Strong Sard Conjecture for analytic sub-Riemannian structures in dimension 3

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Sub-Riemannian Geometry and Interactions

September 07, 2020
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In this talk, we consider:

- A smooth connected manifold $M$ of dimension $n \geq 3$;
- A totally nonholonomic regular distribution $\Delta$ of rank $k < n$;
- A Riemannian metric $g$ over $M$.

**Local meaning:** For every point $x \in M$, there is an open neighbourhood $\mathcal{V}$ where $\Delta$ is locally parametrized by $k$ linearly independent vector fields:

$$\{X^1_x, \ldots, X^k_x\}$$

satisfying the Hörmander bracket generating condition

$$\text{Lie}\left\{X^1_x, \ldots, X^k_x\right\}(y) = T_yM \quad \forall y \in \mathcal{V}.$$
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During the seminar: $\mathcal{V} = M$ and $\Delta = \text{Span}\{X^1, \ldots, X^k\}$. 
Horizontal paths

An absolutely continuous curve $\gamma : [0, 1] \to M$ is called horizontal with respect to $\Delta$ if it satisfies

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \quad \text{for a.e. } t \in [0, 1].$$

Trajectories of a control system: Fix $x \in M$ and denote by $x(\cdot; x, u) : [0, 1] \to M$ the solution of:

$$\dot{x}(t) = \sum_{i=1}^{k} u_i(t)X^i(x(t)) \quad \text{for a.e. } t \in [0, 1] \quad \text{and} \quad x(0) = x$$

where $u = (u_1, \cdots, u_k) \in \mathcal{U}^x \subset L^2([0, 1], \mathbb{R}^k)$.

$\gamma$ is horizontal $\iff$ it is a solution of the Cauchy problem.
The **End-Point Mapping** from $x$ is defined as

$$E^x : U^x \longrightarrow M$$

$$u \quad \longmapsto \quad x(1; x, u).$$

A control $u \in U^x \subset L^2([0, 1], \mathbb{R}^k)$ is said to be **singular** if it is a **critical point** of $E^x$. We consider the set of singular controls

$$S^x = \{ u \in U^x ; u \text{ is singular} \} \subset L^2([0, 1], \mathbb{R}^k)$$

Consider the set of **critical values** of $E^x$:

$$\mathcal{K}^x := E^x (S^x) \subset M.$$
The Sard Conjecture

Sard’s Conjecture

For every \( x \in M \), the set \( \mathcal{X}^x \) has Lebesgue measure zero.

- **Difficulty:** In infinite dimension, Sard’s Theorem is known to fail (Bates and Moreira)

- **If \( \dim(M) \geq 4 \):** Partial results for certain Carnot groups (of prescribed ranks and steps):
  - Boarotto and Vittone (2019)
The Sard Conjecture in dimension 3 (\(\text{dim } M = 3\) and \(\text{rank } \Delta = 2\))

The Martinet surface is the set:

\[
\Sigma := \left\{ x \in M \mid \Delta(x) + [\Delta, \Delta](x) \neq T_x M \right\},
\]

Every singular horizontal path must be contained in \(\Sigma\), that is:

\[\mathcal{X}^x \subset \Sigma, \quad \forall x \in M.\]

Sard’s Conjecture in dimension 3

\(\forall x \in M, \text{ the set } \mathcal{X}^x \text{ has 2-dimensional Hausdorff measure zero.}\)

Strong Sard’s Conjecture in dimension 3

\(\forall x \in M, \text{ the sets } \mathcal{X}^x \text{ have Hausdorff dimension at most one.}\)
Main results when \( \text{dim} \, M = 3 \).

**Theorem (Zelenko and Zhitomirskii, 1995)**

Assume that \( \Delta \) is generic (in respect to the Whitney \( C^\infty \)-topology) than all the sets \( \chi^x \) have Hausdorff dimension at most one.

Furthermore, whenever \( \Delta \) is generic, they prove that the Martinet surface \( \Sigma \) is smooth.

**Theorem (Belotto da Silva and Rifford, 2018)**

Suppose that the Martinet surface \( \Sigma \) is smooth. Then for every \( x \in M \) the set \( \chi^x \) has 2-dimensional Hausdorff measure zero.
Main results when \( \dim M = 3 \) and \( M \) and \( \Delta \) are analytic.

**Theorem (Belotto da Silva and Rifford, 2018)**

Under additional hypothesis on the singular set of \( \Sigma \) (which we recall later), for every \( x \in M \) the set \( \mathcal{X}^x \) has 2-dimensional Hausdorff measure zero.

**Theorem (Belotto da Silva, Figalli, Parusinski and Rifford)**

For every \( x \in M \), the set \( \mathcal{X}^x \) is a countable union of semianalytic curves and it has Hausdorff dimension at most 1.

Combining this result with (Hakavuori and Le Donne 2016) we get:

**Corollary**

Every singular minimizing geodesic \( \gamma : [0, 1] \to M \) is of class \( C^1 \).
Characteristic line foliation

Singular horizontal paths are tangent to $T\Sigma$ and to $\Delta$.

**Lemma**

*Singular horizontal paths are concatenation of leaves (of finite length) of the characteristic line foliation $\mathcal{F}$.***
(Zelenko, Zhimtomirskii, 1995) shows that for $\Delta$ generic:

(i) The Martinet surface $\Sigma$ is smooth;

(ii) The singular points of $\mathcal{F}$ are isolated;

(iii) The singular points of $\mathcal{F}$ can be of the following types:

During the seminar: We will assume (i) and (ii).
Characteristic line foliation (local models)

We may consider the following two local models:

**Vector-fields:** We assume that $\Delta = \text{Span}\{\partial_{x_1}; \partial_{x_2} + A(x)\partial_{x_3}\}$.

$$[X^1, X^2] = h(x)\partial_{x_3}$$

and we conclude that:

Martinet : $\Sigma = \{h(x) = 0\}$
Foliation : $Z = X^1(h)X^2 - X^2(h)X^1$.

**One-form:** We assume that $\Delta = \text{Span}\{\mu\} \subset \Omega^1_M$.

$$\mu \wedge d\mu = h \cdot \text{vol}_M,$$  
$$i_{\Sigma} : \Sigma \to M$$

and we conclude that:

Martinet : $\Sigma = \{h(x) = 0\}$
Foliation : $\eta = i^*_{\Sigma}(\mu)$. 
Let $S$ be a surface. We have the equivalence:

$$Z \leftrightarrow \eta \quad \text{if} \quad \eta = i_Z \text{vol}_S.$$ 

and the divergence of $Z$ may be computed from $\eta$:

$$d\eta = \text{div}_S(Z) \text{vol}_S.$$ 

**Example:** if $S = \mathbb{R}^2$ and $\text{vol}_S = dx \wedge dy$:

$$Z = A(x, y) \partial_x + B(x, y) \partial_y$$

$$\eta = A(x, y)dy - B(x, y)dx$$

And we easily verify that:

$$d\eta = (A_x + B_y)dx \wedge dy$$
Controlled divergence (key property)

Since the generator $\mu$ of $\Delta$ is regular, so:

$$d\mu = \alpha \land \mu + h \langle \mu, \ast \mu \rangle^{-1} (\ast \mu)$$

so if $i : S \to \Sigma \subset M$ is a map (gen. max. rank) into $\Sigma$:

$$d\eta = i^*(d\mu) = i^*(\alpha \land \mu) = i^*(\alpha) \land \eta = i^*(\alpha) \land (i_Z \text{vol}_S)$$

This allows us to conclude that $Z$ has controlled divergence:

$$\text{div}_S(Z) \in Z(O_S).$$

Example: Recalling that $Z = A \partial_x + B \partial_y$:

$$\text{div}_S(Z) dx \land dy = (fdx + gdy) \land (Ady - Bdx)$$

$$= (f \cdot A + g \cdot B)dx \land dy.$$
For the talk: We consider polar coordinate changes

\[ \sigma : S^1 \times \mathbb{R}_{\geq 0} \to \mathbb{R}^2 \]

\[ (\theta, r) \mapsto (r \cos(\theta), r \sin(\theta)) \]

and we denote \( E = \sigma^{-1}(0) = S^1 \times \{0\} \) the exceptional divisor.

Formally: we work with blowings-up

\[ \sigma : (S, E) \to (\mathbb{R}^2, 0) \]

where \( E = \sigma^{-1}(0) = \mathbb{P}^1 \).
Elementary singular points.

A singularity \( p \in \mathbb{R}^2 \) of \( \mathcal{Z} = A \partial_x + B \partial_y \) is elementary if:

\[
\text{Jac}(\mathcal{Z}) = \begin{bmatrix} \partial_x A & \partial_y A \\ \partial_x B & \partial_y B \end{bmatrix}
\]

evaluated at \( p \) admits one eigenvalue with non-zero real part.
Example: Degenerate focus point

After performing two blowings-up, all singular points are saddles.
Reduction of singularities of characteristic foliations.

Theorem (Bendixson-Seidenberg)

Let $S$ be an analytic smooth surface and $\mathcal{F}$ be a line foliation on $S$. There exists a proper analytic morphism

$$\pi : (\tilde{S}, \tilde{E}) \to (S, \text{Sing}(\mathcal{Z}))$$

which is (locally) given by a finite composition of blowings-up, such that the strict transform $\tilde{\mathcal{F}}$ of the line foliation only has isolated elementary singularities.

For the characteristic line foliation, moreover:

(i) $\tilde{\mathcal{F}}$ only has saddle points;

(ii) If $\Sigma$ is smooth, $\tilde{\mathcal{F}}$ is non-dicritical, i.e. tangent to $\tilde{E}$. 
Monodromic convergent singular trajectories.

Proposition

If $\mathcal{Z}$ is analytic and $p$ is a focus point, then either all trajectories have finite length, or all trajectories have infinite length.
Characteristic convergent singular trajectories.

Lemma

If there is one characteristic singular trajectory whose limit is the singular point \( p \), then there are only a finite number of singular trajectories with finite length (all of them characteristic), whose limit is the singular point \( p \).
Proof of Proposition: no singular point after blowing-up

We consider the return map \( T : \Lambda \to \Lambda \).

Denote by \( \gamma_{p, T(p)} \) the trajectory between \( p \) and \( T(p) \) and:

\[
L : \Lambda \to \mathbb{R} \\
p \mapsto \text{length}(\gamma_{p, T(p)})
\]

We can show that \( L \) is an analytic function and w.l.o.g. monotone.

\[
\text{length}(\gamma_p) = \sum_{n \in \mathbb{N}} \text{length}(\gamma_{T^n(p), T^{n+1}(p)}) \implies \text{length}(\gamma_q) \leq \text{length}(\gamma_p)
\]
Proof of Proposition: general case

In general, we consider two types of transitions and length maps:

\[ T_1 : \Lambda_1 \rightarrow \Lambda_2, \quad T_2 : \Lambda_2 \rightarrow \Lambda_3 \]
\[ L_1 : \Lambda_1 \rightarrow \mathbb{R}, \quad L_2 : \Lambda_2 \rightarrow \mathbb{R} \]

\( T_2 \) belongs to a Hardy field (Speissegger, 2018);
Combining with reduction of singularities of metrics:

\[ \exists K > 0, \text{ s.t. } \text{length}(\gamma_q, T_2(q)) \leq K \cdot \text{length}(\gamma_p, T_2(p)) \]
General case: singular Martinet surface

There are $2^N$ distinct singular horizontal paths from $z$ to $x$.

Resolution of singularities: If one path has finite length, all paths with the same “jumps” have finite length.

Symplectic geometry: We obtain a contradiction.
Thank you for the attention!