Stabilization of persistently excited linear systems

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Exposé LJLL

Paris, 28/9/2012

Stabilization & intermittent control

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and a feedback $u = Kx$ stabilizing at 0.

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Let $\alpha : [0, \infty) \to \{0, 1\}$ (or, more generally, $\alpha : [0, \infty) \to [0, 1]$) represent a switching signal which determines whether the feedback $u = Kx$ is active:

$$\dot{x} = Ax + \alpha BKx.$$

The signal α may model

- Unfaithful transmission of the control law ($\alpha(t) \in \{0, 1\}$)
- Approximately periodic or quasi-periodic parameter affecting the control efficiency
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Under which conditions on A, B, K and on α is the non-autonomous system asymptotically stable at 0?

Stabilizable linear control system in $\mathbf{R}^n$

A linear control system

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is stabilizable at the origin if there exists a feedback $u = Kx$ such that $A + BK$ is Hurwitz.

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It is well known that this is true if and only if there exists a system of coordinates in which

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

and A_3 is Hurwitz while (A_1, B_1) is controllable.

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and A_3 is Hurwitz while (A_1, B_1) is controllable.
(Pole shifting theorem:) if (A, B) is controllable, then the system can be stabilized with an arbitrary rate of convergence, i.e., for every $\lambda > 0$ there exist K and $C > 0$ such that

$$\|x(t)\| \leq C\|x(0)\|e^{-\lambda t}$$

for every trajectory x of $\dot{x} = Ax + BKx$.

Persistent excitation

We always assume that for $\alpha \equiv 1$ the system is stabilizable.

Definition ((T,μ)-signal)

Let $0 < \mu \leq T$. A (T,μ)-signal is a function $\alpha \in L^\infty(\mathbf{R},[0,1])$ satisfying

$$\int_t^{t+T} \alpha(s)ds \geq \mu\,, \quad \forall t \in \mathbf{R}\,.$$

$\mathcal{G}(T,\mu)$ = set of all (T,μ)-signals.

Definition ((T,μ)-stabilizer)

Let $0 < \mu \leq T$. The feedback $u = Kx$ is said to be a (T,μ)-stabilizer if there exist $C, \gamma > 0$ such that, for every (T,μ)-signal α, and every $x_0 \in \mathbf{R}^n$, the solution x of $\dot{x} = (A + \alpha BK)x$, $x(0) = x_0$, satisfies

$$\|x(t)\| \leq Ce^{-\gamma t}\|x_0\|\,, \quad \forall t \geq 0\,.$$

A neutrally stable

Lemma

Let (A, B) be stabilizable and A neutrally stable ($Re(\sigma(A)) \leq 0$ and the eigenvalues with real-part equal to zero have trivial Jordan blocks). Then there exists K which is a (T, μ)-stabilizer for every $0 < \mu \leq T$.

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Idea of the proof.

Without loss of generality A skew-symmetric and (A, B) controllable.
Take $K = -B^T$ (independent on (T, μ)).
$V(t) = \|x(t)\|^2$ Lyapunov function.

$$\dot{V} = -2\alpha(t)\|B^T x\|^2 \leq 0\,.$$

Concludes with Lasalle-type argument.

Persistent excitation in the infinite-dimensional case: new phenomena

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Consider the wave equation on a string of finite length L, fixed at both ends and damped on a subset $(a,b) \subsetneq (0,L)$,

$$\begin{aligned} v_{tt}(t,y) &= v_{yy}(t,y) - \alpha(t) 1_{(a,b)}(y) v_t(t,y) \\ v(t,0) &= v(t,L) = 0 \end{aligned}$$

$x(t) = (v(t,\cdot), v_t(t,\cdot)) \in H = H_0^1(0,1) \times L^2(0,1)$
$A(x_1,x_2) = (x_2, \partial_{yy} x_1)$, $B(x_1,x_2) = (0, 1_{(a,b)} x_2)$
A preserves the norm $\|(x_1,x_2)\|_H = \|\partial_y x_1\|_{L^2(0,1)} + \|x_2\|_{L^2(0,1)}$
$A - BB^*$ makes the norm (weakly) decreasing

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Given $T \geq \mu > 0$, it suffices to take a traveling wave with sufficiently small support in order to design α that satisfies the persistent excitation condition and switches off the actuator when the wave arrives.

Persistent excitation in the infinite-dimensional case

[F. Hante, M. Sigalotti, M. Tucsnak, JDE, 2012]

Theorem

- *Exponential stability.* *Let* $\vartheta, c > 0$ *be such that*

$$\int_0^\vartheta \alpha(t)\|B^* e^{tA} z_0\|_H^2 \, dt \geq c\|z_0\|_H^2, \quad \textit{for each } (T,\mu)\textit{-signal } \alpha(\cdot).$$

  Then $-B^*$ *is a* (T,μ)*-stabilizer.*

- *Weak stability.* *Let* $\vartheta > 0$ *be such that*

$$\int_0^\vartheta \alpha(s)\|B^* e^{sA} z_0\|_H^2 \, ds = 0 \quad \Rightarrow \quad z_0 = 0$$

  for every (T,μ)*-signal* $\alpha(\cdot)$*. Then each solution* $t \mapsto z(t)$ *of* $\dot{z} = Az - \alpha BB^* z$ *converges weakly to* 0 *in* H *as* $t \to \infty$ *for any initial data* $z_0 \in H$ *and any* (T,μ)*-signal* $\alpha(\cdot)$*.*

Examples

- Exponential stability: wave equation
  Ω bounded domain of $\mathbf{R}^N$

$$
\begin{aligned}
v_{tt}(t,x) &= \Delta v(t,x) - \alpha(t) d(x)^2 v_t(t,x), && (t,x) \in (0,\infty) \times \Omega, \\
v(0,x) &= y_0(x), v_t(0,x) = y_1(x), && x \in \Omega, \\
v(t,x) &= 0, && (t,x) \in (0,\infty) \times \partial\Omega,
\end{aligned}
$$

with $d \in L^\infty(\Omega)$, $|d(x)| \geq d_0 > 0$.
The generalized observability inequality is satisfied with $\vartheta = T$, $H = H_0^1(\Omega) \times L^2(\Omega)$, $\|(z_1, z_2)\| = \|\nabla z_1\|_{L^2(\Omega)} + \|z_2\|_{L^2(\Omega)}$.

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- Weak stability: Schrödinger equation

$$
\begin{aligned}
y_t(t,x) &= i\Delta y(t,x) - \alpha(t) 1_\omega(x) y(t,x), && (t,x) \in (0,\infty)\times\Omega,\\
y(t,x) &= 0, && t \in (0,\infty)\times\partial\Omega,\\
y(0,x) &= y_0(x), && t \in \Omega,
\end{aligned}
$$

  with $\omega \subset \Omega$ open nonempty.

  Generalized unique continuation principle with $\theta = T$, $H = L^2(\Omega)$.

A stability result in a similar spirit

[Martinez-Vancostenoble, 2002] and [Haraux-Martinez-Vancostenoble, 2005] studied (in particular) the damped wave equation

$$\begin{aligned} v_{tt}(t,x) &= v_{xx}(t,x) - \alpha(t) v_t(t,x) \\ v(t,0) &= v(t,L) = 0. \end{aligned}$$

They proved that if

$$\{t \mid \alpha(t) = 1\} = \cup_{n\in\mathbf{N}} (a_n, b_n)$$

with $b_n \leq a_{n+1}$ and

$$\sum_{n\in\mathbf{N}} (b_n - a_n)^3 = \infty$$

then the solution converges exponentially to zero.

Strong stability in infinite dimension

Theorem

Assume that there exists $\rho > 0$ and a continuous function $c : (0, \infty) \to (0, \infty)$ such that for all T,

$$\int_0^T \alpha(t)\,dt \geq \rho T \Rightarrow \int_0^{\vartheta} \alpha(t)\|B^* e^{tA} z_0\|_H^2\,dt \geq c(T)\|z_0\|_H^2, \quad \forall z_0.$$

Assume moreover that $I_n = (a_n, b_n)$ is a sequence of disjoint intervals in $[0, \infty)$ with $\int_{a_n}^{b_n} \alpha(t)\,dt \geq \rho|b_n - a_n|$ and $\sum_{n=1}^{\infty} \min(c(b_n - a_n), 1) = \infty$. Then each solution of $\dot{z} = Az - \alpha BB^* z$ satisfies $\|z(t)\|_H \to 0$ as $t \to \infty$.

Stability results are then obtained by estimating the asymptotics of $c(T)$ for T small.

EXAMPLE: Wave with damping everywhere: $c(T) \sim T^3$, as proved in [Haraux-Martinez-Vancostenoble, 2005] for the case $\alpha \equiv 1$.

Example: finite-dimensional control systems

Proposition

Let $H = \mathbf{R}^n$. Let A be skew-symmetric and (A, B) controllable. Let r be the minimal non-negative integer such that

$$\operatorname{rank}[B, AB, \ldots, A^r B] = n.$$

Then for every $\rho > 0$ there exists $\kappa > 0$ such that, for every $T \in (0, 1]$ and every $\alpha \in L^\infty([0, T], [0, 1])$, if $\int_0^T \alpha(s) ds \geq \rho T$ then $\int_0^T \alpha(s) \| B^\top e^{sA} z_0 \|^2 ds \geq \kappa T^{2r+1} \| z_0 \|^2$.

Then $c(T) \sim T^{2r+1}$ as proved in [Seidman, 1988] for the case $\alpha \equiv 1$.

Spectra with non-positive real part [A. Chaillet, Y. C., A. Lorìa, M. Sigalotti, MCSS, 2008] & [Y. C., M. Sigalotti, SICON, 2010]

From now on we restrict our attention to systems of the type

$$\dot{x} = Ax + \alpha bu, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}, \quad \alpha \in [0, 1]$$

with (A, b) controllable.

Theorem

Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair and assume that $Re(\sigma(A)) \leq 0$. Then, for every $0 < \mu \leq T$ there exists a (T, μ)-stabilizer.

The uncontrolled system $\dot{x} = Ax$ can have trajectories such that $\|x(t)\| \to \infty$ as $t \to +\infty$.

The proof is based on a compactness argument and a time-contraction procedure, transforming the integral constraint in a pointwise one.

First case: the n-integrator

Proposition

For every $0 < \mu \leq T$ *the system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \vdots \\ \dot{x}_{n-1} = x_{n-2} \\ \dot{x}_n = \alpha u \end{cases}$$

admits a (T, μ)*-stabilizer.*

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First step of the proof:

Lemma (homogeneity and time-rescaling)

Let $\lambda > 0$. Then $(k_1, \ldots, k_n)$ is a (T, μ)-stabilizer if and only if $(\lambda^n k_1, \ldots, \lambda k_n)$ is a $(T/\lambda, \mu/\lambda)$-stabilizer.

$x_\lambda(t) = \mathrm{diag}(1, \lambda, \ldots, \lambda^{n-1}) x(\lambda t)$ solution with $(k_1, \ldots, k_n)$ replaced by $(\lambda^n k_1, \ldots, \lambda k_n)$ and $\alpha(\cdot)$ by $\alpha(\lambda \cdot)$.

(T, μ)-stabilizability of the n-integrator: limit switched systems

We have a one-parameter family of equivalent stabilization problems: we fix K and we assume by contradiction that, for every $\lambda > 0$, K is not a $(T/\lambda, \mu/\lambda)$-stabilizer.

By a compactness argument the proof of the theorem is reduced to the stability of the limit switched system

$$\begin{cases} \dot{x}_1 = x_2 \\ \vdots \\ \dot{x}_{n-1} = x_{n-2} \\ \dot{x}_n = -\beta K^T x, \qquad \beta = \beta(t) \in [\mu/T, 1]. \end{cases}$$

The existence of a common quadratic Lyapunov function follows from a uniform observability result by Dayawansa, Gauthier and Kupka.

Spectra with non-positive real part: the general case

Theorem

Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair and assume that $Re(\sigma(A)) \leq 0$. Then, for every $0 < \mu \leq T$ there exists a (T, μ)-stabilizer.

Assume that $\mathrm{Re}(\sigma(A)) = 0$. Up to a linear change of coordinates

$$\begin{cases} \dot{x}_0 & = & J_{r_0} x_0 + \alpha b^0 u, \\ \dot{x}_j & = & (\omega_j A^{(j)} + J^C_{r_j}) x_j + \alpha b^j u, \quad \text{for } j = 1, \dots, h, \end{cases}$$

where b^0 and b^j have all coordinates equal to zero except the last one that is equal to one and

$$A^{(j)} = \mathrm{Id}_{r_j} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J^C_{r_j} = J_{r_j} \otimes \mathrm{Id}_2$$

Spectra with non-positive real part

Thanks to the dilation, time-rescaling and rotation

$$\begin{aligned} y_0(t) &= D_{r_0,\nu} x_0(\nu t), \\ y_j(t) &= (D_{r_j,\nu} \otimes \mathrm{Id}_2) e^{-\nu t A^{(j)}} x_j(\nu t), \quad \text{for } 1 \le j \le h, \end{aligned}$$

we end up with a limit switched system with a non-scalar switching law

$$\begin{cases} \dot{y}_0 = J_{r_0} y_0 - b^0 \big(C_{00} \mathcal{K}_0 y_0 + \sum_{l=1}^h C_{0l} (\mathcal{K}_l \otimes \mathrm{Id}_2) y_l \big), \\ \dot{y}_j = J_{r_j}^C y_j - (b^j \otimes \mathrm{Id}_2) \Big(C_{0j}^T \mathcal{K}_0 y_0 + \sum_{l=1}^h C_{jl} (\mathcal{K}_l \otimes \mathrm{Id}_2) y_l \Big) \end{cases}$$

with a pointwise (in time) excitation condition of the type

$$(C_{jl})_{j_0 \le j, l \le h} \ge \xi \mathrm{Id}_{2h+1-j_0}$$

with $\xi > 0$.
An extension of the Gauthier-Kupka result allows to conclude.

Arbitrary rate of convergence/divergence

$\lambda^+(\alpha, K) = \sup_{\|x_0\|=1} \limsup_{t\to+\infty} \frac{\log(\|x(t;0,x_0,K,\alpha)\|)}{t}$

$\lambda^-(\alpha, K) = \inf_{\|x_0\|=1} \liminf_{t\to+\infty} \frac{\log(\|x(t;0,x_0,K,\alpha)\|)}{t}.$

Rate of convergence/divergence defined as

$\mathrm{rc}(A, b, T, \mu, K) := -\sup_{\alpha\in\mathcal{G}(T,\mu)} \lambda^+(\alpha, K)$

$\mathrm{rd}(A, b, T, \mu, K) := \inf_{\alpha\in\mathcal{G}(T,\mu)} \lambda^-(\alpha, K).$

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Notice that

- $\mathrm{rc}(A, b, T, \mu, K) \le \min_{\bar\alpha\in[\mu/T,1]} \min\{-\Re(\sigma(A - \bar\alpha b K^T))\}$
- $\mathrm{rc}(A, b, T, \mu, K) = \mathrm{rc}(PAP^{-1}, Pb, T, \mu, (P^{-1})^T K)$

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Define the maximal rate of convergence/divergence as

$$\mathrm{RC}(A, T, \mu) = \sup_{K\in\mathbf{R}^n} \mathrm{rc}(A, b, T, \mu, K),\ \mathrm{RD}(A, T, \mu) = \sup_{K\in\mathbf{R}^n} \mathrm{rd}(A, b, T, \mu, K)$$

- $\mathrm{RC}(A + \lambda \mathrm{Id}_n, T, \mu) = \mathrm{RC}(A, T, \mu) - \lambda$
- $\mathrm{RC}(J_n, T, \rho T) = \mathrm{RC}(J_n, 1, \rho)/T$
- RC and RD monotone with respect to μ

Arbitrary rate of convergence/divergence

Proposition

Let $n = 2$ and (A, b) controllable. Then $\mathrm{RC}(A, T, \mu) = +\infty$ *if and only if* $\mathrm{RD}(A, T, \mu) = +\infty$.

Open question: is this still true for $n > 2$?

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Proposition

There exists $\rho^* \in (0, 1)$ (only depending on n) such that for every controllable pair $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ and every $T > 0$, $\mathrm{RC}(A, T, \rho T) = +\infty$ *for $\rho > \rho^*$ (i.e., the system $\dot{x} = Ax + \alpha bu$ can be (T, μ)-stabilized with an arbitrarily large rate of convergence if $\mu/T > \rho^*$).*

Can the rate of convergence be made arbitrary large by suitably choosing K?

Proposition

There exists $\rho_* \in (0,1)$ such that for every controllable pair $(A,b) \in M_2(\mathbf{R}) \times \mathbf{R}^2$ and every $T > 0$, $\mathrm{RC}(A,T,\rho T) < +\infty$ for $\rho < \rho^*$.

Idea of the proof: take K such that $\min_{\bar{\alpha}\in[\mu/T,1]} \min\{-\Re(\sigma(A - \bar{\alpha} b K^T))\} \gg 1$ and look for destabilizing α

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- In particular, there exist controllable pairs (A,b) that are not (T,μ)-stabilizable for some $T > \mu > 0$.

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- if we add the constraint that the (T,μ)-signal are M-Lipschitz for some fixed $M > 0$ we recover the arbitrary rate of stability [Y. C., G. Mazanti, M. Sigalotti, accepted SICON].

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- Open question: is this still true for $n > 2$?

Bifurcation phenomenon

There is a bifurcation phenomenon at some $\rho = \rho(A, T) \geq 0$.
(Arbitrary rate of convergence if $\mu/T > \rho(A, T)$, bounded rate if $\mu/T < \rho(A, T)$.)
$\rho(A, T) > 0$ if $n = 2$.
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The function $T \mapsto \rho(J_n, T)$ is constant.

Lemma

Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ be a controllable pair. Then
(i) $T \mapsto \rho(A, T)$ is locally Lipschitz;
(ii) there exist $\lim_{T \to +\infty} \rho(A, T) = \sup_{T>0} \rho(A, T)$ and $\lim_{T \to 0} \rho(A, T) = \inf_{T>0} \rho(A, T)$.

Bifurcation phenomenon

There is a bifurcation phenomenon at some $\rho = \rho(A, T) \geq 0$.
(Arbitrary rate of convergence if $\mu/T > \rho(A, T)$, bounded rate if $\mu/T < \rho(A, T)$.)
$\rho(A, T) > 0$ if $n = 2$.
The function $T \mapsto \rho(J_n, T)$ is constant.

Lemma

Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ *be a controllable pair. Then*
(i) $T \mapsto \rho(A, T)$ *is locally Lipschitz;*
(ii) there exist $\lim_{T \to +\infty} \rho(A, T) = \sup_{T>0} \rho(A, T)$ *and* $\lim_{T \to 0} \rho(A, T) = \inf_{T>0} \rho(A, T)$.

OPEN PROBLEMS

- is $T \mapsto \rho(A, T)$ monotone? constant?
- are $A \mapsto \lim_{T \to 0} \rho(A, T)$ or $A \mapsto \lim_{T \to \infty} \rho(A, T)$ constant?