Control for the Grushin-Schrödinger equation

"Une excursion a travers les paysages de l’analyse semi-classique"

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Outline

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   - Controllability and observability
   - The geometric control condition
   - Main results and motivations

2. The different regimes
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   - The different regimes: second microlocalisation
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     - Transversal propagation

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   - The half wave regime
   - Transversal propagation
     - Pure transversal
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2D Grushin-Schrödinger equation

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the 1D torus, and $\Omega = (-1, 1)_x \times \mathbb{T}_y$

$\partial\Omega = \{\pm 1\}_x \times \mathbb{T}_y$

$\Delta_G = \partial_x^2 + x^2 \partial_y^2$ the Grushin operator with domain

$$D(\Delta_G) = \{f \in L^2(\Omega) : \Delta_G f \in L^2(\Omega), f|_{\partial\Omega} = 0\}.$$

$$\Delta_G = X_1^2 + X_2^2: \text{ type I}$$

and

$X_1 = \partial_x, X_2 = x\partial_y; [X_1, X_2] = \partial_y, \text{span}\{X_1, X_2, [X_1, X_2]\} = \mathbb{R}^2 = T_x\mathbb{R}^2.$

Objective: Observability and Controlability for Grushin-Schrödinger equation with Dirichlet boundary condition:

$$\begin{cases}
    i\partial_t u - \Delta_G u = 0, & (t, x, y) \in \mathbb{R} \times \Omega \\
    u|_{t=0} = u_0 \\
    u|_{\mathbb{R} \times \partial\Omega} = 0
\end{cases}$$
2D Grushin-Schrödinger equation

- **Conservation of mass:**

\[
M(u) = \int_{\Omega} |u(t, x, y)|^2 \, dx \, dy.
\]

- **Conservation of energy:**

\[
\|u\|_{H_{G}^{1\frac{1}{2}}}^2 = \int_{\Omega} (|\partial_x u(t, x, y)|^2 + |x \partial_y u(t, x, y)|^2) \, dx \, dy.
\]

- **Subellipticity:** Energy controls \(H^{1/2}\) norm

\[
[\partial_x, x \partial_y] = \partial_y \Rightarrow \|u\|_{H_{V}^{1/2}}^2 \leq \|u\|_{H_{G}^{1}}^2.
\]
Controllability

Let $\omega \subset \Omega$ be a non-empty open set, and $T > 0$.

**Definition (Exact-controllability on $\omega$ at time $T$)**

$\forall u_0, u_1 \in L^2(\Omega), \exists f \in L^2([0, T] \times \omega)$ such that the solution $u$ of

$$(i\partial_t - \Delta_G)u = f, \quad u|_{\partial\Omega} = 0, u|_{t=0} = u_0$$

satisfies $u|_{t=T} = u_1$

**Definition (Observability on $\omega$ at time $T$)**

$\forall u_0 \in L^2(\Omega), \exists C > 0$ such that the solution $u$ of

$$(i\partial_t - \Delta_G)u = 0, \quad u|_{t=0} = u_0, \quad u = e^{it\Delta_G}u_0,$$

$$\|(e^{iT\Delta}u_0)\|_{L^2(\Omega)}^2 \leq C \int_0^T \|(e^{it\Delta_G}u_0)\|_{L^2(\omega)}^2 dt$$

For time-reversible evolution equations, the two concepts concide.
Observability results for three typical PDE’s

- **Heat:** (Lebeau-Robbiano, Fursikov-Imanuvilov): \(\omega \neq \emptyset, \ T > 0,\)
  \[
  \| e^{T\Delta} u_0 \|_{L^2(\Omega)}^2 \leq C_T \int_0^T \| e^{t\Delta} u_0 \|_{L^2(\omega)}^2 dt.
  \]
  Spectral inequality (Carleman) + very fast dissipation of HF.

- **Wave:** (80’ Rauch-Taylor, Bardos-Lebeau-Rauch, Burq-Gérard)
  \(\omega\) satisfies the geometric control condition (GCC), \(T > T_{GCC},\)
  \[
  \| u_0 \|_{L^2(\Omega)}^2 \leq C_T \int_0^T \| e^{it\sqrt{-\Delta}} u_0 \|_{L^2(\omega)}^2 dt.
  \]
  Melrose-Sjöstrand propagation of singularities. Finite propagation speed. Wave-Grushin Letrouit 20’

- **Schrödinger:** Infinite propagation speed
  - (Lebeau 90’) \(\omega\) satisfies (GCC), then \(\forall T > 0,\)
    \[
    \| u_0 \|_{L^2(\Omega)}^2 \leq C_T \int_0^T \| e^{it\Delta} u_0 \|_{L^2(\omega)}^2 dt.
    \]
GCC: pictures

ω satisfies (GCC) if there exists $T_{GCC} > 0$, such that all generalized geodesics of length $T > T_{GCC}$ intersects with ω.

GCC is satisfied

GCC is not satisfied
Observability for Schrödinger equation beyond GCC

Schrödinger: GCC not necessary (stability/instability) geodesic flow

**Theorem (N.B.-Zworski 03, 12, Anantharaman-Macià 14, Bourgain-N.B.-Zworski 13, N.B.-Zworski 17)**

\[ \Omega = \mathbb{T}^2, \forall E, |E| > 0 \text{ and } T > 0 \text{+ stable perturbations } V \in L^2(\mathbb{T}^2) \]

and let \( P = -\Delta + V(x) \).

\[ \exists C(T, E) > 0, \forall u_0 \in L^2(\mathbb{T}^2), \]

\[ \| u_0 \|_{L^2(\mathbb{T}^2)}^2 \leq C(T, E) \int_0^T \| e^{itP} u_0 \|_{L^2(E)}^2 dt. \]

**Theorem (Ikawa 80', Gaspard-Rice 80', N.B. 93, Schenck 10, Anantharaman-Rivière 11, Bourgain-Dyatlov 16, Jin 17, Dyatlov-Jin-Nonnenmacher 19)**

\( \Omega \) surface neg curvature, \( \forall \omega, T > 0. \exists C > 0; \forall u_0 \in L^2(\Omega), \)

\[ \| u_0 \|_{L^2(\Omega)}^2 \leq C(T, \omega) \int_0^T \| e^{it\Delta} u_0 \|_{L^2(\omega)}^2 dt. \]
Bibliographic for the results of parabolic-Grushin:

**Question**: Hypoelliptic geometry?

- **Heat type equation**: (Alabau, Beauchard, Cannarsa, Duprez, Guglielmi, Koenig, Pravda-Starov, ...)
  for different operators $A$ and control domains $\omega$,
  new phenomena happen: observability false for some $T > 0$ or even for all finite $T > 0$!

In particular, for Grushin (heat) equation

$$\partial_t u - \Delta_G u = 0$$

on $\Omega = (-1, 1)_x \times \mathbb{T}_y$, the following striking result holds:

**Theorem (A. Koenig ’17)**

Assume that there exists a horizontal strip $(-1, 1)_x \times (a, b)_y$ which does not encounter $\omega$. Then for any $T > 0$, the heat-observability is untrue (as well as null-controllability).
Schrödinger: No $T \leq \mathcal{L}(\omega)$ observability!

Let $\omega$ be of the form $(-1, 1)_x \times I$, where $I \subset \mathbb{T}$ is a finite union of intervals. For such $\omega$, we define $\mathcal{L}(\omega)$:

$$\mathcal{L}(\omega) := \sup\{ s : \exists y_1, y_2 \in \mathbb{T}, \text{dist}_{\mathbb{T}}(y_1, y_2) = s, [(0, y_1), (0, y_2)] \cap \omega = \emptyset \}$$

the length of largest interval in $\Omega \setminus \omega \cap \{x = 0\}$. 

\[ \begin{array}{c}
\omega \\
\mathcal{L}(\omega) \\
\omega \\
\omega
\end{array} \]
Schrödinger: No $T \leq \mathcal{L}(\omega)$ observability!

Let $\omega$ be of the form $(-1, 1) \times I$, where $I \subset \mathbb{T}$ is a finite union of intervals. For such $\omega$, we define $\mathcal{L}(\omega)$:

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the lenghth of largest interval in $\Omega \setminus \omega \cap \{x = 0\}$. 
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the length of largest interval in $\Omega \setminus \omega \cap \{x = 0\}$.

**Theorem (N. B, Chenmin Sun. '19)**

If $T \leq \mathcal{L}(\omega)$, the observability by $(T, \omega)$ is false.
**Observability** $T > \mathcal{L}(\omega)$

**Theorem (N. B, Chenmin Sun ’19)**

Assume that $T > \mathcal{L}(\omega)$. There exists $C_T > 0$, such that for all $u_0 \in L^2(\Omega)$,

$$
\|u_0\|_{L^2(\Omega)}^2 \leq C_T \int_0^T \|e^{it\Delta_G}u_0\|_{L^2(\omega)}^2 dt.
$$

**Corollary (Exact-controllability)**

Assume that $T > \mathcal{L}(\omega)$. For any $u_0, v_0 \in L^2(\Omega)$, there exists $f \in L^2([0, T] \times \omega)$, such that the solution of

$$
i \partial_t u + \Delta_G u = 1_\omega f, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0
$$

satisfies that $u|_{t=T} = v_0$.

**Remark**

First result where Schrödinger equation is exact controllable while heat equation is *not exact controllable*.
A harmonic oscillator

Take fourier transform w.r.t. $y$

$$(i \partial_t u + \partial_x^2 u - x^2 \eta^2) \hat{u}(t, x, \eta) = 0,$$

The harmonic oscillator

$$-\partial_x^2 u + x^2 \eta^2 = \eta^2(-\eta^{-2} \partial_x^2 u + x^2)$$

has a fundamental (ground) state, with eigenvalue $\lambda_{\eta,1}$

$$e_{\eta,1}(x) \sim e^{-\frac{|\eta|x^2}{2}}, \quad \lambda_{\eta,1} \sim |\eta|,$$

Take initial data distributed on these ground state

$$u_0 = \sum_{\eta \in \mathbb{Z}} \alpha_n e^{i\eta y} e_{\eta,1}(x),$$

Then solution of Grushin-Schrödinger with initial data $u_0$ is a solution of (half) wave equation

$$\left(i \partial_t + |D_y|\right)u \sim 0.$$
Harmonic oscillator: consequences

- Finite speed of propagation for Half wave equation responsible for the requirement $T \geq L(\omega)$.
- If $|D_y| \sim \epsilon^{-1}$, then $-H \geq \epsilon^{-1}$.
- Semi-classical reduction: prove observability only for initial data

$$u_0 = \chi(-h^2 H)u_0, \chi \in C_0^\infty(\frac{1}{2}, 2), \chi \big|_{(\frac{1}{\sqrt{2}}, \sqrt{2})} = 1$$

- Then use that errors $O(h^\delta) = O(H^{-\delta/2})$ are compact to conclude by compactness uniqueness.
- Work on Characteristic manifold after (anisotropic) semi-classical scaling, $\tau \sim h^{-2}, -\Delta_G \sim h^{-2}$

$$\{(t, \tau, x, \xi, y, \eta); \tau = \xi^2 + x^2 \eta^2, \tau \in (\frac{1}{2}, 2)\}$$

- Second microlocalize w.r.t. $\eta$ variable and assume with $h^2 \leq \epsilon \leq 1$.

$$u_0 = \chi(-h^2 H)\chi(\epsilon|D_y|)u_0, \chi \in C_0^\infty(\frac{1}{2}, 2), \chi \big|_{(\frac{1}{\sqrt{2}}, \sqrt{2})} = 1$$
The semi-classical regimes \((\xi = hD_x, \eta = hD_y)\)

- **The Half wave regime** \(\xi = 0, x = 0, \eta = +\infty,\)

\[|D_x| \ll h^{-1}, \quad |x| \ll 1, \quad |D_y| \gg h^{-1} \text{ (but } |D_y| \leq h^{-2})\]

\(|D_y| \sim h^{-2}\) responsible for finite time observation. Careful positive commutator estimates

- **The Semi-classical propagation regime** \(0 < \eta < +\infty\)

\((x, hD_x)\) bounded, \(ch^{-1} < |D_y| < Ch^{-1}\)

semi-classical propagation \(\Rightarrow\) arbitrary small time observation

- **The Transversal propagation regime** \(\eta = 0\)
  - **Rapid propagation regime** \(h^{-\delta} \leq |D_y| \ll h^{-1}, 0 < \delta < 1/4:\)
    semi-classical propagation + positive commutator
  - **Normal form regime**: \(|D_y| \leq h^{-\delta}:\) normal form + positive commutator
Semi-classical propagation: Lebeau’s method

Theorem (Lebeau ’92)

**Non degenerate Laplace.** Assume geometric control condition.

\[ \forall T > 0, \exists C_T > 0, \quad \| u_0 \|^2_{L^2(\Omega)} \leq C_T \int_0^T \| e^{it\Delta} (t, \cdot) \|^2_{L^2(\omega)} dt, \]

- **Unique continuation** + semi-classical observation \(0 < h \ll 1\) \((\psi \in C_c^\infty (1/2 \leq |r| \leq 2)).\)

\[ \| \psi (h^2 \Delta) u_0 \|^2_{L^2(\Omega)} \leq C_T \int_0^T \| \psi (h^2 \Delta) u(t, \cdot) \|^2_{L^2(\omega)} dt, \]

- **Rescaling in time** \(v(s, x) = u(hs, x)\): semi-classical Schrödinger equation:

\[ ih \partial_s v_h + h^2 \Delta v_h = 0, \text{ where } v_h(s, x) = \psi (h^2 \Delta) u(hs, x). \]
Semi-classical propagation: Lebeau’s method II

• Propagation of singularities: for semi-classical Schrödinger

\[ i\hbar \partial_s v_h + \hbar^2 \Delta v_h = 0, \]

\[ \text{WF}_h (v_h) \subset \text{Char} (P_h) = \{(s, x; \tau, \xi) \in T^* (\mathbb{R}_s \times \Omega_x) : \tau - |\xi|^2_g = 0 \}. \]

\[ \text{WF}_h (v_h) \text{ invariant under Hamiltonian (geodesic) flow of } p = \tau - |\xi|^2_g. \]

• (GCC) assumption

\[(\star) \quad \exists \alpha > 0, \quad \|v_h\|_{t=0}^2_{L^2(\Omega_x)} \leq C \int_0^\alpha \|v_h(s, x)\|_{L^2(\omega)}^2 ds. \]

• Back to the classical time scale \( t \): From \((\star)\)

\[ \|u_h\|_{t=0}^2_{L^2(\Omega_x)} \leq \frac{C}{h} \int_0^{\alpha h} \|u_h(t, x)\|_{L^2(\omega)}^2 dt. \]

Write for \( t = 0, t = \alpha h, \cdots, t = T \), combine with conservation \( L^2 \)

norm, \( \exists 0 < h_0 = h_0(T, a) \ll 1, \forall 0 < h < h_0, \)

\[ \frac{1}{h} \|u_h(0)\|_{L^2(\Omega)}^2 \leq \frac{C_T}{h} \int_0^T \|u_h(t, x)\|_{L^2(\omega)}^2 dt. \]
Equation of geodesic flow

\( \dot{\eta} = 0, \quad \dot{y} = 2x^2\eta, \quad \dot{x} = 2\xi^2, \quad \dot{\xi} = -2x\eta^2 = -2x\eta^2 \)

- In \( x, \xi \) equation of an ellipse (with reflexions at the boundary)
- \( x \) takes rarely the value 0,
- \( y \) increases (if \( \eta_0 > 0 \)) or decreases (if \( \eta_0 < 0 \)).
- Geodesic intersects the domain \( \omega \) in finite time \( \alpha < T_{GCC} \)
- Apply the semi-classical propagation. Implies the semi-classical observation estimate in time \( \alpha > T_{GCC} \) and consequently the classical observation estimate in arbitrary small time.
The half wave regime $x = 0$, $\xi = 0$, $\eta = +\infty$

- Impossible to apply the semi-classical propagation because $\eta = +\infty$!
- Key point

\[(\ast) \quad [i\partial_t - \Delta_G, x\partial_x + y\partial_y] = -2\Delta_G\]

- Idea: compute

\[\langle [i\partial_t - \Delta_G, x\partial_x + y\partial_y]u, u \rangle_{L^2}\]

integrate by parts and use coercivity to control $H^1_H$ norm of $u$ from $(\ast)$

- Problems: boundary terms in $x$, $t$, and $y\partial_y$ is not $y$ periodic
- Solution introduce cut of in $x$, $t$, $y$ (and deal with the additional terms)
A priori estimates

- Hypoelliptic estimate

\[ \|f\|_{L^2(\Omega)}^2 + \|D_y|^{1/2}f\|_{L^2(\Omega)}^2 \leq \|f\|_{H^1_G}^2 = ((-\Delta_G)f, f)_{L^2(\Omega)}. \]

- Elliptic estimate: Characteristic manifold

\[ \text{Char} = \{ \tau = \xi^2 + x^2\eta^2 \in (\frac{1}{2}, 2) \} \]

We deduce that if \( |D_y| \geq Ch^{-1} \) then \( |x| \leq \frac{\sqrt{2}}{C} \). i.e. for any \( \chi \in C_0^\infty(\mathbb{R}) \) equal to 1 on \((-1, 1)\),

\[ \|(1 - \chi(\frac{Cx}{\sqrt{2}})u\|_{H^1} = O(h^\infty). \]
Positive commutator

Let $\varphi_T \in C_0^\infty(\mathbb{R}_t)$ with support in $(-0, T)$ equal to 1 on $\varepsilon, T - \varepsilon$ and $\zeta \in C_0^\infty(\mathbb{R}_y)$. Compute

$$\ast = \left( [i\partial_t - \Delta_G, x\partial_x + y\partial_y] \varphi_T(t)\chi(x)\zeta(y)v, v \right)_{L^2_{t,x,y}}$$

$$= \left( -2\Delta_G \varphi_T(t)\chi(x)\zeta(y)u, u \right)_{L^2_{t,x,y}}$$

$$= 2 \int_{t,x,y} \varphi_T(t)\chi(x)\zeta(y)(|\partial_x u|^2 + x^2|\partial_y u|^2) \, dx \, dy \, dt + O(\|u\|_{H^1_G} \|u\|_{L^2})$$

$$\ast = \left( (x\partial_x + y\partial_y)[i\partial_t - \partial_x^2 - x^2\partial_y^2, \varphi_T(t)\chi(x)\zeta(y)]u, u \right)_{L^2_{t,x,y}}$$

$$= \int_{t,x,y} \varphi_T'(t)\chi(x)\zeta(y)uy\bar{u}\partial_y \, dx \, dy \, dt$$

$$+ O(\|u\|_{H^1_G} \|u\|_{L^2}) + O(h^\infty)\|u\|^2_{L^2} + O(\|u\|^2_{H^1_G(\omega)})$$

$$2T\|u\|^2_{H^1_G} \leq \text{Observation} + 2\mathcal{L}(\omega)\|u\|^2_{H^1_G} + l.o.t.$$
Transversal propagation regime $\eta = 0$: The pure transversal propagation (from N.B. Zworski 2003)

Method in this regime is inspired from control for Schrödinger on $\mathbb{T}^2$

- Semi-classical propagation does not give result because geodesic are horizontal!
- Step 1: apply semi-classical propagation to escape set $\{x = 0\}$
- Step 2 apply positive commutator $[-\Delta_G, y \partial_y] = -x^2 \partial_y^2 \geq -c \partial_y^2$
  (away from $\{x = 0\}$)
- Since $u$ microlocalized $|D_y| \sim \epsilon^{-1}$, $u_0 = \chi(-h^2 H)\chi(\epsilon |D_y|)u_0$

$$\forall T > 0; \exists C > 0, h_0 > 0, \epsilon_0 > 0, \forall 0 < h \leq h_0, \forall h^2 < \epsilon \leq \epsilon_0,$$

$$\|u_0\|_{L^2(\Omega)}^2 \leq C_T \int_0^T \|u(t, x)\|_{L^2(\omega)}^2 dt$$

Problem: How to glue the second semi-classical estimates? Defect of compactness! (i.e. errors $O(|D_y|^{-\infty})$ are not necessarily compact!)

Solution: stop at $\epsilon \sim h^\delta$
Transversal propagation: The normal form regime

\[ |D_y| \lesssim h^{-\delta} \] (inspired from N.B.-Zworski 12)

\[ i \partial_t u + \partial_x^2 u + x^2 \partial_y^2 u = 0 \]

\[ v = (\text{Id} + hQ(x, hD_x)\partial_y^2)u. \]

\[ (i \partial_t u + \partial_x^2 u + x^2 \partial_y^2 - h[\Delta_G, Q]\partial_y^2)v = 0 \]

\[ h[\Delta_G, Q] = 2(i \xi \partial_x q)(x, hD_x)\partial_y^2 + h(\partial_x^2 q)(x, hD_x)\partial_y^2 + h[x^2, Q]\partial_y^4 \]

\[ = 2(i \xi \partial_x q)(x, hD_x) + O_{L^2}(h^{1-2\delta} + h^{1-4\delta}). \]

Choose

\[ q(x, \xi) = \frac{1}{2i \xi} \int_{-1}^{x} (z^2 - M)dx \iff x^2 - 2i \xi \partial_x q(x, \xi) = M \]

\[ i \partial_t v + \partial_x^2 v + \underbrace{M}_{\text{average of } x^2 \text{ along } x = \text{const.}} \cdot \partial_y^2 v = O_{L^2}(h^{1-2\delta}) \]

Conclude from the observability for Schrödinger on \( \mathbb{T}^2 \).