Stochastic processes on surfaces in 3D contact sub-Riemannian manifolds

Talk by Karen Habermann on joint work with Davide Barilari, Ugo Boscain and Daniele Cannarsa


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9 September 2020
Setting

three-dimensional smooth manifold, \((D, g)\) sub-Riemannian structure on \(M\), contact structure on \(M\), that is, \(D = \ker \omega\) for one-form \(\omega\) on \(M\) with \(\omega \wedge d\omega \neq 0\). Orientable surface embedded in \(x, y, z\).
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Wikipedia
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- $S$ orientable surface embedded in $M$
Adjusted setting (for notational convenience)
- $M$ three-dimensional smooth manifold,
- $\mathcal{D}$ distribution spanned by $X_1$ and $X_2$,
- $g$ given by requiring $(X_1, X_2)$ to be an orthonormal frame,
- $X_0$ the unique vector field on $M$ such that $[X_1, X_2] = X_0 + c_{12} X_1 + c_{21} X_2$,
- $X_0$ the Reeb vector field for the contact form $\omega$ normalised such that $\delta\omega|_{\mathcal{D}} = -\text{vol}_g$,
- $S$ embedded surface in $M$ given by $S = \{ x \in M : u(x) = 0 \}$ for $u \in C^2(M)$ with $\delta u \neq 0$ on $S$. 
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Definition

Set $\Gamma(S)$ of characteristic points on $S$

$$x \in \Gamma(S) \text{ if and only if } (X_1 u)(x) = (X_2 u)(x) = 0$$
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**Theorem (Barilari, Boscain, Cannarsa, H)**

For $f \in C^2_c(S \setminus \Gamma(S))$, we have

\[ \Delta_\varepsilon f \to \Delta_0 f \]

uniformly on $S \setminus \Gamma(S)$ as $\varepsilon \to 0$. 

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Note that $b\hat{X}_S^\perp - X_0$ is a vector field on $S \setminus \Gamma(S)$. 
Sketch of the proof:

Consider vector fields on $S \setminus \Gamma(S)$ given by

$$F_1 = (X_2 u)X_1 - (X_1 u)X_2 \sqrt{(X_1 u)^2 + (X_2 u)^2} + (X_2 u)^2$$

and

$$F_2 = (X_0 u)(X_1 u)X_1 + (X_0 u)(X_2 u)X_2 \sqrt{(X_1 u)^2 + (X_2 u)^2} - X_0.$$ 

They satisfy $g_\varepsilon(F_1, F_2) = 0$ with

$$g_\varepsilon(F_1, F_1) = 1$$

and

$$g_\varepsilon(F_2, F_2) = (X_0 u)^2 (X_1 u)^2 + (X_2 u)^2 + 1.$$ 

Proposition (Barilari, Boscain, Cannarsa, H)

We have

$$K_0 := \lim_{\varepsilon \to 0} K_\varepsilon = -\hat{X}_S(b) - b^2$$

uniformly on compact subsets of $S \setminus \Gamma(S)$. 
Sketch of the proof: Consider vector fields on $S \setminus \Gamma(S)$ given by

$F_1 = \frac{X_2u \cdot X_1 - (X_1u) \cdot X_2}{\sqrt{(X_1u)^2 + (X_2u)^2}}$ and

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Characteristic foliation described by logarithmic spirals

In coordinates \((s, \psi)\) with \(s > 0\) and \(\psi \in [0, 2\pi)\) on \(S \setminus \Gamma(S)\)

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\hat{X}_S = \frac{\partial}{\partial s} \quad \text{and} \quad b(s, \psi) = \frac{2}{s},
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Characteristic foliation described by loxodromes

In coordinates $(s, \varphi)$ on $S \setminus \Gamma(S)$

$$\frac{1}{2} \Delta_0 = \frac{1}{2} \frac{\partial^2}{\partial s^2} + \left( \cot (\theta(s)) \frac{d\theta}{ds} \right) \frac{\partial}{\partial s},$$

where $\varphi \in [0, 2\pi)$ and $s$ is given in terms of the polar angle $\theta$ as a multiple of an elliptic integral of the second kind.
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Theorem (Barilari, Boscain, Cannarsa, H)

Let $k \geq 0$ be such that $|\kappa| = 4k^2$. 

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S = \{ \exp(r \cos \theta X_1 + r \sin \theta X_2) : r \in I \text{ and } \theta \in [0, 2\pi) \},
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we have \( \Delta_0 = \frac{\partial^2}{\partial r^2} + b(r) \frac{\partial}{\partial r} \), where

\[
b(r) = \begin{cases} 
2k \cot(kr) & \text{if } \kappa = 4k^2 \\
\frac{2}{r} & \text{if } \kappa = 0 \\
2k \coth(kr) & \text{if } \kappa = -4k^2
\end{cases}.
\]
This gives rise to three classes of familiar stochastic processes.

- Bessel process of order 3 has generator \( \frac{1}{2} \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \) for \( r > 0 \).

- Legendre process of order 3 has generator \( \frac{1}{2} \frac{\partial^2}{\partial \theta^2} + k \cot(k \theta) \frac{\partial}{\partial \theta} \) for \( \theta \in (0, \pi/k) \).

- Hyperbolic Bessel process of order 3 has generator \( \frac{1}{2} \frac{\partial^2}{\partial r^2} + k \coth(kr) \frac{\partial}{\partial r} \) for \( r > 0 \).
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  \]
Criterion for accessibility of characteristic points

Definition

Characteristic point $x \in \Gamma(S)$ is called elliptic if $\det((\text{Hess}u)(x)) > 0$, hyperbolic if $\det((\text{Hess}u)(x)) < 0$.

Theorem (Barilari, Boscain, Cannarsa, H)

For the canonical stochastic process with generator $\Delta$, elliptic characteristic points are inaccessible, while hyperbolic characteristic points are accessible from the separatrices.
Criterion for accessibility of characteristic points in terms of

\[ \text{Hess } u = \begin{pmatrix} X_1X_1u & X_1X_2u \\ X_2X_1u & X_2X_2u \end{pmatrix} \]
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Sketch of the proof:
Write $\Delta_0$ as $\partial^2 / \partial s^2 + b(\gamma(s)) \partial / \partial s$ near $x \in \Gamma(S)$. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $(X_1 X_2 u - X_1 X_1 u X_2 X_2 u - X_2 X_1 u)(x)$ subject to $(X_0 u)(x) = 1$. In particular, we have $\lambda_1 + \lambda_2 = 1$, and if $x \in \Gamma(S)$ is elliptic or hyperbolic then $\lambda_1, \lambda_2 \neq 0$. 
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For the canonical stochastic process with generator $\frac{1}{2} \Delta_0$

- elliptic characteristic points are inaccessible, while
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Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of

$$
\begin{pmatrix}
X_1X_2u & -X_1X_1u \\
X_2X_2u & -X_2X_1u
\end{pmatrix}
\begin{pmatrix}
x
\end{pmatrix}
$$

subject to $$(X_0u)(x) = 1.$$
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$$

In particular, we have

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\lambda_1 + \lambda_2 = 1.
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X_2X_2u & -X_2X_1u
\end{pmatrix}(x) \quad \text{subject to} \quad (X_0u)(x) = 1.
$$

In particular, we have

$$
\lambda_1 + \lambda_2 = 1,
$$

and if $x \in \Gamma(S)$ is elliptic or hyperbolic then $\lambda_1, \lambda_2 \neq 0$. 
\( b(\gamma(s)) \sim 2s \)

\( b(\gamma(s)) \sim \lambda_i s \)
\[ \gamma(s) \sim 2 \quad \gamma(s) \sim 1 \quad \lambda_i \quad s \]
\[ b(\gamma(s)) \sim 2s b(\gamma(s)) \sim 1 \lambda_i s \]
\( \gamma(s) \sim 2s \)
\( b(\gamma(s)) \sim \frac{2}{s} \)
\[ b(\gamma(s)) \sim \frac{2}{s} \quad b(\gamma(s)) \sim \frac{1}{\lambda_i s} \quad b(\gamma(s)) \sim \frac{1}{\lambda_i s} \]