_V g(\mathcal{H}, \mathcal{H}) = 0 \]

- **Totally geodesic foliation:** A foliation is said to be totally geodesic if the geodesics of the fibers are embedded geodesics of the total space. This is equivalent to
  \[ \mathcal{L}_\mathcal{H} g(\mathcal{V}, \mathcal{V}) = 0 \]
J Map

On \((\mathbb{M}, \mathcal{H}, g)\) we can associate to each vector field \(Z \in \Gamma(\mathbb{T}\mathbb{M})\) an endomorphism \(J_Z\) of \(\mathbb{T}\mathbb{M}\) defined by

\[
\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle
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\]

If \(\mathcal{V}\) is integrable,

\[
\begin{cases} 
J_Z X \in \mathcal{H} & \text{if } Z \in \mathcal{V}, X \in \mathcal{H} \\
J_Z X = 0 & \text{otherwise}
\end{cases}
\]
On \((M, \mathcal{H}, g)\) we can associate to each vector field \(Z \in \Gamma(TM)\) an endomorphism \(J_Z\) of \(TM\) defined by

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\end{cases}
\]

We thus take the perspective

\[
J: \mathcal{V} \to \text{End}(\mathcal{H}), \quad Z \mapsto J_Z
\]
H-type Foliations

Definition

Let $(\mathcal{M}, \mathcal{H}, g)$ be a sRmc-manifold. We say that $(\mathcal{M}, \mathcal{H}, g, J)$ is an $H$-type foliation if
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**Definition**

Let $\left(M, H, g\right)$ be a sRmc-manifold. We say that $\left(M, H, g, J\right)$ is an H-type foliation if

1. $\left(M, \mathcal{V}, g\right)$ is a totally geodesic foliation with bundle-like metric, and
2. for all $X, Y \in \Gamma(H), Z \in \Gamma(\mathcal{V})$,

$$\langle J_Z X, J_Z Y \rangle_H = \|Z\|^2 \langle X, Y \rangle_H$$
Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion $T$. 
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- If $\nabla_{\mathcal{H}} T = 0$ we say $\mathcal{M}$ has horizontally parallel torsion, and
Parallel Torsion

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- If $\delta_\mathcal{H} T = 0$ we say $\mathcal{M}$ is of Yang-Mills type,
- If $\nabla_\mathcal{H} T = 0$ we say $\mathcal{M}$ has horizontally parallel torsion, and
- If $\nabla T = 0$ we say $\mathcal{M}$ has completely parallel torsion.
Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion $T$.

- If $\delta_H T = 0$ we say $\mathcal{M}$ is of \underline{Yang-Mills type},
- If $\nabla_H T = 0$ we say $\mathcal{M}$ has \underline{horizontally parallel torsion}, and
- If $\nabla T = 0$ we say $\mathcal{M}$ has \underline{completely parallel torsion}.

**Lemma**

All H-type foliations are Yang-Mills.
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**Definition**

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that it satisfies the $J^2$ condition if for every $Z_1, Z_2 \in \mathcal{V}$ with $\langle Z_1, Z_2 \rangle = 0$ there exists $Z_3 \in \mathcal{V}$ such that

$$JZ_1 JZ_2 = JZ_3$$
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$$J_{Z_1} J_{Z_2} = J_{Z_3}$$

The H-type groups with this property were classified by (M. Cowling, A.H. Dooley, A. Korányi, and F.Ricci ’91 [4])
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but this isn’t true for general connections, or in particular the Bott connection.
Adjoint Connections and the Jacobi Equation

For an arbitrary connection $\nabla$ with torsion

$$\text{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

define its adjoint connection to be

$$\hat{\nabla}_X Y = \nabla_X Y - \text{Tor}(X, Y)$$
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notice, $\hat{\nabla} = \nabla$.

In general, the adjoint of a metric connection is not metric. As a consequence, terms involving the torsion of $\nabla$ are introduced to the Jacobi equation.
However, in the special case that both $\nabla, \hat{\nabla}$ are metric, the Jacobi equation along a geodesic $\gamma$ is

$$\hat{\nabla}_{\dot{\gamma}} \nabla_{\dot{\gamma}} W + \hat{R}(W, \dot{\gamma})\dot{\gamma} = 0$$
However, in the special case that both $\nabla, \hat{\nabla}$ are metric, the Jacobi equation along a geodesic $\gamma$ is

$$\hat{\nabla}_\gamma \nabla_\gamma W + \hat{R}(W, \gamma) \dot{\gamma} = 0$$

This is a consequence of the commutation $\nabla_V \dot{\gamma} = \hat{\nabla}_\gamma V$ for a Jacobi field $V$ along a geodesic $\gamma$. 
For any $\varepsilon > 0$ the connection

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From this we recover a Jacobi equation for all $\varepsilon > 0$. 
The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier ‘17 [1])

Let $x, y \in \mathbb{M}$,
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- Let \( x, y \in \mathbb{M} \),
- \( \gamma: [0, r_\varepsilon] \to \mathbb{M} \) a unit speed \( g_\varepsilon \)-geodesic connecting \( x, y \), and
- \[ \sum_{i=1}^{k} \int_{0}^{r_\varepsilon} \langle \hat{\nabla} \varepsilon \dot{\gamma}, \nabla \varepsilon \dot{\gamma} \rangle \langle W_i, W_i \rangle \geq 0 \]
- with equality if and only if the \( W_i \) are Jacobi fields.
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- $W_1, \cdots, W_k$ be a collection of vector fields along $\gamma$ such that

$$\sum_{i=0}^{k} \int_{0}^{r_\varepsilon} \langle \hat{\nabla}_\gamma^\varepsilon \nabla^\varepsilon_{\gamma} W_i + \hat{R}^\varepsilon (W_i, \dot{\gamma}) \dot{\gamma}, W_i \rangle_\varepsilon \geq 0$$
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then at $y = \gamma(r_{\varepsilon})$ it holds that

$$\sum_{i=0}^{k} \text{Hess}^{\varepsilon}(d_{\varepsilon}^{\text{p}})(W_i, W_i) \leq \sum_{i=0}^{k} \left\langle W_i, \hat{\nabla}^{\varepsilon}_{\dot{\gamma}} W_i \right\rangle_{\varepsilon}$$
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with equality if and only if the \( W_i \) are Jacobi fields.
Along a geodesic $\gamma$ let $V$ satisfy the Jacobi equation

$$\hat{\nabla}_{\dot{\gamma}} \nabla_{\dot{\gamma}} V - \hat{R}(V, \dot{\gamma})\dot{\gamma} = 0$$

and initial conditions $V(0) = 0$, $V(r) = X$
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Then it can be shown

$$
\hat{\nabla}^2 d_p(q)(X, X) = I(V, V)
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I(V, V) = \int_0^r \langle \hat{\nabla}_{\dot{\gamma}}^\varepsilon V, \nabla_{\dot{\gamma}}^\varepsilon V \rangle - \hat{R}^\varepsilon(V, \dot{\gamma}, \dot{\gamma}, V) \, dt
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Along a geodesic $\gamma$ let $V$ satisfy the Jacobi equation
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This gives bounds on the behavior of $\text{Hess}^{\hat{\nabla}^\varepsilon}(r_\varepsilon)$
Horizontal Splitting

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Lemma

Denoting $n = \text{rk}(\mathcal{H}), m = \text{rk}(\mathcal{V})$, we will have

$$\text{dim}(\mathcal{H}_{Sas}) = m, \quad \text{dim}(\mathcal{H}_{Riem}) = n - m - 1$$
Comparison Functions

Similarly to the Riemannian case, we consider the comparison functions

\[ F_{Riem}(r, \kappa) = \begin{cases} \sqrt{\kappa} \cot(\sqrt{\kappa} r) & \text{if } \kappa > 0 \\ \frac{1}{r} & \text{if } \kappa = 0 \\ \sqrt{|\kappa|} \coth(\sqrt{|\kappa|} r) & \text{if } \kappa < 0 \end{cases} \]
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\[ F_{Sas}(r, \kappa) = \begin{cases} \frac{\sqrt{\kappa}(\sin(\sqrt{\kappa}r) - \sqrt{\kappa}r \cos(\sqrt{\kappa}r))}{2 - 2 \cos(\sqrt{\kappa}r) - \sqrt{\kappa}r \sin(\sqrt{\kappa}r)} & \text{if } \kappa > 0 \\ \frac{4}{r} & \text{if } \kappa = 0 \\ \frac{\sqrt{\kappa}(\sqrt{\kappa}r \cosh(\sqrt{\kappa}r) - \sinh(\sqrt{\kappa}r))}{2 - 2 \cosh(\sqrt{\kappa}r) + \sqrt{\kappa}r \sinh(\sqrt{\kappa}r)} & \text{if } \kappa < 0 \end{cases} \]
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\frac{4}{r} & \text{if } \kappa = 0 \\
\sqrt{\kappa} \left(\sqrt{\kappa} r \cosh(\sqrt{\kappa} r) - \sinh(\sqrt{\kappa} r)\right) & \text{if } \kappa < 0 \\
2 - 2 \cosh(\sqrt{\kappa} r) + \sqrt{\kappa} r \sinh(\sqrt{\kappa} r)
\end{cases}
\end{align*}

These comparison functions will correspond to the splitting of $\mathcal{H}$. 
Hessian Comparisons

Theorem (Baudoin, Grong, M., & Rizzi ‘19 [3])

Let \( \gamma : [0, r_\varepsilon] \to \mathbb{M} \) be a \( g_\varepsilon \)-geodesic. Then

\[
\text{Hess}(r_\varepsilon)(\dot{\gamma}, \dot{\gamma}) \leq \frac{\|\dot{\gamma}\|^2 (1 - \|\dot{\gamma}\|^2)}{r_\varepsilon}
\]

- If \( \text{Sec}(X \wedge Y) \geq \rho > 0 \) for all unit \( X, Y \in \mathcal{H}_{\text{Riem}}(\dot{\gamma}) \), then
  \[
  \text{Hess}(r_\varepsilon)(X, X) \leq F_{\text{Riem}}(r_\varepsilon, K)
  \]

- If \( \text{Sec}(X \wedge J_Z X) \geq \rho > 0 \) for all unit \( X \in \mathcal{H}_{\text{Sas}}(\dot{\gamma}) \), then
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  \text{Hess}(r_\varepsilon)(X, X) \leq F_{\text{Sas}}(r_\varepsilon, K)
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Where \( K \) is a constant depending on \( \rho, \varepsilon, \|\nabla_{\mathcal{Y}}r_\varepsilon\|, \) and \( \|\nabla_{\mathcal{H}}r_\varepsilon\| \).
Horizontal Ricci Curvature

We define the horizontal Ricci curvature as the horizontal trace of the Riemann tensor,

\[
\text{Ric}_H(X, X) = \sum_{i=0}^{n} \langle R^\nabla(W_i, X)X, W_i \rangle
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= \langle R^\nabla(Y, X)X, Y \rangle + \text{Ric}_{Sas}(X, X) + \text{Ric}_{Riem}(X, X)
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\mathcal{H} = \text{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}
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Bonnet-Meyers Estimates

Theorem (Baudoin, Grong, M., & Rizzi ‘19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

\[
\frac{\text{Ric}_{\text{Riem}}(X, X)}{n - m - 1} \geq \rho \implies \text{diam}_0(M) \leq \frac{\pi}{\sqrt{\rho}}
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2. $\text{Sec}(X \wedge J_Z X) \geq \rho \implies \text{diam}_0(\mathcal{M}) \leq \frac{2\pi}{\sqrt{\rho}}$

and in each case the fundamental group of $\mathcal{M}$ must be finite.

The first two of these are sharp, as they are achieved in the complex, quaternionic, and octonionic Hopf fibrations.
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3. $\frac{\text{Ric}_{\text{Sas}}(X, X)}{m} \geq \rho \implies \text{diam}_0(\mathbb{M}) \leq \frac{2\pi \sqrt{3}}{\sqrt{\rho}}$

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$$= \text{Hess}(r_\varepsilon)(Y, Y) + \sum_{i=0}^{m} \text{Hess}(r_\varepsilon)(JZ_i Y, JZ_i Y) + \sum_{i=0}^{n-m-1} \text{Hess}(r_\varepsilon)(W_i, W_i)$$

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for appropriate bases $\{W_i\}$ of $\mathcal{H}$ and $\{Z_i\}$ of $\mathcal{V}$. This splitting corresponds again to the decomposition

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Laplacian Comparisons

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Let $(\mathcal{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure and satisfying the $J^2$ condition, and with nonnegative horizontal Bott curvature. Then there exists a $C > 4$ such that

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n - m + 3 + C(m - 1)}{r_0}$$

This is not sharp, but we can recover sharp estimates in each subspace.
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Thank you for your attention!