NOTIONS OF RECTIFIABILITY
IN HEISENBERG GROUPS

(joint work with Daniela Di Donato & Tuomas Orponen)

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Sub-Riemannian Geometry and Interactions
I) Rectifiability

Codimension-1 rectifiable set

~ measure-theoretic generalization of "smooth hypersurface", nice enough to do analysis and geometry
Federer's definition for $k \in \mathbb{N}$:

**Def.** ($k$-rechifiable)

$S \subset (X,d)$ st.

$\exists \ f_j : A_j \subset (\mathbb{R}^k, 1 \cdot 1) \to (X,d), \ j \in \mathbb{N}, \ \text{Lipschitz st.}$

$\mathcal{H}^k (S \setminus \bigcup_{j \in \mathbb{N}} f_j (A_j)) = 0$
For us:

\[ \mathcal{X} = (X, d) = (\mathbb{H}^n = (\mathbb{R}^{2n+1}, \cdot), \text{left-inv.} d_{\mathbb{H}} = d_{\text{sub-Riem}}) \]

the \( n \)-th Heisenberg group with its Lie algebra generated by horizontal vectors \( \{X_1, \ldots, X_{2n}\} \)

\[ \{X_i, X_{n+i}\} = T, \quad i = 1, \ldots, n \]

\( k = 2n+1 \) codim-1,

recall: \( \dim_{\text{Haus}}(\mathbb{H}^n, d_{\text{SR}}) = 2n+2 \)
Federer's definition does not yield a useful class of sets in this setting:

**Thm.** (Ambrosio-Kirchheim n=1, Magnani ...)

Let \( k, n \in \mathbb{N}, \quad k > n \).

If \( f: AC(R^k, l^1) \rightarrow (H^n, d_{SR}) \) is Lipschitz,
then

\[ H^k(f(A)) = 0 \]

In particular: \( H^k(S) = 0 \quad \forall S \subset H^n \) \( k \)-rectif.
Alternatives to Federer's definition for codimension-1 rectifiability in $(\mathbb{H}^n, d_{sr})$?

1. Modify Federer's definition [Pauls]

   OR

2. Use definition motivated by applications (e.g., calc. of var.) [Franchi, Serapioni, Serra Cassano]

Connections to other notions:
- Intrinsic Lipschitz graphs [Franchi, Serapioni, Serra Cassano]
- Approximate tangent subgroups [Mabila, Serapioni, Serra Cassano]
- "Strong approximate tangent cone" [Pauls]
- Tangent measures & upper/lower density [Merlo]
- Density [Merlo] ...
Two definitions

$SC(H^n, ds) \text{ is codimension-1 rectifiable}$

if $\exists \Gamma_j, j \in \mathbb{N}$ s.t.

$H^{2n+1}(S \setminus \bigcup_{j \in \mathbb{N}} \Gamma_j) = 0$

where:

1. $\Gamma_j = \text{Lipschitz image of a subset in codim-1 homog. subgroup of } H^n$

2. $\Gamma_j = \text{H-regular hypersurface in } H^n$
Lipschitz image rectifiability due to Pauls

\[ \Gamma_j = f_j(A_j) \text{ for } f_j : A_j \subset (\mathbb{W}, d_{SR}|_{\mathbb{W}}) \overset{\text{Lipschitz}}{\hookrightarrow} (\mathbb{H}^n, d_{SR}) \]

\[ \mathbb{W} = \text{codim}.1 \text{ homog. subgraph of } \mathbb{H}^n \]

\[ \mathbb{W} = \{ (x_1, \ldots, x_n, t) : x_1 = 0 \} \]

'vertical hyperplane'
\( \Gamma_j = \{ \text{H-regular hypersurface} \} \)

\( \Gamma = \{ \text{H-regular hypersurface} \} : \)
\[ \forall p \in \Gamma \quad \exists B(p, r) \]
\[ \exists f \in C^1_{\text{H}}(B(p, r)) \]
\[ \nabla_{\text{H}} f(p) \neq 0 \]
\[ \Gamma \cap B(p, r) = \{ q \in B(p, r) : f(q) = 0 \} \]

Ex: \( \Gamma = W \) (with \( f(x_1, \ldots, x_n, c) = x_1, \nabla_{\text{H}} f = (1, 0, \ldots, 0) \))
Corresponding notions of rectifiability with

1. Lipschitz images
2. $C^1$ surfaces

in Euclidean spaces are equivalent.
Are the notions of rectifiability in $H^n$ equivalent?

In particular:

**Question** Are $H^1$-regular hypersurfaces (à la Franchi - Serapioni - Serra Cassano) metric Lipschitz image rectifiable (à la Pauls)?
Towards Metric Rectifiability

Thm (Coles, Pauls, Bigolin, Vittone)

Let \( S \subset H^1 \) Euclidean \( C^1 \) hypersurface

\[ p \in S \text{ non-characteristic} \]

(i.e. \( T_p S \neq \text{span} \{ X_1, X_2 \} \))

Then \( \exists B(p, r) \) s.t.

\[
(S \cap B(p, r), d_{SR} \mid_{S \cap B(p, r)}) \sim \text{open subset of (W, d_{SR}|_W)}
\]
Bi-Lipschitz map on open subset maps Lipschitz (horizontal) curves onto Lipschitz (horizontal) curves!

- Problem at characteristic points:

- Problem for $H^1$-regular surfaces without Euclidean $C^1$

cf. Rectifiability result for $H^1$-regular surfaces w/ additional metric/intrinsic condition in $H^1$ by T. Orponen (2015).
What about higher dimensions?

Thm (Antonelli, Le Donne, 2019)

Let $n \geq 2$.

If $S$ is a $C^\infty$ hypersurface in $H^n$, then it is rectifiable by bi-Lipschitz maps from compact subsets of $W \approx H^{m-1} \times \mathbb{R}$. 
Proof reduces to corresponding result for

\[ S = \{ \omega \cdot \varnothing(\omega) : \omega \in W \} \]

\[ \varnothing : W \to (W)^\perp \]

\[ \infty \text{ "intrinsic Lipschitz graph"} \]

\[ \vartriangle \]

\[ \Phi : W \to S, \quad \Phi(\omega) = \omega \cdot \varnothing(\omega) \]

is in general not metrically Lipschitz.

Interpret \( S \) as sub-Riemannian manifold with tangents \( W \) a.e.

Apply rectifiability criterion for sub-Riemannian manifolds by Le Donne-Yang.
**Review on Earlier Results**

1. **Assumptions on dimension of $H^n$**
   - **Cole-Pauls, Bigolin-Vitale:**
     \[ n = 1 \to \text{have distinguished horizontal direction} \]
   - **Antonelli-Le Donne:**
     \[ n \geq 2 \to \text{W = Carnot group (} H^{n-1} \times \mathbb{R}) \]
     \[ S = \text{sub-Riemannian mfd (locally at a.e. pt)} \]
     \[ \to \text{rectifiability criterion (Le Donne - Young) applies.} \]

2. **Regularity assumptions on $S$**
   - **Cole-Pauls, Bigolin-Vitale:**
     \[ C^1 \to \text{have local foliation by horiz. curves} \]
   - **Antonelli-Le Donne:**
     \[ C^\infty \to \text{apply Le Donne - Young criterion} \]

[cf. also result by Pauls under assumptions for strong approx. tangent cones]
Approach that
- treats all $n \in \mathbb{N}$ simultaneously,
- works for $S = H$-regular hypersurface?

open, but ...

Approach that
- treats all $n \in \mathbb{N}$ simultaneously,
- works without Euclidean regularity assumption for $S$

[joint work with D. Di Donato, T. Orponen]
RECTIFIABILITY OF INTRINSIC $C^{1,\alpha}$ SURFACES

Definition

$C^{1,\alpha}_{H^n}$-surface = $H$-regular hypersurface with
$\alpha$-Hölder continuous (wrt $d_H$)
horizontal normal
(think: $\nabla f \cdot \nu = -\frac{\nabla f}{|\nabla f|}$)

Thm

Let $n \in \mathbb{N}$, $\alpha > 0$.

$C^{1,\alpha}_{H^n}$-hypersurfaces in $H^n$
are Pauls rectifiable

(by bi-Lipschitz images of
subsets of codim-1 vertical
subgroups $W$ of $H^n$)

[For $n \geq 2$, this recovers the Antociu-Le Donne result for $C^{\infty}$ surfaces]

Proof uses the following:
A GENERAL CRITERION FOR FINDING BILIPSCHITZ MAPS BETWEEN "BIG PIECES" OF METRIC SPACES

- \((G, d_G, \mu)\) complete metric space w/ non-triv. doubling measure \(\mu\), \(\text{diam } G \geq 1\)
  
  [think: \((G, d_G) = (\mathbb{W}, d_{\text{H}2^n_{1,1}})\)]
  \(\mu = \text{Haar measure on } \mathbb{W} = \mathbb{H}2^n_{1,1}\)

- \((M, d_M)\) complete metric space
  
  [think: \((M, d_M) = (S, d_{\text{H}2^n_{1,1}})\)]

\[
G = \mathbb{W} \xrightarrow{F \text{ bi-lip}} S = M
\]

Inspired by Le Donne - Young, but approach (due to Orponen) does not refer to curves.
**Thm.**

**Assumptions:** \( \exists \alpha > 0, \ L \geq 1, \ A \geq 1, \ x_0 \in G, \ p_0 \in M \) s.t.

\( \forall x \in B(x_0, 1) \subset G, \ p \in B(p_0, 1) \subset M, \ n \in \mathbb{N}, \)

\( \exists \ i_{x \to p}^n : G \to M \)

(i) **Local approx. bi-Lipschitz correspondence at scale** \( 2^{-n}, \ n \in \mathbb{N} \)

(a) \( i_{x \to p}^n(x) = p \)

(b) \( \frac{1}{L} d_G(y,z) - A 2^{-n(1+\alpha)} \leq d_M(i_{x \to p}^n(y), i_{x \to p}^n(z)) \leq L d_G(y,z) + A 2^{-n(1+\alpha)} \)

for \( y, z \in B(x, 2^{-n}) \)

(ii) **Compatibility condition for consecutive scales**

If \( x, y \in B_G(x_0, 1) \), \( p, q \in B_M(p_0, 1) \)

\( d_G(x,y) \leq 2^{-n}, \ i_{x \to p}^n(y) = q, \)

then

\( d_M(i_{x \to p}^n(z), i_{y \to q}^n(z)) \leq A 2^{-n(1+\alpha)} \)

for \( z \in B_G(x, 2^{-n}) \)

**Conclusion:**

\( \exists \ \delta = \delta((G,d_G,\mu), \ \alpha, \ L, \ A) > 0 \)

\( \exists \ \ K \subset B(x_0, 1) \subset G \) c.p.d., \( \mu(K) \geq \delta \mu(B(x_0, 1)) \)

\( \exists \ \ 2L-\text{bi-Lipschitz} \ F : K \to M \) with \( F(K) \subset B(p_0, 1) \).
Proof idea

$$F_n \big|_Q = \mathcal{L}_{c_a}^{n} \rightarrow F_{n-1}(c_a) \big|_Q$$ for $Q \subset Ch(Q_{n-1}) \subset \mathbb{D}$, 'dyadic cubes' of side length $\sim 2^{-n}$

$$K = \bigcap_{n\geq 0} \bigcup_{Q \in \mathbb{D}_n} Q \subset B(x,1)$$

$$\mu(K) > \delta \mu(B(x,1)) > 0.$$
To-Do List:

1. $d_{\mu}(F_n(w), F_{n+1}(w)) \leq A 2^{-n(1+\alpha)}$, weak for $n$ large enough (dep. on L and A)

2. $\implies F = \lim F_n$ exists by completeness of $(M, d_{\mu})$

3. $F$ is bi-Lip. [using separation of cubes!]
APPLICATION TO INTRINSIC $C^{1,\alpha}$ SURFACES

Let

- $G \hat{\in} \mathcal{W} = \text{codim} - 1$ vertical subgroup in $H^n$
- $\mu = \text{Haar measure on } \mathcal{W} = H^{2n+1}$
- $M = \Gamma \varphi = \{ w \cdot \varphi(w) : w \in \mathcal{W} \}$, where

$\varphi : \mathcal{W} = \{ x = 0 \} \rightarrow \mathcal{V} = (\mathcal{W})^\perp$ Compactly supported

(i) $M = \{ w \cdot \varphi(w) : w \in \mathcal{W} \}$ is $C^{1,\alpha}$ - hypersurface

(ii) horiz. normal $\nu_H = (\nu_{H1}^1, \ldots, \nu_{H1}^{2n})$

assoc. to

$\{ w \cdot v : v < \varphi(w), w \in \mathcal{W} \}$

satisfies

$\nu_{H1}(w) < 0 \forall \rho \in M$
If abstract theorem applies

⇒ fat Cantor set \( K \subset W \)

2L-bi-Lipschitz \( F : K \hookrightarrow \Gamma^\circ \cap B(\rho, 1) \), \( \rho \in \Gamma^\circ \)

\[ H^{2n+1}(\Gamma^\circ \cap B(\rho, 1) \cap F(K)) \cong H^{2n+1}(K) \geq \delta \]

⇒ \( \Gamma^\circ \) has "big pieces of bi-Lip. images at unit scale"

Smaller scales follow by blowing up to unit scale,

larger scales follow from compact support assumption

⇒ \( \Gamma^\circ \) has big pieces of bi-Lipschitz images of compact subset of \( W \),

in particular:

\( \Gamma^\circ \) is Pauls rectifiable -
Why does the abstract theorem apply in the $C^{1,a}$ case?
Toy example

If \( f: \mathbb{R} \to \mathbb{R} \) is \( C^{\alpha} \) (i.e. \( f' \) is \( \alpha \)-Hölder), then \( f \) is very well approximated by the affine map

\[
s \mapsto f'(x)(s-x) + f(x)
\]

in a neighborhood of \((x, f(x))\):

\[
\left| f(s) - [f'(x)(s-x) + f(x)] \right| = \left| f(x) \right|_{s-x} \int_x^s f'(t) \, dt - f'(x)(s-x) \right|
\leq \int_x^s \left| f'(t) - f'(x) \right| \, dt
\leq \int_x^s |t-x|^{1+\alpha} \, dt \leq L|s-x|^{1+\alpha}
\]

for \( s > x \).

\[
d(q, \Psi_p(y-x)+\rho) = d(S, \Psi_p(y-x)+\rho)
\]

\[
M = \text{graph}(f)
\]

\[
\ell_{\alpha} \to \gamma_y = q \quad \text{independent of generation } \gamma
\]
Improvement in the first Heisenberg G.

Recall:
- toy example
- $\omega \cdot \mathcal{O}(\omega) : \omega \in W^2 \subset H^1$
  for intrinsic $\mathcal{O}_{1/2}$

Used:
- approximation of graph by tangent planes
- bi-Lipschitz parametrizations of tangent planes

Replace:

(vertical) tangent plane

by

“flag surfaces”
in $H^1$

bi-lip
Replace:

(Vertical) tangent plane by "flag surfaces" in $H^1$

Intrinsic graph of intrinsic gradient $(y,t) \rightarrow \nabla \phi(\omega_y)y$

Intrinsic graph of $(y,t) \rightarrow \Psi_{\omega_y}(y)$ for Euclidean Lip. $\Psi_{\omega}$

$\phi \in C^{1,\alpha}_H$

by

$\phi$ intrinsic Lipschitz with improved vertical Hölder regularity (weaker than $C^{1,\alpha}_H$).
**Thm**: Let $\alpha > 0$.

Let $W = \text{vertical subgroup in } H^1$

$\varnothing : W \rightarrow \mathbb{V}$ intrinsic Lipschitz

compactly supported

$|\varnothing(y, \varepsilon) - \varnothing(y', \varepsilon)| \leq H \varepsilon^{\frac{1 + \alpha}{2}}$

$\Rightarrow M = \{ w : \varnothing(w) : w \in W \}$

has big pieces of bi-Lipschitz images of

$(W, d_{SR}|_W) \approx \text{parabolic plane}$

This recovers the Pauls rectifiability of $C^{1,\alpha}_H$ surfaces and Euclidean $C^1$ surfaces in the first Heisenberg group.
Merci !