Analysis and control of a conservation law with nonlocal velocity

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February 3, 2012
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Acknowledgements

- Séminaire du Groupe de Travail Contrôle
- Laboratoire Jacques-Louis Lions
- Fondation Sciences Mathématiques de Paris
- 2009 Benasque Summer School
- 2010 Triemestre IHP

Joint work with J.-M. Coron, M. Gröschel, M. Gugat, M. Kawski, A. Keimer, G. Leugering and P. Shang

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Semiconductor wafer manufacture system

Semiconductor wafer manufacture system

The semiconductor manufacturing system which has a highly re-entrant character.

- very high volume (number of parts manufactured per unit time) at each process
- very large number of consecutive production steps

Cycle time: 6 weeks Work steps: 500 WIP: 60,000 wafers

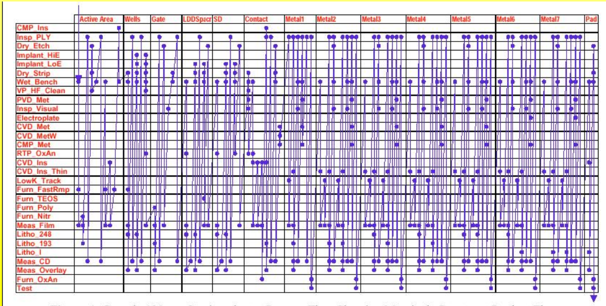

Figure 1. Generic 130-nm Semiconductor Process Flow Showing Massively Reentrant Product Flows

Model

1-D hyperbolic system (conservation law with nonlocal velocity) [Armbruster et al., 2006]

$$\begin{cases} \rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, \quad (t,x) \in (0,T) \times (0,1), \\ \rho(0,x) = \rho_0(x), \quad x \in (0,1), \\ \rho(t,0)\lambda(W(t)) = u(t), \quad t \in (0,T). \end{cases} \tag{1}$$

with $W(t) = \int_0^1 \rho(t,x)dx$.
$\lambda(\cdot) \in C^1([0,+\infty);(0,+\infty))$ and is usually decreasing.
A typical special case: $\lambda(W) = \frac{1}{1+W}$.

- The space variable $x \in [0, 1]$ represents the whole manufacture process. $x = 0$ corresponds to the entry of the factory (manufacture line) while $x = 1$ corresponds to the exit of the factory (manufacture line).
- $\rho(t, x) \geq 0$ is the density of product at time t and stage x. $\rho(t, \cdot)$ $(t \geq 0)$ is called the state in exact controllability problem.
- The control $u(t)$ of this system is acted on the influx $\rho(t, 0)\lambda(W(t))$, the rate of products entering the factory. The natural output is the outflux $y(t) := \rho(t, 1)\lambda(W(t))$, the rate of products exiting the factory.

Some control problems (I)

- Exact (state) Controllability Problem
  Possibility to drive arbitrary initial state ρ_0 to arbitrary final state ρ_1 over some $[0, T]$.
- Time-optimal Transition Problem
  Find time-optimal control to drive given initial state ρ_0 to given final state ρ_1.
- Stabilization Problem
  Possibility to obtain asymptotic stability of the closed-loop system by constructing a feedback control. Natural feedback $u(t) = ky(t)$. $\|\rho(t, \cdot) - \overline{\rho}\| \to 0$ as $t \to \infty$.

Some control problems (II)

- Nodal Profile (Outflux) Controllability Problem
  Possibility to drive the system, which starts from arbitrary initial state ρ_0, to reach exactly some given value y_d for the outflux y over $[T_1, T_2]$.
- Demand Tracking Problem
  Minimize the *Error Signal* $\|y - y_d\|$,
  the difference between the actual outflux $y(t)$ and a given demand forecast $y_d(t)$.
- Backlog Problem
  Minimize the *Accumulated Error Signal* $\|\beta\|$,
  $\beta(t) = \int_0^t y(s)\,ds - \int_0^t y_d(s)\,ds$.

Related work

1. D. Armbruster, D. Marthaler, C. Ringhofer, K. Kempf and T.-C. Jo, *A continuum model for a re-entrant factory*, Oper. Res., 2006.
2. M. La Marca, D. Armbruster, M. Herty, and C. Ringhofer, *Control of continuum models of production systems*, IEEE Trans. Automatic Control, 2010.
3. R. Colombo, M. Herty, and M. Mercier, *Control of the continuity equation with a non local flow*, ESAIM Control Optim. Calc. Var., 2010.

Definition of weak solution in L^1

Let $T > 0$, $\rho_0 \in L^1(0,1)$ and $u \in L^1(0,T)$ be given. A weak solution of the system (1) is a function $\rho \in C^0([0,T]; L^1(0,1))$ such that, for every $\tau \in [0,T]$ and every $\varphi \in C^1([0,\tau]\times[0,1])$ such that

$$\varphi(\tau,x) = 0, \forall x \in [0,1] \quad \text{and} \quad \varphi(t,1) = 0, \forall t \in [0,\tau],$$

one has

$$\int_0^\tau \int_0^1 \rho(t,x)(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt + \int_0^\tau u(t)\varphi(t,0)dt + \int_0^1 \rho_0(x)\varphi(0,x)dx = 0.$$

Well-posedness in L^1

Theorem

If $\rho_0 \in L^1_+(0,1)$ *and* $u \in L^1_+(0,T)$*, then system* (1) *admits a unique weak solution* $\rho \in C^0([0,T]; L^1_+(0,1))$*, which is also nonnegative almost everywhere in* $Q = [0,T] \times [0,1]$.

Proof sketch.

1. Existence and uniqueness of local solution
   Characteristic method + fixed point argument by contraction mapping principle.
   Explicit expression of ρ in terms of $\xi(t) := \int_0^t \lambda(W\tau))d\tau$.
2. Existence of global solution
   Uniform a priori estimate of $\|\rho(t,\cdot)\|_{L^1(0,1)}$.

□

Regularity

Remark (Hidden regularity)

$\rho \in C^0([0,T]; L^1(0,1)) \subset L^1(0,1; L^1(0,T))$
$\rightsquigarrow \rho \in C^0([0,1]; L^1(0,T))$.

Remark (L^p Regularity)

$\rho_0 \in L^p_+(0,1)$ and $u \in L^p_+(0,T)$ $(1 \le p < \infty)$
$\Longrightarrow \rho \in C^0([0,T]; L^p_+(0,1)) \cap C^0([0,1]; L^p_+(0,T))$.

Remark ($W^{1,p}$ Regularity)

$\rho_0 \in W^{1,p}_+(0,1)$ and $u \in W^{1,p}_+(0,T)$ $(1 \le p < \infty)$ with
$\rho_0(0) = u(0)$
$\Longrightarrow \rho \in C^0([0,T]; W^{1,p}_+(0,1)) \cap C^0([0,1]; W^{1,p}_+(0,T))$.

Adjoint system

Suppose that $\rho \in C^0([0,T]; W_+^{1,p}(0,1)) \cap C^0([0,1]; W_+^{1,p}(0,T))$ is solution to (1). The adjoint system is defined as

$$\begin{cases} q_t(t,x) + (q(t,x)\lambda(W(t)))_x \\ \qquad = -\lambda(W(t)) \int_0^1 \rho(t,s) q_x(t,s) ds, \quad (t,x) \in (0,T) \times (0,1), \\ q(T,x) = q_0(x), \quad x \in (0,1), \\ q(t,1) = v(t), \quad t \in (0,T). \end{cases} \tag{2}$$

where $W(t) = \int_0^1 \rho(t,x) dx$.

Well-posedness of adjoint system

Theorem

Let $p \in [1, \infty)$, $q_0 \in W^{1,p}(0,1)$ *and* $v \in W^{1,p}(0,T)$ *with* $q_0(0) = v(0)$, *then* (2) *has a unique solution* $\rho \in C^0([0,T], W^{1,p}(0,1)) \cap C^0([0,1], W^{1,p}(0,T))$.

Proof sketch (I)

Step 1

It suffices to study

$$\begin{cases} q_t(t,x) + \alpha(t) q_x(t,x) \\ \qquad = -\int_0^1 \gamma(t,s) q_x(t,s) ds, \quad (t,x) \in (0,T) \times (0,1), \\ q(0,x) = q_0(x), \quad x \in (0,1), \\ q(t,0) = v(t), \quad t \in (0,T). \end{cases} \tag{3}$$

where $\alpha \in C^0([0,T];(0,\infty))$ and $\gamma \in C^0_+([0,T] \times [0,1])$.

Proof sketch (II)

Step 2

Differentiate (3) with respect to x:

$$\begin{cases} q_{xt}(t,x) + \alpha(t) q_{xx}(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ q_x(0,x) = q_0'(x), & x \in (0,1), \\ q_x(t,0) == -\frac{1}{\alpha(t)} \cdot \left(v'(t) + \int_0^1 \gamma(t,s) q_x(t,s) ds \right), & t \in (0,T). \end{cases} \quad (4)$$

Let $f(t) := \int_0^1 \gamma(t,s) q_x(t,s) ds$.

Proof sketch (III)

Step 3

$$f(t) = g(t) + \int_0^t k(t,\tau) f(\tau) d\tau, \tag{5}$$

where $k(t,\tau) := -\gamma\left(t, \int_\tau^t \alpha(r)dr\right)$ and

$$g(t) := -\int_0^t \gamma\left(t, \int_\tau^t \alpha(r)dr\right) v'(\tau) d\tau + \int_0^{1-\int_0^t \alpha(r)dr} \gamma(t, s + \int_\tau^t \alpha(r)dr) q_0'(s) ds$$

are continuous.

$$(5) \Longrightarrow f \xrightarrow{(4)} q_x \xrightarrow{\int_0^x} q.$$

Definition of exact (state) controllability

Exact (State) Controllability

State $\rho(t,\cdot) \in L^p_+(0,1)$.
Does there exists Control $u(t) \in L^p_+(0,T)$ to drive the hyperbolic system (1): $\rho_0 \in L^p_+(0,1) \rightsquigarrow \rho_1 \in L^p_+(0,1)$ on $[0,T]$, i.e.,

$$\rho(T,x) = \rho_1(x), \quad x \in (0,1).$$

Classical result can not be applied

Li-Rao's constructive method.
Key idea: $u_t + A(u)u_x = 0 \Longleftrightarrow u_x + A^{-1}(u)u_t = 0$ when $A(u)$ has no zero eigenvalues.

- Tatsien Li, Bopeng Rao, *Exact boundary controllability for quasi-linear hyperbolic systems*, SIAM J. Control Optim., 41(6):1748–1755 (electronic), 2003.
- Tatsien Li, *Controllability and observability for quasilinear hyperbolic systems*, volume 3 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2010.

Local controllability around $\overline{\rho}$

Theorem

Let $\overline{\rho} \geq 0$ be the given constant equilibrium and $T_0 := \frac{1}{\lambda(\overline{\rho})}$. Then, for any $T > T_0$, and any $p \geq 1$, the system (1) *is locally controllable.*

Proof sketch.

1. Controllability for linear control system
   $\xi \Longrightarrow u \Longrightarrow \rho$
2. Fixed point argument by contraction mapping principle
   $\xi \Longrightarrow u \Longrightarrow \rho \Longrightarrow F(\xi) = \xi$

□

Global controllability

Theorem

For any $p \in [1, +\infty)$, any $\rho_0 \in L^p_+(0,1)$ and any $\rho_1 \in L^p_+(0,1)$, there exists $T_1 > 0$ (depending on ρ_0 and ρ_1) such that for any $T \geq T_1$, the system (1) is controllable.

Proof sketch.

1. Drive the state from ρ_0 at $t = 0$ to some equilibrium $\overline{\rho}$ at $t = T_f$.
   Input control $u(t)$ can be induced by natural state control $\rho(t, 0) \equiv \overline{\rho}$.
2. Drive the system from $\overline{\rho}$ at $t = T_f$ to ρ_1 at $t = T := T_f + T_b$ by using the reversibility of the hyperbolic system $(t, x, \rho(t, x)) \to (T - t, 1 - x, \rho(T - t, 1 - x))$.

□

Transition between equilibria

Exact controllability

Drive the system from one equilibrium ρ_0 to another ρ_1.
Admissible control set.

Time-optimal control

The global controllability time is not optimal.
What is the optimal control time to drive the system (1) from one equilibrium state ρ_0 to another equilibrium state ρ_1.
What is the control?

An intuitive time-optimal control

Choose the intuitive density control

$$\begin{cases} \rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, & (t,x) \in (0,T) \times (0,1), \\ \rho(0,x) = \rho_0, & x \in (0,1), \\ \rho(t,0) = \rho_1, & t \in (0,T) \end{cases}$$

for the special case: $\lambda(W) = \frac{1}{1+W}$.

$\rho(T,x) \equiv \rho_1$ for $T \geq 1 + \frac{\rho_0+\rho_1}{2}$
$u(t) = \frac{\rho_1}{\sqrt{(1+\rho_0)^2 + 2t(\rho_1 - \rho_0)}}, t \in (0,T)$.

Time-optimal control of transition between equilibria

Theorem

The minimum time to transfer the state from one equilibrium ρ_0 to the other equilibrium ρ_1 using nonnegative influx control $u \in L^1_+(0, \infty)$ is $T = 1 + \frac{\rho_0+\rho_1}{2}$. The time-optimal control is indeed the natural one $u(t) = \frac{\rho_1}{\sqrt{(1+\rho_0)^2+2t(\rho_1-\rho_0)}}, t \in (0, T)$.

Proof.

Direct computations. □

Definition of nodal profile controllability

Where it comes?

Nodal profile controllability was originally introduced by M. Gugat et al. for gas demanding control. This kind of controllability was later named by T. Li and generalized for first order quasilinear hyperbolic systems.

Definition

For any given initial data ρ_0, boundary data y_d and any T_1, T with $0 < T_1 < T$, to find suitable suitable control $u : (0, T) \mapsto [0, +\infty)$ such that the solution ρ to the system (1) satisfies also the nodal profile condition:

$$\rho(t, 1)\lambda(W(t)) = y_d(t), \quad t \in (T_1, T). \tag{6}$$

Nodal profile controllability

Theorem

Let $\overline{\rho} \geq 0$ *be the given constant equilibrium and let* $T_0 := \frac{1}{\lambda(\overline{\rho})}$. *For any* $p \in [1, +\infty)$, *and any* T_1, T *with* $T_0 < T_1 < T$, *there exists* $\nu > 0$ *such that the following holds: For any* $\rho_0 \in L^p_+(0,1)$ *and any* $y_d \in L^p_+(T_1, T)$

$$\|\rho_0(\cdot) - \overline{\rho}\|_{L^p(0,1)} \leq \nu, \ \|y(\cdot) - \overline{\rho}\lambda(\overline{\rho})\|_{L^p(T_1,T)} \leq \nu,$$

there exists $u \in L^p_+(0,T)$ *such that the weak solution* $\rho \in C^0([0,T]; L^p(0,1))$ *to the system* (1) *satisfies the out-flux condition* (6).

Proof sketch.

Nodal profile controllability for linear system
Fixed point argument. □

Output feedback stabilization

Definition of output feedback stabilization

$$\begin{cases} \rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, \quad (t,x) \in (0,\infty) \times (0,1), \\ \rho(0,x) = \rho_0(x), \quad x \in (0,1), \\ u(t) - \overline{\rho}\lambda(\overline{\rho}) = k(y(t) - \overline{\rho}\lambda(\overline{\rho})), \quad t \in (0,\infty). \end{cases} \tag{7}$$

with $W(t) = \int_0^1 \rho(t,x)dx, u(t) = \rho(t,0)\lambda(W(t)), y(t) = \rho(t,1)\lambda(W(t))$.

Can we find k such that $\|\rho - \overline{\rho}\| \to 0$ as $t \to \infty$?

Simple observation

Does $k = 0$ work?

$u(t) = \overline{\rho}\lambda(\overline{\rho})$.

- For $\overline{\rho} = 0$, YES, it works!
  $u(t) = 0 \Longleftrightarrow \rho(t, 0) = 0$
- While for $\overline{\rho} \neq 0$, NOT clear!

Linearization near constant $\overline{\rho} \geq 0$

Linearized system

Let

$$d := \frac{\overline{\rho}\lambda'(\overline{\rho})}{\lambda(\overline{\rho})}. \tag{8}$$

The difference function $\widetilde{\rho} := \rho - \overline{\rho}$. Omitting $\widetilde{}$, then ρ satisfies

$$\begin{cases} \rho_t(t,x) + \rho_x(t,x) = 0, \quad t \in (0,\infty), x \in (0,1), \\ \rho(0,x) = \rho_0(x), \quad x \in (0,1), \\ \rho(t,0) = k\rho(t,1) + (k-1)dW(t), \quad t \in (0,\infty), \end{cases} \tag{9}$$

where $W(t) = \int_0^1 \rho(t,x)dx$ and

$$\lambda(\overline{\rho}) = 1$$

without loss of generality.

IFF condition for stabilization

Theorem

Let $\overline{\rho} \geq 0$. Then, $\overline{\rho} \in L^2(0,1)$ is exponentially stable in $L^2(0,1)$ for the control system (9) if and only if $d > -1$ and $|k| < 1$. That is to say: if and only if $d > -1$ and $|k| < 1$, there exist constants $C = C(d,k) > 0$ and $\alpha = \alpha(d,k) > 0$ such that the following holds: For any $\rho_0 \in L^2(0,1)$, the weak solution $\rho \in C^0([0,\infty); L^2(0,1))$ to the system (9) satisfies

$$\|\rho(t,\cdot)\|_{L^2(0,1)} \leq Ce^{-\alpha t} \cdot \|\rho_0\|_{L^2(0,1)}, \quad \forall t \in [0,\infty). \tag{10}$$

Proof by spectral analysis

Characteristic equation

Let $\mu \in \mathbb{C}$ and $\phi \neq 0$ be an eigen pair s.t.

$$\begin{cases} \mu\phi(x) + \phi'(x) = 0, \quad x \in (0,1), \\ \phi(0) = k\phi(1) + (k-1)d \cdot \int_0^1 \phi(x)dx. \end{cases}$$

which implies

$$\begin{cases} \phi(x) = e^{-\mu x}, \\ 1 - ke^{-\mu} + (1-k)d \cdot \int_0^1 e^{-\mu x}dx = 0. \end{cases} \tag{11}$$

Spectral condition

Proposition

(11) *has no solution* (μ, ϕ) *such that* $\Re(\mu) \geq 0$ *and* $\phi \neq 0$ *if and only if* $d > -1$ *and* $|k| < 1$*.*

Proof sketch.

Degree theory homotopic functions

1. $d = -1$ and $k \in \mathbb{R}$ or $d \neq -1$ and $k = 1$. $\exists \mu = 0$
2. $d \neq -1$ and $|k| > 1$. $\exists \Re(\mu) > 0$
3. $d < -1$ and $|k| < 1$ or $k = -1$. $\exists \mu \in (0, \infty)$
4. $d > -1$ and $k = -1$. $\exists \mu = ib, b \in \mathbb{R}, b \neq 0$
5. $d > -1$ and $|k| < 1$. No $\Re\mu \geq 0$

□

Lyapunov function approach

Case $|d| < 1$ and $|k| < 1$

Define

$$L(t) := \int_0^1 e^{-\beta x} \rho^2(t,x)dx + aW^2(t), \tag{12}$$

By letting $\beta \to 0^+$, $a := \frac{e^{-\beta}-k}{1-k} d > 0$,

$$\dot{L}(t) \leq -\frac{\beta}{C}\left[1 - d^2(e^{\beta}-1)^2 e^{-\beta}\beta^{-2}\right] \cdot L(t) \leq -\alpha L(t),$$

since $1 - d^2(e^{\beta}-1)^2 e^{-\beta}\beta^{-2} \to 1 - d^2 > 0$ as $\beta \to 0^+$.

Lyapunov function approach

Case $d \geq 1$ and $|k| < 1$

Define

$$V_1(t) := \int_0^1 \rho^2(t,x)dx + bW^2(t), \tag{13}$$

Choose $b = d$, then

$$\dot{V}_1(t) = (k^2 - 1)\xi^2(t,1) \leq 0, \quad \forall t \geq 0,$$

where

$$\xi(t,x) := \rho(t,x) + dW(t)$$

satisfies

$$\begin{cases} \xi_t(t,x) + \xi_x(t,x) = d\dot{W}(t), & t \in (0,\infty), x \in (0,1), \\ \xi(0,x) = \rho_0(x) - \overline{\rho} + dW(0), & x \in (0,1), \\ \xi(t,0) = k\xi(t,1), & t \in (0,\infty). \end{cases} \tag{14}$$

Lyapunov function approach

Case $d \geq 1$ and $|k| < 1$

Define

$$V_2(t) := \int_0^1 e^{-x} \xi^2(t,x)dx, \tag{15}$$

then

$$\dot{V}_2(t) \leq -\frac{1}{2} V_2(t) + A\xi^2(t,1).$$

Let

$$V(t) := \frac{A}{1-k^2} V_1(t) + V_2(t). \tag{16}$$

Then

$$\dot{V}(t) = \frac{A}{1-k^2} \dot{V}_1(t) + \dot{V}_2(t) \leq -\frac{1}{2} V_2(t) \leq -\alpha V(t), \quad \forall t \geq 0.$$

Stabilization to $\overline{\rho} = 0$ for nonlinear system

$\overline{\rho} = 0 \Longrightarrow d = 0.$

Theorem

For any $k \in (-1, 1)$ and any $R > 0$, there exist constants $C = C(k, R) > 0$ and $\alpha = \alpha(k, R)$ such that for any $\rho_0 \in L^2((0, 1); [0, \infty))$ with

$$\|\rho_0\|_{L^1(0,1)} \leq R, \tag{17}$$

the solution $\rho \in C^0([0, \infty); L^2(0, 1))$ to the system with feedback

$$u(t) = ky(t)$$

satisfies

$$\|\rho(t, \cdot)\|_{L^2(0,1)} \leq Ce^{-\alpha t} \cdot \|\rho_0\|_{L^2(0,1)}, \quad \forall t \in [0, \infty). \tag{18}$$

Proof by Lyapunov approach

Proof.

Define

$$L(t) := \int_0^1 e^{-\beta x} |\rho(t,x)|^2 dx, \quad \forall t \in [0,\infty),$$

then

$$\begin{aligned} \dot{L}(t) = & -\beta\lambda(W(t))L(t) + (\lambda(W(t)))^{-1}(k^2 - e^{-\beta})y^2(t), \\ \leq & -\alpha L(t), \quad t \in (0,\infty) \end{aligned}$$

by $\beta \to 0^+$ and

$$0 \leq W(t) \leq \|\rho_0\|_{L^1(0,1)} \leq R, \quad \forall t \in [0,\infty).$$

□

Stabilization to $\overline{\rho} > 0$ for nonlinear system

Theorem

Let $d > -1$. For any $k \in (-1, 1)$, there exist constants $\varepsilon = \varepsilon(d, k) > 0$, $C = C(d, k) > 0$ and $\alpha = \alpha(d, k) > 0$ such that the following holds: For any $\rho_0 \in L^2((0, 1); [0, \infty))$ with

$$\|\rho_0(\cdot) - \overline{\rho}\|_{L^2(0,1)} \leq \varepsilon,$$

the weak solution $\rho \in C^0([0, \infty); L^2(0, 1))$ to the system with

$$u(t) - \overline{\rho}\lambda(\overline{\rho}) = k(y(t) - \overline{\rho}\lambda(\overline{\rho})), \quad t \in (0, \infty)$$

satisfies

$$\|\rho(t, \cdot) - \overline{\rho}\|_{L^2(0,1)} \leq Ce^{-\alpha t} \cdot \|\rho_0(\cdot) - \overline{\rho}\|_{L^2(0,1)}, \quad \forall t \in [0, \infty).$$

Proof by Lyapunov approach

Case $d \in (-1, 1)$ and $|k| < 1$.

Define $L(t)$ as (12), then

$$\begin{aligned}\dot{L}(t) &\leq -\frac{\beta}{C_2}\left(1 - d^2(e^{\beta} - 1)^2 e^{-\beta}\beta^{-2} + O(\varepsilon)\right) \cdot L(t)\\ &\leq -\alpha L(t),\end{aligned}$$

letting first $\beta = \beta(d, k) \to 0^+$ and then $\varepsilon = \varepsilon(d, k) \to 0^+$, since $1 - d^2(e^{\beta} - 1)^2 e^{-\beta}\beta^{-2} \to 1 - d^2 > 0$ as $\beta \to 0^+$.

□

Proof by Lyapunov approach

Case $d \geq 1$ and $|k| < 1$.

Define $V_1(t)$, $V_2(t)$, $V(t)$ as (13),(15),(16), then

$$\dot{V}(t) \leq (-\frac{1}{2} + O(\varepsilon))V_2(t) \leq -\alpha V(t), \quad \forall t \geq 0$$

by letting $\varepsilon = \varepsilon(d,k) \to 0^+$.

□

Optimal Control Problem (I)

Optimal Control Problem without constrains

For any fixed *Demand Forecast* $y_d \in L^2_+(0, T)$ and initial data $\rho_0 \in L^2_+(0, 1)$, we look for solution to the optimal control problem:

$$\min_{u \in L^2_+(0,T)} J(u) := \|u\|^2_{L^2(0,T)} + \|y - y_d\|^2_{L^2(0,T)},$$

where $y(t) = \rho(t, 1)\lambda(W(t))$ is the outflux corresponding to the influx $u \in L^2_+(0, T)$ and initial data $\rho_0 \in L^2_+(0, 1)$.

Solution to optimal control problem

Theorem

The infimum of the functional $J(u)$ in $L^2_+(0,T)$ is achieved, i.e., there exists $u_\infty \in L^2_+(0,T)$ such that $J(u_\infty) = \inf_{u \in L^2_+(0,T)} J(u)$.

Proof sketch.

1. Choose a minimizing sequence $\{u_n\}_{n=1}^{\infty}$.
   Take a subsequence $u_n \rightharpoonup u_\infty$ weakly in $L^2_+(0,T)$.
2. Solve system (1) corresponding to influx u_n (resp., u_∞) to obtain solution ρ_n (resp., ρ_∞).
   Prove that $y_n(\cdot) := \lambda(W_n(\cdot))\rho_n(\cdot,1)$
   $\rightharpoonup y_\infty(\cdot) := \lambda(W_\infty(\cdot))\rho_\infty(\cdot,1)$ weakly in $L^2(0,T)$.

□

Optimal Control Problem (II)

Optimal control problem with constrains

$$\min_{u \in U, y \in Y} J(u, y)$$

subject to

$$\begin{cases} \rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, \quad (t,x) \in (0,T) \times (0,1), \\ \rho(0,x) = \rho_0(x), \quad x \in (0,1), \\ \rho(t,0)\lambda(W(t)) = u(t), \quad t \in (0,T), \\ \rho(t,1)\lambda(W(t)) = y(t), \quad t \in (0,T), \end{cases}$$

with $W(t) = \int_0^1 \rho(t,x)dx$.

Basic setting of cost functional J

Basic setting of J

For given $U \subseteq L^2_+(0,T)$ and $Y \subseteq L^2_+(0,T)$, $J : U \times Y \to \mathbb{R}$ is assumed to be Fréchet differentiable in $L^2(0,T)$ and $\frac{\partial J(u,y)}{\partial u}, \frac{\partial J(u,y)}{\partial y} \in H^1(0,T)$.

Example

$U = Y \subseteq H^1_+(0,T)$

$J(u,y) = \frac{\kappa}{2}\|u-u_d\|^2_{L^2(0,T)} + \frac{\nu}{2}\|y-y_d\|^2_{L^2(0,T)}$, with

$u_d, y_d \in H^1(0,T)$ and $\kappa, \nu \in (0,\infty)$.

$\frac{\partial J(u,y)}{\partial u} = \kappa(u-u_d)$, $\frac{\partial J(u,y)}{\partial y} = \nu(y-y_d)$.

Lagrange approach

Definition of Lagrange functional

$$\mathfrak{L} : (u, y, \rho, q) \in U \times Y \times H^1(\Omega_T) \times H^1(\Omega_T) \mapsto \mathbb{R}$$

with

$$\begin{aligned}\mathfrak{L}(u, y, \rho, q) :=& J(u, y) - < q_t + \lambda(W(\cdot))q_x, \rho >_{L^2(\Omega_T)} \\ &+ < q(T, \cdot), \rho(T, \cdot) >_{L^2(0,1)} - < q(0, \cdot), \rho_0 >_{L^2(0,1)} \\ &+ < q(\cdot, 1), y >_{L^2(0,T)} - < q(\cdot, 0), u >_{L^2(0,T)} .\end{aligned}$$

KKT conditions

Theorem

The minimizer $(u, y) \in U \times Y$ of J satisfies problems

$$\begin{cases} \rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, & (t,x) \in (0,T) \times (0,1), \\ \rho(0,x) = \rho_0(x), & x \in (0,1), \\ \rho(t,0)\lambda(W(t)) = u(t), & t \in (0,T), \\ \rho(t,1)\lambda(W(t)) = y(t), & t \in (0,T), \end{cases}$$

$$\begin{cases} q_t(t,x) + (q(t,x)\lambda(W(t)))_x \\ \qquad = -\lambda(W(t)) \int_0^1 \rho(t,s) q_x(t,s) ds, & (t,x) \in (0,T) \times (0,1), \\ q(T,x) = 0, & x \in (0,1), \\ q(t,1) = -\frac{\partial J(u,y)}{\partial y}, & t \in (0,T) \\ q(t,0) = \frac{\partial J(u,y)}{\partial u}, & t \in (0,T). \end{cases}$$

KKT conditions tell what?

How to find optimal control

$$u \overset{(1)}{\Longrightarrow} \rho \Longrightarrow W, y \overset{(2)}{\Longrightarrow} q \Longrightarrow q(t,0) = \frac{\partial J}{\partial u}(u^*, y) \Longrightarrow u^*$$

Example

$U = Y \subseteq H^1_+(0,T)$
$J(u,y) = \frac{\kappa}{2}\|u - u_d\|^2_{L^2(0,T)} + \frac{\nu}{2}\|y - y_d\|^2_{L^2(0,T)}$, with
$u_d, y_d \in H^1(0,T)$ and $\kappa, \nu \in (0,\infty)$.

$$\frac{\partial J}{\partial u}(u^*, y) = \kappa(u^* - u_d) \Longrightarrow u^* = u_d + \frac{1}{\kappa} q(t,0)$$

Our articles related to this work

- J.-M. Coron, M. Kawski, Z. Wang, Analysis of a conservation law modeling a highly re-entrant manufacturing system, Discrete Contin. Dyn. Syst. Ser. B, 2010.
- P. Shang, Z. Wang, Analysis and control of a scalar conservation law modeling a highly re-entrant manufacturing system, J. Differential Equations, 2011.
- J.-M. Coron, Z. Wang, Controllability for a scalar conservation law with nonlocal velocity, J. Differential Equations, 2012.

Work in progress

- J.-M. Coron, Z. Wang, Output feedback stabilization for a scalar conservation law with nonlocal velocity, in prepare.
- M. Gröschel, A. Keimer, G. Leugering, Z. Wang, Regularity theory and adjoint based optimality conditions for a nonlinear transport equation with nonlocal velocity, in prepare.
- M. Gugat, A. Keimer, Z. Wang, Optimal control of a network of re-entrant factories with input and capacity constraints, in prepare.

Future work

- Other application
  Particle synthesis process, Follicle ovulation
- Generalization
  Networks, Coupled systems, Higher dimension
- Numerical experiments
- Stabilization with other types of feedback
- Optimal control problems with constrains

Thank you for your attention!

谢谢!

GBKkai