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Transposition method for BSDEs / BSEEs and applications

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1. Backgroud from control theory

• Review on LQ problems

Given $T > 0$, consider the following control system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

with a cost functional

$$J(u(\cdot)) = \frac{1}{2}\int_0^T [\langle\, Cx(t), x(t)\,\rangle + \langle\, Du(t), u(t)\,\rangle]dt \\ + \frac{1}{2}\langle\, Fx(T), x(T)\,\rangle, \quad u(\cdot) \in L^2(0, T; \mathbb{R}^m). \tag{2}$$

Here $C = C^*$, $D = D^*$ and $F = F^*$.

LQ Problem: Minimize (2) over $L^2(0,T;\mathbb{R}^m)$.

Any $\overline{u}(\cdot) \in L^2(0,T;\mathbb{R}^m)$ satisfying

$$J(\overline{u}(\cdot)) = \inf_{u(\cdot)\in L^2(0,T;\mathbb{R}^m)} J(u(\cdot))$$

is called an *optimal control*, the corresponding state trajectory $\overline{x}(\cdot) \stackrel{\Delta}{=} x(\cdot\,;\overline{u}(\cdot))$ is called an *optimal state trajectory*.

How to "identify" $\overline{u}(\cdot)$? For the above LQ, this is easy.

Indeed, for any $\varepsilon \in \mathbb{R}$ and $u(\cdot) \in L^2(0, T; \mathbb{R}^m)$, write

$$u^\varepsilon(\cdot) = \overline{u}(\cdot) + \varepsilon u(\cdot), \qquad x^\varepsilon(\cdot) = x(\cdot\,; x_0, u^\varepsilon(\cdot)).$$

Then,

$$x^\varepsilon(\cdot) = \overline{x}(\cdot) + \varepsilon r(\cdot), \qquad r(\cdot) = \int_0^\cdot e^{A(\cdot - s)} B u(s) ds.$$

By

$$\begin{aligned} & J(x^\varepsilon(\cdot)) \\ & = J(\overline{x}(\cdot)) + \varepsilon \Big\{ \int_0^T [\langle\, C\bar{x}(t), r(t)\,\rangle + \langle\, D\bar{u}(t), u(t)\,\rangle] dt \\ & \qquad + \langle\, F\bar{x}(T), r(T)\,\rangle \Big\} + O(\varepsilon^2) \geq J(\overline{x}(\cdot)), \end{aligned}$$

We get

$$D\overline{u}(t) = -B^*\Big[\int_t^T e^{A^*(s-t)}C\overline{x}(s)ds + e^{A^*(T-t)}F\overline{x}(T)\Big].$$

Write

$$\psi(t) = -\int_t^T e^{A^*(s-t)}C\overline{x}(s)ds - e^{A^*(T-t)}F\overline{x}(T)$$

We obtain

$$D\overline{u}(t) = -B^*\psi(t),$$

where $\psi(\cdot)$ solves the following "backward" ODE:

$$\begin{cases} \dot{\psi}(t) = -A^*\psi(t) + C\overline{x}(t), \\ \psi(T) = -F\overline{x}(T). \end{cases}$$

• Review on deterministic optimal control problems

Now, consider the following controlled ODE under standard assumptions:

$$\begin{cases} \dot{x}(t) = b(t, x(t), u(t)), & \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{3}$$

with a cost functional

$$J(u(\cdot)) = \int_0^T f(t, x(t), u(t))dt + h(x(T)). \tag{4}$$

Here $\mathcal{V}[0, T] = \{u(\cdot) : [0, T] \to U \;\big|\; u(\cdot) \text{ measurable}\}$ and $U \subset \mathbb{R}^m$.

Deterministic optimal control problem can be stated as follows.

Problem (D): Minimize (4) over $\mathcal{V}[0, T]$.

Any $\overline{u}(\cdot) \in \mathcal{V}[0, T]$ satisfying

$$J(\overline{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{V}[0,T]} J(u(\cdot))$$

is called an *optimal control*, the corresponding state trajectory $\overline{x}(\cdot) \stackrel{\Delta}{=} x(\cdot\,; \overline{u}(\cdot))$ and $(\overline{x}(\cdot), \overline{u}(\cdot))$ are called an *optimal state trajectory* and *optimal pair*, respectively.

The following result gives a (first-order) necessary condition for optimal pairs.

Pontryagin's Maximum Principle. *Let* $(\overline{x}(\cdot), \overline{u}(\cdot))$ *be an optimal pair for Problem (D), and let* $p(\cdot) : [0, T] \to \mathbb{R}^n$ *solve the "backward" ODE*

$$\begin{cases} \dot{p}(t) = -b_x(t, \overline{x}(t), \overline{u}(t))^\top p(t) + f_x(t, \overline{x}(t), \overline{u}(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in [0, T], \\ p(T) = -h_x(\overline{x}(T)). \end{cases} \tag{5}$$

Then, for a.e. $t \in [0, T]$,

$$H(t, \overline{x}(t), \overline{u}(t), p(t)) = \max_{u \in U} H(t, \overline{x}(t), u, p(t)),$$

where $H(t, x, u, p) \stackrel{\Delta}{=} \langle\, p, b(t, x, u)\, \rangle - f(t, x, u)$.

Key to the proof: Spike Variation Technique. That is,

Let $(\overline{x}(\cdot), \overline{u}(\cdot))$ be the given optimal pair. Let $\varepsilon > 0$ and $E_\varepsilon \subseteq [0, T]$ be a measurable set whose Lebesgue measure $|E_\varepsilon| = \varepsilon$. For any $u(\cdot) \in \mathcal{V}[0, T]$, define

$$u^\varepsilon(t) = \begin{cases} \overline{u}(t), & \text{if } t \in [0, T] \setminus E_\varepsilon, \\ u(t), & \text{if } t \in E_\varepsilon. \end{cases}$$

Then $u^\varepsilon(\cdot) \in \mathcal{V}[0, T]$. Note that, U does not necessarily have a linear structure. Thus, in general, a perturbation like $\overline{u}(t) + \varepsilon u(t)$ is meaningless. We refer to $u^\varepsilon(\cdot)$ as a *spike* (or *needle*) *variation* of the control $\overline{u}(\cdot)$.

• Review on stochastic optimal control problems in $\mathbb{R}^n$

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability s-pace with $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, on which a 1-dimensional standard Brownian motion $\{W(t)\}_{t\in[0,T]}$ is defined. Denote by $\mathbb{W}$ the natural filtration generated by $\{W(t)\}$ and augmented by all the $\mathbb{P}$-null sets.

The filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ plays a crucial role, and it represents the "information" that one has at each time t. For stochastic differential equation (in the Itô sense), one needs to use adapted processes $X(\cdot)$, i.e., for each give t, the r.v. $X(t)$ is at least $\mathcal{F}_t$-measurable.

We consider the following stochastic controlled equation under standard assumptions:

$$\begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{6}$$

with a cost functional

$$J(u(\cdot)) = E\Big\{ \int_0^T f(t, x(t), u(t))dt + h(x(T)) \Big\}. \tag{7}$$

The control $u(\cdot)$ belongs to the following

$$\mathcal{U}[0, T] \stackrel{\Delta}{=} \{u : [0, T] \times \Omega \to U \bigm| u \text{ is } \{\mathcal{F}_t\}_{t\geq 0}\text{-adapted}\}.$$

The usual optimal control problem for (6) can be stated as follows.

Problem (S): Minimize (7) over $\mathcal{U}[0,T]$.

Any $\overline{u}(\cdot) \in \mathcal{U}[0,T]$ satisfying

$$J(\overline{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} J(u(\cdot)),$$

is called an *optimal control*, the corresponding $\overline{x}(\cdot) \equiv x(\cdot\,;\overline{u}(\cdot))$ and $(\overline{x}(\cdot),\overline{u}(\cdot))$ are called an *optimal state process/trajectory* and *optimal pair*, respectively.

To solve Problem (S), one needs to introduce the following BSDE:

$$\begin{cases} dp(t) = -\Big\{ b_x(t,\overline{x}(t),\overline{u}(t))^\top p(t) + \sigma_x(t,\overline{x}(t),\overline{u}(t))^\top q(t) \\ \qquad\qquad - f_x(t,\overline{x}(t),\overline{u}(t)) \Big\} dt + q(t)dW(t), \quad t \in [0,T], \\ p(T) = -h_x(\overline{x}(T)). \end{cases} \tag{8}$$

Here the unknown is a *pair* of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $(p(\cdot), q(\cdot))$.

Why two unknowns for a single equation? Without the corrected term $q(\cdot)$, it is impossible to find an adapted solution to the following simple equation:

$$\begin{cases} dp(t) = 0, \quad t \in [0,T], \\ p(T) = p_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}). \end{cases}$$

Since the control variable $u(\cdot)$ appears in the diffusion term, S. Peng (1990) introduced an additional adjoint equation as follows:

$$\begin{cases} dP(t) = -\Big[b_x(t,\overline{x}(t),\overline{u}(t))^\top P(t) + P(t)b_x(t,\overline{x}(t),\overline{u}(t)) \\ \qquad\qquad +\sigma_x(t,\overline{x}(t),\overline{u}(t))^\top P(t)\sigma_x(t,\overline{x}(t),\overline{u}(t)) \\ \qquad\qquad +\sigma_x(t,\overline{x}(t),\overline{u}(t))^\top Q(t) + Q(t)\sigma_x(t,\overline{x}(t),\overline{u}(t)) \\ \qquad\qquad\quad +H_{xx}(t,\overline{x}(t),\overline{u}(t),p(t),q(t))\Big]dt + Q(t)dW(t), \\ P(T) = -h_{xx}(\overline{x}(T)), \end{cases} \tag{9}$$

where

$$H(t,x,u,p,q) = \langle\, p, b(t,x,u)\,\rangle + \langle\, q, \sigma(t,x,u)\,\rangle - f(t,x,u).$$

Write $\mathcal{S}^n = \{A \in \mathbb{R}^{n\times n} \mid A^\top = A\}$. Equation (9) is an $\mathcal{S}^n$-valued BSDE.

Define an $\mathcal{H}$*-function*:

$$\begin{aligned}&\mathcal{H}(t,x,u)\\&\equiv \frac{1}{2}\mathrm{tr}\left[\sigma(t,x,u)^\top P(t)\sigma(t,x,u)\right]+\langle\, p(t),b(t,x,u)\,\rangle - f(t,x,u)\\&+\mathrm{tr}\left[q(t)^\top\sigma(t,x,u)\right]-\mathrm{tr}\left[\sigma(t,x,u)^\top P(t)\sigma(t,\overline{x}(t),\overline{u}(t))\right].\end{aligned}$$

Peng's Stochastic Maximum Principle. *Assume* $\mathbb{F}=\mathbb{W}$*. Let* $(\overline{x}(\cdot),\overline{u}(\cdot))$ *be an optimal pair of Problem (S). Then there are pairs of processes* $(p(\cdot),q(\cdot))$ *and* $(P(\cdot),Q(\cdot))$ *satisfying the first-order and second-order adjoint equations (8) and (9), respectively, such that*

$$\mathcal{H}(t,\overline{x}(t),\overline{u}(t))=\max_{u\in U}\mathcal{H}(t,\overline{x}(t),u),\quad \text{a.e. } t\in[0,T],\quad \mathbf{P}\text{-a.s.}$$

• Stochastic optimal control problems in infinite dimensions

Consider the following controlled forward stochastic evolution equation

$$\begin{cases} dx(t) = \big[Ax(t) + a(t, x(t), u(t))\big]dt \\ \qquad\qquad +b(t, x(t), u(t))dW(t), \quad t \in (0, T], \\ x(0) = x_0, \end{cases} \tag{10}$$

where A is an unbounded linear operator (on a Hilbert space H), generating a C_0-semigroup. Let U be a metric space. Put

$$\mathcal{U}[0, T] \triangleq \Big\{u(\cdot) : [0, T] \to U \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-adapted}\Big\}.$$

Define a cost functional $\mathcal{J}(\cdot)$ as follows:

$$\mathcal{J}(u(\cdot)) \triangleq \mathbb{E}\Big[\int_0^T g(t, x(t), u(t))dt + h(x(T))\Big].$$

We consider the following optimal control problem:

Problem (P): Find $\overline{u}(\cdot) \in \mathcal{U}[0, T]$ such that

$$\mathcal{J}(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} \mathcal{J}(u(\cdot)). \tag{11}$$

Any $\overline{u}(\cdot) \in \mathcal{U}[0, T]$ satisfying (11) is called an *optimal control*, the corresponding $\overline{x}(\cdot) \equiv x(\cdot\,; \overline{u}(\cdot))$ and $(\overline{x}(\cdot), \overline{u}(\cdot))$ are called an *optimal state process/trajectory* and *optimal pair*, respectively.

Our goal is to give a Pontryagin-type maximum principle for the above stochastic optimal control problem.

• The case when $\dim H < \infty$ is now well-understood, see S. Peng (1990).

• The case when the control does NOT appear in the diffusion term or the control set is convex: A. Bensoussan (1983), Y. Hu and S. Peng (1992), V.V. Anh, W. Grecksch and J. Yong (2010), A. Al-Hussein (2010, 2011), etc.

• The case when the control appears in the diffusion term and the control set is nonconvex: Only two previous references (i.e., X.Y. Zhou (1993) addressing the linear problem, and S. Tang and X. Li (1994) for the problem with very special data).

Main difficulty: How to define the solution to operator-valued BSEE?

• When $H = \mathbb{R}^n$, an $\mathbb{R}^{n\times n}$ (matrix)-BSDE can be regarded as an $\mathbb{R}^{n^2}$(vector)-valued BSDE.

• When $\dim H = \infty$, $\mathcal{L}(H)$ (with the uniform operator topology) is still a Banach space. Nevertheless, it is neither reflexive nor separable even if H itself is separable.

• There exist no satisfactory stochastic integration/evolution equation theories in general Banach spaces, say how to define the Itô integral $\int_0^T Q(s)dW(s)$ (for an operator-valued process $Q(\cdot)$)? The existing result on stochastic integration/evolution equation in UMD Banach spaces does not fit the present case because, if a Banach space is UMD, then it is reflexive.

In this talk, we shall present a new method to solve BSDEs/BSEEs.

The main idea comes from the transposition method for deterministic non-homogeneous boundary value problems by J.-L. Lions and E. Magenes (1972), and especially the boundary controllability problem for hyperbolic equations (e.g., J.-L. Lions (1988)).

2. The classical transposition method in PDEs

We now recall the main idea in the classical transposition method to solve the following wave equation with non-homogeneous Dirichelt boundary conditions:

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } Q \equiv (0,T) \times G, \\ y = u & \text{on } \Sigma \equiv (0,T) \times \Gamma, \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } G, \end{cases} \tag{12}$$

where $T > 0$, G is a nonempty open bounded domain in $\mathbb{R}^d$ ($d \in \mathbb{N}$) with C^2 boundary Γ, $(y_0, y_1) \in L^2(G) \times H^{-1}(G)$ and $u \in L^2((0,T) \times \Gamma)$.

When $u \equiv 0$, one can use Semigroup Theory to show the well-posedness of (12) in the solution space $(y \in)$ $C([0,T];L^2(G)) \bigcap C^1([0,T];H^{-1}(G))$.

When $u \not\equiv 0$, one needs to use the transposition method because $y|_\Sigma = u$ does NOT make sense by the usual trace theorem. For this purpose, for any $f \in L^1(0,T;L^2(G))$ and $g \in L^1(0,T;H_0^1(G))$, consider the following adjoint problem of (12):

$$\begin{cases} \zeta_{tt} - \Delta\zeta = f + g_t, & \text{in } Q, \\ \zeta = 0, & \text{on } \Sigma, \\ \zeta(T) = \zeta_t(T) = 0, & \text{in } G. \end{cases} \tag{13}$$

Equation (13) admits a unique solution $\zeta \in C([0,T];H_0^1(G)) \bigcap C^1([0,T];L^2(G))$, which enjoys the regularity $\frac{\partial\zeta}{\partial\nu} \in L^2(\Sigma)$.

In order to give a reasonable definition for the solution to the non-homogenous boundary problem (12) by the transposition method, we consider first the case when y is sufficiently smooth. The following result holds:

Assume $g \in C_0^\infty(0,T;H_0^1(G))$ and that $y \in H^2(Q)$ satisfies (12). Then, multiplying the first equation in (12) by ζ, integrating it in Q, and using integration by parts, we find that

$$\begin{aligned} &\int_Q fy\,dxdt - \int_Q gy_t\,dxdt \\ &= \int_G \zeta(0)y_1\,dx - \int_G \zeta_t(0)y_0\,dx - \int_\Sigma \frac{\partial\zeta}{\partial\nu}u\,d\Sigma. \end{aligned} \tag{14}$$

Note that (14) still makes sense even if the regularity of y is relaxed as $y \in C([0,T];L^2(G)) \bigcap C^1([0,T];H^{-1}(G))$.

Definition 1. We call $y \in C([0,T];L^2(G)) \bigcap C^1([0,T];H^{-1}(G))$ a transposition solution to (12), if $y(0) = y_0$, $y_t(0) = y_1$, and for any $f \in L^1(0,T;L^2(G))$ and $g \in L^1(0,T;H_0^1(G))$, it holds that

$$
\begin{aligned}
&\int_Q fydxdt - \int_0^T \langle\, g, y_t \,\rangle_{H_0^1(G),H^{-1}(G)} dt \\
&= \langle\, \zeta(0), y_1 \,\rangle_{H_0^1(G),H^{-1}(G)} + \int_\Omega \zeta_t(0) y_0 dx - \int_\Sigma \frac{\partial \zeta}{\partial \nu} u d\Sigma,
\end{aligned}
$$

where ζ is the unique solution to (13).

One can show the well-posedness of (12) in the sense of transposition. The main idea of this method is to interpret the solution to a forward wave equation with non-homogeneous Dirichlet boundary conditions in terms of another backward wave equation with non-homogeneous source terms.

We shall use this idea to interpret BSDEs/BSEEs in terms of SDEs/SEEs. This enables us

- To provide a new method for solving BSDEs/BSEEs with general filtration;
- To give a new numerical schemes for solving BSDEs (even with the natural filtration);
- To establish a general Pontryagin-type stochastic maximum principle in general filtration spaces.

The transposition method is a variant of duality method.

3. Motivation and definition for transposition solution of BSDEs

Consider the following semilinear BSDE:

$$\begin{cases} dy(t) = f(t, y(t), Y(t))dt + Y(t)dw(t) \text{ in } [0, T], \\ y(T) = y_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n), \end{cases} \tag{15}$$

where $f(t, \cdot, \cdot)$ satisfies the global Lipschitz condition (uniformly w.r.t. t) and $f(\cdot, 0, 0) \in L^2_{\mathbb{F}}(\Omega; L^1(0, T; \mathbb{R}^n))$.

Recall that, $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T; \mathbb{R}^n))$ is called a (strong) solution to the equation (15) if for any $t \in [0,T]$,

$$y(t) = y_T - \int_t^T f(s, y(s), Y(s))ds - \int_t^T Y(s)dw(s).$$

The first step to establish the well-posedness of the semilinear BSDE (15) is to study the same problem but for the following linear BSDE with a non-homonomous term $f(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T; \mathbb{R}^n))$:

$$\begin{cases} dy(t) = f(t)dt + Y(t)dW(t), \ t \in [0,T), \\ y(T) = y_T. \end{cases} \tag{16}$$

The case $\mathbb{F} = \mathbb{W}$ is now well-understood (J.-M. Bismut (1978) and E. Pardoux & S. Peng (1990)). In this case, the main idea to solve (16) is as follows

Define a square integrable $\{\mathcal{F}_t\}$-martingale

$$M(t) = \mathbb{E}\Big(y_T - \int_0^T f(s)ds \,\Big|\, \mathcal{F}_t\Big). \tag{17}$$

By MRT, $\exists\, Y(\cdot) \in L^2_{\mathbb{W}}(\Omega; L^2(0,T;\mathbb{R}^n))$ so that

$$M(t) = M(0) + \int_0^t Y(s)dW(s). \tag{18}$$

Putting

$$y(t) = M(t) + \int_0^t f(s)ds, \tag{19}$$

one then find the strong solution $(y(\cdot), Y(\cdot))$ for (16).

MRT plays a crucial role in the above. In the general case when the filtration $\mathbb{F}$ is not equal to the natural one, $\mathbb{W}$ might be a proper sub-class of $\mathbb{F}$, and therefore, the MRT fails.

Only a very few works addressing the well-posedness for BSDEs with the general filtration, say N. El Karoui and S.-J. Huang (1997), and G. Liang, T. Lyons and Z. Qian (2011).

The main idea of N. El Karoui and S.-J. Huang (1997) for solving (16) is as follows. Since the filtration $\mathbb{F}$ is not equal to the natural one, the following

$$\begin{aligned}&\mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)\\&\equiv\left\{\int_0^{\cdot} g(s)dW(s)\;\middle|\; g(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))\right\}\end{aligned}$$

is a proper subspace of $\mathcal{M}^2_{0,\mathbb{F}}([0,T];\mathbb{R}^n)$.

Then one has the following (unique) orthogonal decomposition:

$$M(\cdot) - M(0) = P(\cdot) + Q(\cdot), \qquad (20)$$

for some $P(\cdot) \in \mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)$ and $Q(\cdot) \in \big(\mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)\big)^{\perp}$. Hence, there is a $Y(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ such that

$$P(t) = \int_0^t Y(s)dW(s). \qquad (21)$$

Still, we define $y(\cdot)$ as in (19). It is easy to check that $(y(\cdot), Q(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \times \big(\mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)\big)^{\perp} \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ is the unique solution of the following equation

$$\begin{aligned} y(t) = {} & y_T + Q(t) - Q(T) \\ & - \int_t^T f(s)ds - \int_t^T Y(s)dw(s), \\ & \forall\, t \in [0,T]. \end{aligned} \tag{22}$$

This means that (22) is another reasonable “modification” of the linear BSDE (16) (by adding another corrected term $Q(\cdot)$).

Note that the appearance of the extra term $Q(\cdot)$ makes the rigorous analysis on the properties of $y(\cdot)$ and $Y(\cdot)$ much more complicated than the case of natural filtration. Indeed,

1) One needs to use some deep results in martingale theory to establish the duality relationship between this sort of modified BSDEs and the usual (forward) stochastic differential equations although it is not difficult to give the desired relationship formally;

2) Meanwhile, one knows very little about the space $\mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)$ (which is actually introduced to replace the use of MRT), and therefore, it seems very difficult to "compute" the above $Y(\cdot)$ in (21).

Recently, by replacing $Y(t)dW(t)$ (in (16)) by $dM(t)$ (with $M(\cdot)$ being a square-integrable martingale), G. Liang, T. Lyons and Z. Qian (2011) developed another approach for the well-posedness of BSDEs with the general filtration. Their main idea to solve the equation (16) (with general filtration) is as follows:

Although formula (18) does not make sense any more, $M(\cdot) \in \mathcal{M}^2_{\mathbb{F}}([0,T];\mathbb{R}^n)$ and $y(\cdot) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n))$ are still well-defined respectively by (17) and (19), and verifies $M(0) = y(0)$, $\mathbb{P}$-a.s. Then, it is easy to check that the above $(y(\cdot), M(\cdot))$ is the unique solution of the following equation

$$y(t) = y_T - \int_t^T f(s)ds + M(t) - M(T), \ \forall\, t \in [0,T] \tag{23}$$

in the solution space

$$\Upsilon = \Big\{ (h(\cdot), N(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \\ \times \mathcal{M}^2_{\mathbb{F}}([0,T];\mathbb{R}^n) \;\Big|\; N(0) = h(0) \Big\}.$$

This means that (23) is a reasonable “modification” of the linear BSDE (16).

The advantage of this approach is that MRT is not required. But the cost is that the adjusting term $Y(\cdot)$ in (16) (or more generally, in (15)) is suppressed. Note that this term plays a crucial role in some problems, say the Pontryagin-type maximum principle for stochastic optimal control problems. Also, comparison theorem is unclear in this setting because the usual duality analysis is not available.

We now present a different approach to treat the well-posedness of BSDEs with general filtration. The idea is as follows. Fixing $t \in [0,T]$, we start from the following linear (forward) stochastic differential equation

$$\begin{cases} dz(\tau) = u(\tau)d\tau + v(\tau)dw(\tau), \ \tau \in (t,T], \\ z(t) = \eta. \end{cases} \tag{24}$$

It is clear that, for given $u(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(t,T;\mathbb{R}^n))$, $v(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n))$ and $\eta \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, the equation (24) admits a unique strong solution $z(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$.

Now, if the equation (15) admits a strong solution $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T; L^2(\Omega;\mathbb{R}^n))$ (say, when $\mathbb{F} = \mathbb{W}$), then, applying Itô's formula to $\langle\, z(t), y(t)\,\rangle$, it follows

$$\begin{aligned}
&\mathbb{E}\,\langle\, z(T), y_T\,\rangle - \mathbb{E}\,\langle\, \eta, y(t)\,\rangle \\
&= \mathbb{E}\int_t^T \langle\, z(\tau), f(\tau, y(\tau), Y(\tau))\,\rangle\, d\tau \\
&\quad + \mathbb{E}\int_t^T \langle\, u(\tau), y(\tau)\,\rangle\, d\tau \\
&\quad + \mathbb{E}\int_t^T \langle\, v(\tau), Y(\tau)\,\rangle\, d\tau.
\end{aligned} \tag{25}$$

This inspires us to introduce the following new notion for the solution to (15).

Definition 2. We call $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ a transposition solution to (15) if for any $t \in [0,T]$, $u(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(t,T;\mathbb{R}^n))$, $v(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n))$ and $\eta \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, the identity (25) holds.

Clearly, any transposition solution of the equation (15) coincides with its strong solution whenever the filtration $\mathbb{F}$ is natural.

The main advantage of our approach consists in the fact that the duality analysis is contained in the definition of solutions, and therefore, one can easily deduce a similar comparison theorem for transposition solutions of (15) by using almost the same approach as in N. El Karoui, S. Peng and M. C. Quenez (1997).

Also, it is even easier to establish a Pontryagin-type maximum principle for stochastic optimal control problems in general filtration spaces than to solve the same problem with the natural filtration because, again, the desired duality analysis is contained in our definition of solutions.

On the other hand, our definition can be used as a basis for numerical solutions to BSDEs (even for natural filtration). To see this, we recall the main idea of the classical finite element method to find the solution $y \in H_0^1(G)(\cap H^2(G))$ to the following elliptic equation:

$$\begin{cases} -\Delta y = f, & \text{in } G, \\ y = 0 & \text{on } \partial G, \end{cases} \tag{26}$$

where G is a bounded smooth domain in $\mathbb{R}^n$, $f \in L^2(G)$. A weak (or variational) formulation of (26) reads

$$\int_G \nabla y \cdot \nabla \phi dx = \int_G f \phi dx, \quad \forall\, \phi \in H_0^1(G). \tag{27}$$

The key variational formulation (27) reminds one to choose a sequence of finite dimensional spaces $\{H_m\} \subset H_0^1(G)$ (tending to $H_0^1(G)$ in some sense), called finite element subspaces, and to find approximate solutions $\{y_m\}$ of the equation (26) so that $y_m \in H_m$ and

$$\int_G \nabla y_m \cdot \nabla \phi dx = \int_G f \phi dx, \quad \forall\, \phi \in H_m. \tag{28}$$

In "P. Wang and X. Zhang, CRAS, 2011", we have used a similar idea to give a numerical scheme solving BSDEs.

4. Well-posedness of BSDEs in the transposition sense

Consider first the linear case.

Theorem 1. (Q. Lü and X. Zhang, JDE, 2013) For any $f(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n))$ and $y_T \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, the equation (16) admits a unique transposition solution $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$. Furthermore,

$$
\begin{aligned}
&|(y(\cdot),Y(\cdot))|_{L^2_{\mathbb{F}}(\Omega;D([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))}\\
&\le C\left[|f(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))}+|y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\right], \forall\, t\in[0,T].
\end{aligned}
\tag{29}
$$

Sketch of the proof: *Step 1*. Define a linear functional ℓ on $\mathcal{H}_t \equiv L^2_{\mathbb{F}}(\Omega; L^1(t,T;\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n)) \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ as follows:

$$
\begin{aligned}
&\ell\big(u(\cdot),v(\cdot),\eta\big) \\
&= \mathbb{E}\,\langle\, z(T), y_T \,\rangle - \mathbb{E}\int_t^T \langle\, z(\tau), f(\tau)\,\rangle\, dt, \\
&\quad \forall\ \big(u(\cdot),v(\cdot),\eta\big) \in \mathcal{H}_t,
\end{aligned}
$$

where $z(\cdot)$ solves (24). Then,

$$
\begin{aligned}
&\big|\ell\big(u(\cdot),v(\cdot),\eta\big)\big| \\
&\le C\Big[|f(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))} + |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\Big] \qquad (30)\\
&\quad \times \big|\big(u(\cdot),v(\cdot),\eta\big)\big|_{\mathcal{H}_t}, \qquad \forall\, t\in[0,T].
\end{aligned}
$$

From (30), we know ℓ is a bounded linear functional on $\mathcal{H}_t$. By a Riesz-type representation theorem in Q. Lü, J. Yong and X. Zhang (JEMS, 2012), there exist $y^t(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^\infty(t,T;\mathbb{R}^n))$, $Y^t(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n))$ and $\varsigma^t \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ such that

$$\begin{aligned}
&\mathbb{E}\langle z(T), y_T\rangle - \mathbb{E}\int_t^T \langle z(\tau), f(\tau)\rangle\, d\tau \\
&= \mathbb{E}\int_t^T \langle u(\tau), y^t(\tau)\rangle\, d\tau + \mathbb{E}\int_t^T \langle v(\tau), Y^t(\tau)\rangle\, d\tau \\
&\quad + \mathbb{E}\langle \eta, \varsigma^t\rangle .
\end{aligned} \tag{31}$$

It is clear that $\varsigma^T = y_T$.

Step 2 (The time consistency). Note that the "solution" $(y^t(\cdot), Y^t(\cdot))$ may depend on t. It is easy to show that, for any $0 \leq t_2 \leq t_1 \leq T$, for $(\tau, \omega) \in [t_1, T] \times \Omega$ a.e.

$$\big(y^{t_2}(\tau,\omega), Y^{t_2}(\tau,\omega)\big) = \big(y^{t_1}(\tau,\omega), Y^{t_1}(\tau,\omega)\big). \quad (32)$$

Put $y(t,\omega) = y^0(t,\omega)$ and $Y(t,\omega) = Y^0(t,\omega)$. Then,

$$\begin{aligned} &\mathbb{E}\,\langle\, z(T), y_T \,\rangle - \mathbb{E}\,\langle\, \eta, \varsigma^t \,\rangle \\ &= \mathbb{E}\int_t^T \langle\, z(\tau), f(\tau)\,\rangle\, d\tau + \mathbb{E}\int_t^T \langle\, u(\tau), y(\tau)\,\rangle\, d\tau \\ &\qquad + \mathbb{E}\int_t^T \langle\, v(\tau), Y(\tau)\,\rangle\, d\tau. \end{aligned} \quad (33)$$

Step 3. We show in this step that ς^t has a càdlàg modification. For this, clearly, it suffices to show that

$$X(t) \equiv \varsigma^t - \int_0^t f(s)ds, \qquad t \in [0,T] \tag{34}$$

is an $\{\mathcal{F}_t\}$-martingale.

The key observation to show the above martingale property is the following: For each $t \in [0,T]$,

$$\mathbb{E}\left(y_T - \int_t^T f(s)ds \,\middle|\, \mathcal{F}_t\right) = \varsigma^t, \text{ a.s.} \tag{35}$$

Step 4. In this step, we show that, for a.e. $t \in [0, T]$,

$$\varsigma^t = y(t) \ \text{ a.s.} \tag{36}$$

Indeed, for any fixed any $\gamma \in L^2_{\mathcal{F}_{t_2}}(\Omega; \mathbb{R}^n)$,

$$\begin{aligned}\mathbb{E}\left\langle \gamma, \varsigma^{t_2} \right\rangle = \frac{1}{t_1 - t_2}\mathbb{E}\int_{t_2}^{t_1} \left\langle (\tau - t_2)\gamma, f(\tau) \right\rangle d\tau \\ - \int_{t_2}^{t_1} \left\langle \gamma, f(\tau) \right\rangle d\tau \\ + \frac{1}{t_1 - t_2}\int_{t_2}^{t_1} \mathbb{E}\left\langle \gamma, y(\tau) \right\rangle d\tau.\end{aligned}$$

Hence, for $t_2 \in [0, T]$ a.e. ,

$$\mathbb{E}\left\langle \gamma, y(t_2) \right\rangle = \mathbb{E}\left\langle \gamma, \varsigma^{t_2} \right\rangle, \ \ \forall\, \gamma \in L^2_{\mathcal{F}_{t_2}}(\Omega; \mathbb{R}^n).$$

Next, we consider the case of semilinear BSDEs.

By Theorem 1 and Banach fixed point theorem, we deduce that

Theorem 2. (Q. Lü and X. Zhang, JDE, 2013) For any given $y_T \in L^2_{\mathcal{F}_T}(\Omega)$, the equation (15) admits a unique transposition solution $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T; \mathbb{R}^n))$. Furthermore,

$$
\begin{aligned}
&|(y(\cdot), Y(\cdot))|_{L^2_{\mathbb{F}}(\Omega;D([0,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))}\\
&\leq C\left[|f(\cdot,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))} + |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\right].
\end{aligned}
\tag{37}
$$

5. Transposition solution to vector-valued BSEEs

Consider the following BSEE valued in H:

$$\begin{cases} dy = -A^* y dt + f(t, y, Y)dt + Y dw \quad \text{in } [0, T), \\ y(T) = y_T \in L^p_{\mathcal{F}_T}(\Omega; H). \end{cases} \tag{38}$$

Here, $p \in (1, 2]$. In order to give the definition of transposition solution to (38), we introduce the following forward stochastic differential equation:

$$\begin{cases} dz = (Az + v_1)dt + v_2 dw \quad \text{in } (t, T], \\ z(t) = \eta. \end{cases} \tag{39}$$

Here $v_1(\cdot) \in L^1_{\mathbb{F}}(t, T; L^q(\Omega; H))$, $v_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^q(\Omega; H))$, $\eta \in L^q_{\mathcal{F}_t}(\Omega; H)$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 3. We call $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0,T]; L^p(\Omega; H)) \times L^2_{\mathbb{F}}(0,T; L^p(\Omega; H))$ a transposition solution to (38) if for any $t \in [0,T]$, $v_1(\cdot) \in L^1_{\mathbb{F}}(t,T; L^q(\Omega; H))$, $v_2(\cdot) \in L^2_{\mathbb{F}}(t,T; L^q(\Omega; H))$ and $\eta \in L^q_{\mathcal{F}_t}(\Omega; H)$, it holds that

$$
\begin{aligned}
&\mathbb{E}\big\langle z(T), y_T\big\rangle_H - \mathbb{E}\int_t^T \big\langle z(s), f(s, y(s), Y(s))\big\rangle_H \\
&= \mathbb{E}\big\langle \eta, y(t)\big\rangle_H + \mathbb{E}\int_t^T \big\langle v_1(s), y(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle v_2(s), Y(s)\big\rangle_H ds.
\end{aligned}
$$

Theorem 3. (Q. Lü and X. Zhang, 2012) *For any* $y_T \in L^p_{\mathcal{F}_T}(\Omega; H)$, *and* $f(\cdot,\cdot,\cdot) : [0,T] \times H \times H \to H$ *satisfying some assumptions, the equation* (38) *admits one and only one unique transposition solution* $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0,T]; L^p(\Omega; H)) \times L^2_{\mathbb{F}}(0,T; L^p(\Omega; H))$. *Furthermore,*

$$
\begin{aligned}
&|(y(\cdot), Y(\cdot))|_{D_{\mathbb{F}}([t,T];L^p(\Omega;H))\times L^2_{\mathbb{F}}(t,T;L^p(\Omega;H))} \\
&\le C\left[|f(\cdot,0,0)|_{L^1_{\mathbb{F}}(t,T;L^p(\Omega;H))} + |y_T|_{L^p_{\mathcal{F}_T}(\Omega;H)}\right], \quad (40) \\
&\quad \forall\, t \in [0,T].
\end{aligned}
$$

6. Well-posedness of an operator-valued BSEE

Further, we consider the following operator-valued backward stochastic evolution equation:

$$\begin{cases} dP = -(A^* + J^*(t))Pdt - P(A + J(t))dt - K^*PKdt \\ \qquad\qquad -(K^*Q + QK)dt + Fdt + Qdw \quad \text{in } [0, T), \\ P(T) = P_T. \end{cases} \tag{41}$$

Here $F \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))$, $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))$, and $J, K \in L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))$.

The equation (41) appeared in the study of general stochastic maximum principle in infinite dimensions.

In order to define the transposition solution to the equation (41), we introduce the following two stochastic differential equation:

$$\begin{cases} dx_1 = (A+J)x_1 ds + u_1 ds + Kx_1 dw + v_1 dw \quad \text{in } (t,T], \\ x_1(t) = \xi_1, \end{cases} \tag{42}$$

$$\begin{cases} dx_2 = (A+J)x_2 ds + u_2 ds + Kx_2 dw + v_2 dw \quad \text{in } (t,T], \\ x_2(t) = \xi_2. \end{cases} \tag{43}$$

Here $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1, u_2 \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$ and $v_1, v_2 \in L^4_{\mathbb{F}}(t,T; L^4(\Omega; H))$.

Definition 4. We call $(P(\cdot), Q(\cdot)) \in D_{\mathbb{F},w}([0,T]; L^2(\Omega; \mathcal{L}(H))) \times L^2_{\mathbb{F},w}(0,T; L^2(\Omega; \mathcal{L}(H)))$ a transposition solution to (41) if for any $t \in [0,T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^4_{\mathbb{F}}(t,T; L^4(\Omega; H))$, it holds that

$$
\begin{aligned}
&\mathbb{E}\big\langle P_T x_1(T), x_2(T)\big\rangle_H - \mathbb{E}\int_t^T \big\langle F(s)x_1(s), x_2(s)\big\rangle_H ds \\
&= \mathbb{E}\big\langle P(t)\xi_1, \xi_2\big\rangle_H + \mathbb{E}\int_t^T \big\langle P(s)u_1(s), x_2(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle P(s)x_1(s), u_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)K(s)x_1(s), v_2(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), Kx_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), v_2(s)\big\rangle_H ds \\
&\quad + \mathbb{E}\int_t^T \big\langle Q(s)v_1(s), x_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle Q(s)x_1(s), v_2(s)\big\rangle_H ds.
\end{aligned}
$$

Denote by $\mathcal{L}_2(H)$ the set of the Hilbert-Schmidt operators on H.

Theorem 4. (Q. Lü and X. Zhang, 2012) *Assume that H is a separable Hilbert space and $L^p_{\mathcal{F}_T}(\Omega)$ ($1 \leq p < \infty$) is a separable Banach space. Then, for any $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}_2(H))$, $F \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}_2(H)))$ and $J, K \in L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))$, the equation (41) admits one and only one transposition solution (P, Q) with the regularity $\big(P(\cdot), Q(\cdot)\big) \in D_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{L}_2(H))) \times L^2_{\mathbb{F}}(0, T; \mathcal{L}_2(H))$. Furthermore,*

$$\begin{aligned} &|(P, Q)|_{D_{\mathbb{F}}([0,T];L^2(\Omega;\mathcal{L}_2(H)))\times L^2_{\mathbb{F}}(0,T;\mathcal{L}_2(H))} \\ &\leq C\left[|F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}_2(H)))} + |P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}_2(H))}\right]. \end{aligned} \tag{44}$$

• Theorems 4 indicates that, in some sense, the transposition solution introduced in Definition 4 is a reasonable notion for the solution to (41).

• Unfortunately, we are unable to prove the existence of transposition solution to (41) in the general case.

• We shall introduced below a weaker version of solution, i.e., relaxed transposition solution (to (41)), which looks awkward but it suffices to establish the Pontryagin-type stochastic maximum principle for Problem (P) in the general setting.

Definition 5. We call $\big(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}\big) \in D_{\mathbb{F},w}([0,T]; L^{\frac{4}{3}}(\Omega; \mathcal{L}(H))) \times \mathcal{Q}[0,T]$ a relaxed transposition solution to (41) if for any $t \in [0,T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t,T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^4_{\mathbb{F}}(t,T; L^4(\Omega; H))$, it holds that

$$
\begin{aligned}
&\mathbb{E}\big\langle P_T x_1(T), x_2(T)\big\rangle_H - \mathbb{E}\int_t^T \big\langle F(s)x_1(s), x_2(s)\big\rangle_H ds\\
&= \mathbb{E}\big\langle P(t)\xi_1, \xi_2\big\rangle_H + \mathbb{E}\int_t^T \big\langle P(s)u_1(s), x_2(s)\big\rangle_H ds\\
&\quad +\mathbb{E}\int_t^T \big\langle P(s)x_1(s), u_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)K(s)x_1(s), v_2(s)\big\rangle_H ds\\
&\quad +\mathbb{E}\int_t^T \big\langle P(s)v_1(s), Kx_2(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle P(s)v_1(s), v_2(s)\big\rangle_H ds\\
&\quad +\mathbb{E}\int_t^T \big\langle v_1(s), \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s)\big\rangle_H ds + \mathbb{E}\int_t^T \big\langle Q^{(t)}(\xi_1, u_1, v_1)(s), v_2(s)\big\rangle_H ds.
\end{aligned}
$$

• It is easy to see that, if $\big(P(\cdot), Q(\cdot)\big)$ is a transposition solution to (41), then one can find a relaxed transposition solution $\big(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}\big)$ to the same equation (from $\big(P(\cdot), Q(\cdot)\big)$). Indeed, they are related by

$$Q(s)x_1(s) = Q^{(t)}(\xi_1, u_1, v_1)(s),$$
$$Q(s)^* x_2(s) = \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s).$$

This means that, we know only the action of $Q(s)$ (or $Q(s)^*$) on the solution processes $x_1(s)$ (or $x_2(s)$).

• However, it is unclear how to obtain a transposition solution $\big(P(\cdot), Q(\cdot)\big)$ to (41) by means of its relaxed transposition solution $\big(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}\big)$. It seems that this is possible but we cannot do it at this moment.

• Well-posedness result for the equation (41) in the general case

Theorem 5. (Q. Lü and X. Zhang, 2012) *Assume that H is a separable Hilbert space, and $L^p_{\mathcal{F}_T}(\Omega;\mathbb{C})$ $(1\le p<\infty)$ is a separable Banach space. Then, for any $P_T\in L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))$, $F\in L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))$ and $J,K\in L^4_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H)))$, the equation (41) admits one and only one relaxed transposition solution $\big(P(\cdot),Q^{(\cdot)},\widehat{Q}^{(\cdot)}\big)$. Furthermore,*

$$\begin{aligned}&\|P\|_{\mathcal{L}(L^2_{\mathbb{F}}(0,T;L^4(\Omega;H)),\ L^2_{\mathbb{F}}(0,T;L^{\frac{4}{3}}(\Omega;H)))}\\&\quad+\sup_{t\in[0,T]}\big\|\big(Q^{(t)},\widehat{Q}^{(t)}\big)\big\|_{\big(\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega;H)\times L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))\times L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)),\ L^2_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H))\big)^2}\\&\le C\left[|F|_{L^1_{\mathbb{F}}(0,T;\ L^2(\Omega;\mathcal{L}(H)))}+|P_T|_{L^2_{\mathcal{F}_T}(\Omega;\ \mathcal{L}(H))}\right].\end{aligned}\tag{45}$$

7. Pontryagin-type stochastic maximum principle

For $(t, x, u, k_1, k_2) \in [0, T] \times H \times U \times H \times H$, write

$$
\begin{aligned}
&\mathbb{H}(t, x, u, k_1, k_2)\\
&= \big\langle k_1, a(t, x, u)\big\rangle_H + \big\langle k_2, b(t, x, u)\big\rangle_H - g(t, x, u).
\end{aligned}
$$

Theorem 6. (Q. Lü and X. Zhang, 2012) *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (P), and let $(y(\cdot), Y(\cdot))$ be the transposition solution to (38) with $p = 2$, and y_T and $f(\cdot, \cdot, \cdot)$ given by*

$$
\begin{cases}
y_T = -h_x\big(\bar{x}(T)\big),\\
f(t, y_1, y_2) = -a_x(t, \bar{x}(t), \bar{u}(t))^* y_1 - b_x\big(t, \bar{x}(t), \bar{u}(t)\big)^* y_2\\
\qquad\qquad\qquad +g_x\big(t, \bar{x}(t), \bar{u}(t)\big).
\end{cases}
\tag{46}
$$

Assume $b_x(\cdot,\bar{x}(\cdot),\bar{u}(\cdot)) \in L^4_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathcal{L}(D(A))))$, *and* $(P(\cdot),Q^{(\cdot)},\widehat{Q}^{(\cdot)})$ *is the relaxed transposition solution to* (41) *with* P_T, $J(\cdot)$, $K(\cdot)$ *and* $F(\cdot)$ *given by*

$$\begin{cases} P_T = -h_{xx}\big(\bar{x}(T)\big), \quad J(t) = a_x(t,\bar{x}(t),\bar{u}(t)), \\ K(t) = b_x(t,\bar{x}(t),\bar{u}(t)), \quad F(t) = -\mathbb{H}_{xx}\big(t,\bar{x}(t),\bar{u}(t),y(t),Y(t)\big). \end{cases}$$

Then

$$\begin{aligned} &\mathrm{Re}\,\mathbb{H}\big(t,\bar{x}(t),\bar{u}(t),y(t),Y(t)\big) - \mathrm{Re}\,\mathbb{H}\big(t,\bar{x}(t),u,y(t),Y(t)\big) \\ &-\frac{1}{2}\Big\langle P(t)\Big[b\big(t,\bar{x}(t),\bar{u}(t)\big) - b\big(t,\bar{x}(t),u\big)\Big], b\big(t,\bar{x}(t),\bar{u}(t)\big) - b\big(t,\bar{x}(t),u\big)\Big\rangle_H \\ &\geq 0, \qquad\qquad \forall\, u \in U, \text{ a.e. } [0,T]\times\Omega. \end{aligned}$$

Thank You