Feedback Stabilization and Conservation Laws.

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Outline

Conservation Laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0.$$

- Many physical systems: gas dynamics, magneto-hydrodynamic, electromagnetism, shallow water theory, combustion theory, petroleum reservoir engineering.
- Scalar case: fewer physical systems (network flows...) but also first step toward more complex systems.
- Second order terms (diffusion, viscosity ...) effects neglected during modelization.

Bibliography free evolution

- Scalar Cauchy: Oleinik (59), Kruzkov (70).
- Systems Cauchy: Lax (57), Glimm (65), Bressan (92..).
- Vanishing Viscosity: Bianchini Bressan (05).
- Scalar initial boundary value: Leroux (76), Bardos Leroux Nedelec (77), Otto(96).
- Systems initial boundary value: Dubois Lefloch (88), Sable-Tougeron (93), Amadori (96), Ancona Bianchini (2003), Donadello Marson (2007).

Generalities

- Short time H^s solutions.
- Classical solutions 'generally' blow up in finite time.
- Weak solution in L^∞ but then no uniqueness.
- Entropy weak solutions, remembering neglected terms.
- Solutions not time reversible anymore.

Characteristics Method

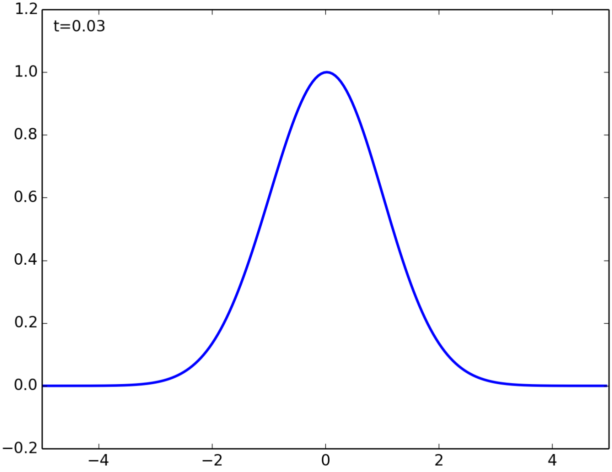
$$u_t + (f(u))_x = 0, \quad \Leftrightarrow \quad u_t + f'(u)u_x = 0$$

$$\Leftrightarrow \begin{cases} \frac{d}{dt} u(t, \psi(t,x)) = 0 \\ \frac{d}{dt}\psi(t,x) = u(t, \psi(t,x)). \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{d}{dt}\psi(t,x) = u_0(x), \\ u(t, \psi(t,x)) = u_0(x) \end{cases}$$

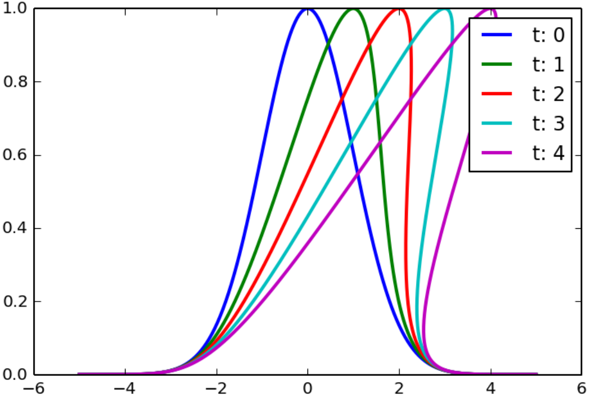
$$\Leftrightarrow \begin{cases} \psi(t,x) = x + tu_0(x), \\ u(t, \psi(t,x)) = u_0(x) \end{cases}$$

Multivalued Solution.



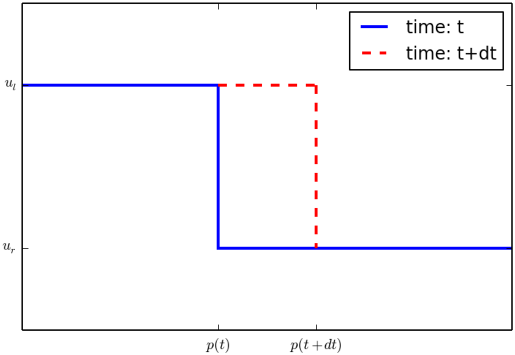
Gradient blow up

$$\partial_t(u_x) + f'(u)\partial_x(u_x) = -(u_x)^2.$$



Rankine Hugoniot

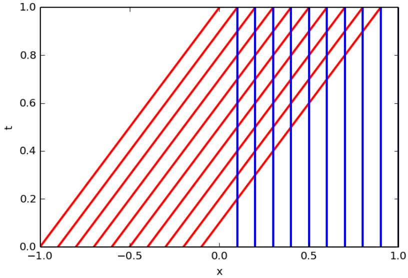
$$\frac{d}{dt}\int_a^b u(t,x)dx = f(u(a)) - f(u(b)).$$

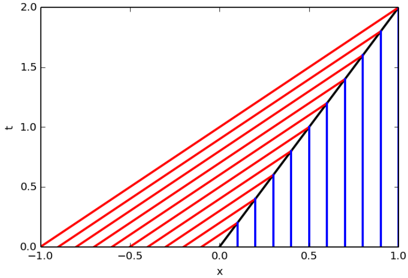


$$\dot{p}(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Characteristics: Shock Wave

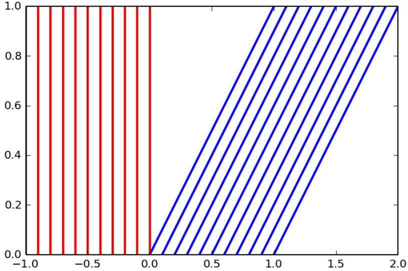
$$u_0 = \begin{cases} 1.0 & \text{if } x < 0 \\ 0.0 & \text{if } x > 0. \end{cases}$$

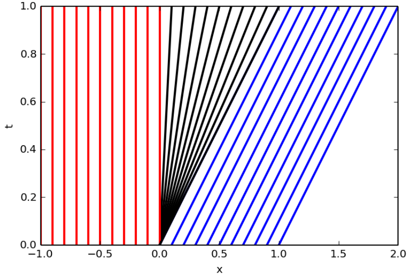




Characteristics: Rarefaction Wave

$$u_0 = \begin{cases} 0.0 & \text{if } x < 0 \\ 1.0 & \text{if } x > 0. \end{cases}$$



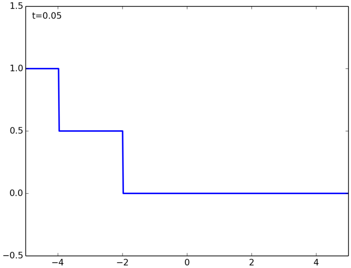


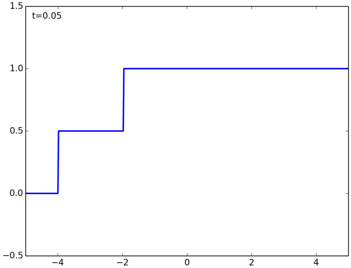
Lax Entropy Condition

- u solution and $u(t, p(t)^-) \neq u(t, p(t)^+$

$$\Rightarrow f'(u(t, p(t)^-)) > \dot{p}(t) > f'(u(t, p(t)^+)).$$

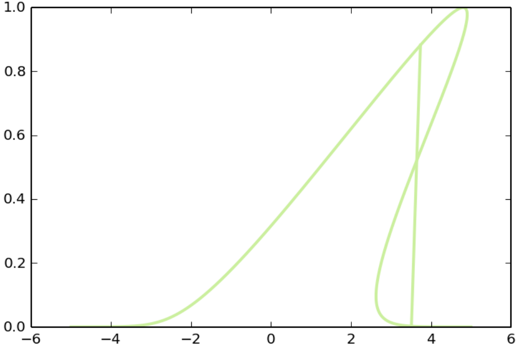
- Stability condition:



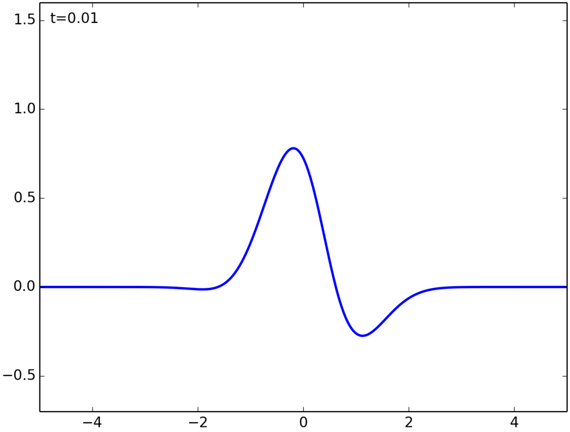


Equal Area Rule

From Lagrangian Description, how to get the entropy solutions



Simulation



Entropy Solution with boundary terms:

Viscosity method $\Rightarrow$ boundary condition, following Leroux(1D scalar, 1976) and Bardos Leroux Nedelec (scalar multiD 1977) we define
A function u in $L^\infty((0,+\infty),\mathrm{BV}(0,1))$ is an entropy solution of:

$$\partial_t u + \partial_x(f(u)) = g(t)$$
$$u(0,x) = u_0 \quad {}''u(.,0) = u_l'' \quad {}''u(.,1) = u_r''$$

If for every number k in $\mathbb{R}$ and every positive function ϕ in $\mathcal{C}_c^1(\mathbb{R}^2)$ we have:

$$\int_0^{+\infty}\int_0^1 |u-k|\phi_t + \mathrm{sgn}(u-k)(f(u)-f(k))\phi_x$$
$$+\mathrm{sgn}(u-k)g(t)\phi dtdx + \int_0^1 |u_0(x)-k|\phi(0,x)dx$$
$$+\int_0^{+\infty} \mathrm{sgn}(u_r(t)-k)(f(k)-f(u(t,1^-)))\phi(t,1)$$
$$-\mathrm{sgn}(u_l(t)-k)(f(k)-f(u(t,0^+))\phi(t,0)dt \geq 0$$

Boundary Conditions

- At $x = 1$: for almost all t and for any number k between $u_r(t)$ and $u(t, 1^-)$

$$\mathrm{sgn}(u(t, 1^-) - u_r(t))(f(u(t, 1^-) - f(k)) \geq 0.$$

We will say:

$$u(t, 1^-) \in \mathcal{A}_r(u_r(t)).$$

- At $x = 0$: for almost all t and for any number k between $u_l(t)$ and $u(t, 0^+)$

$$\mathrm{sgn}(u(t, 0^+) - u_l(t))(f(u(t, 0^+)) - f(k)) \leq 0.$$

We will say:

$$u(t, -1^+) \in \mathcal{A}_l(u_l(t)).$$

Control Bibliography

- Huge litterature for regular solutions, control MUST prevent blow up.
- Scalar exact controllability: Ancona Marson (98), Horsin (98), Adimurthi Ghoshal Gowda (11), P. (11), Andreianov Donadello Ghoshal Razafison (14), Corghi Marson (14).
- System exact controllability: Bressan Coclite (02), Ancona Coclite (05), Glass (07), Ancona Nguyen Priuli (12), Glass (13).
- Exact Controllability and vanishing viscosity: Glass Guerrero (07), Leautaud (12).
- Feedback Stabilization: P. (13).

Problem setting

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad t > 0,\ x \in (-1, 1).$$

2 familles de solutions stationnaires:

- $\forall \bar{u} \in \mathbb{R}$,
$$S^1_{\bar{u}}(x) = \bar{u}, \qquad \forall x \in (-1, 1).$$
- $\forall \bar{u} > 0,\ \forall p \in (-1, 1)$,
$$S^2_{p,\bar{u}}(x) = \begin{cases} \bar{u} & \text{if } x < p \\ -\bar{u} & \text{if } x > p. \end{cases}$$

Stability Result

Theorem

For any initial data u_0 in $\mathrm{BV}(-1,1)$, for any boundary data u_l, u_r constant in or all time, there exists a unique entropy weak solution u in the space $L^\infty(0,+\infty;\mathrm{BV}(-1,1)) \cap \mathrm{Lip}(0,+\infty;L^1(-1,1))$ of

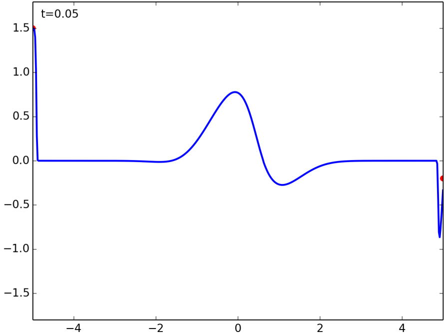
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ u(t,-1^+) \in \mathcal{A}_l(u_l), \\ u(t,1^-) \in \mathcal{A}_r(u_r), \qquad dt \ a.e. \end{cases}$$

And it satisfies the following asymptotic properties:

Finite time Stability I

If $u_l > 0$ and $u_r > -u_l$, there is a time $T(u_l)$ such that:

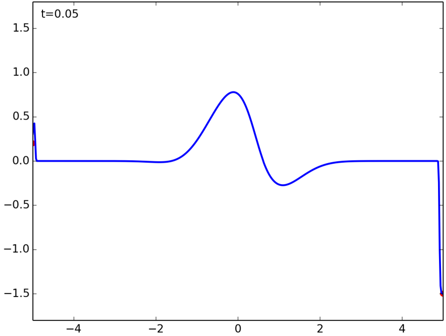
$$\forall t \geq T, \qquad u(t,.) = S^1_{u_l}.$$



Finite time stability II

If $u_r < 0$ and $u_l < -u_r$ there is a time $T(u_r)$ such that:

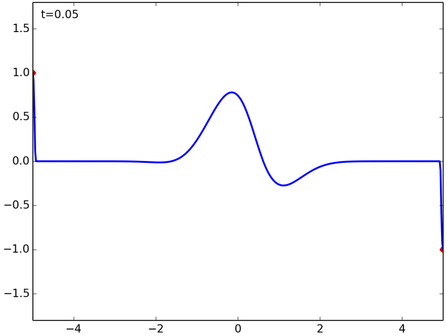
$$\forall t \geq T, \qquad u(t,.) = S^1_{u_r}.$$



Finite time Stability III

If $u_l = -u_r > 0$, there is a time $T(u_l)$ and a point $p \in (-1, 1)$ such that:

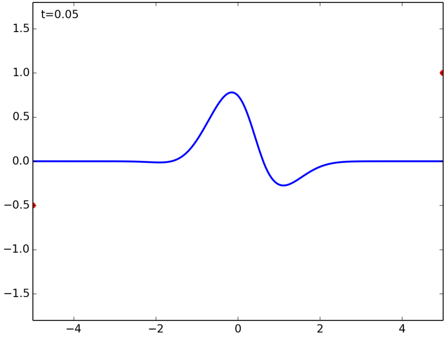
$$\forall t \geq T, \qquad u(t,.) = S^2_{p,u_l}.$$



Asymptotic Stability

If $u_l \leq 0 \leq u_r$, we have:

$$||u(t,.)||_{L^\infty(-1,1)} \leq \frac{1}{t}.$$



Control Point of view

- The families S^1 is finite time stabilizable.
- The actuating for family S^1 is robust.
- The family S^2 is stable.
- The actuating for family S^2 is not robust.

Generalized Characteristics

- extend method of characteristics to low regularity
- differential inclusion in the sense of Filippov (1960) (existence OK but no uniqueness)
- extend Dafermos (1977) to intervals
- many “genuine” characteristics $\Rightarrow$ estimates on u

Definition

a function $\gamma(t)$ absolutely continuous is a generalized characteristic if

$$\dot{\gamma}(t) \in [u(t,\gamma(t)^+), u(t,\gamma(t)^-)] \quad dt \ a.e.$$

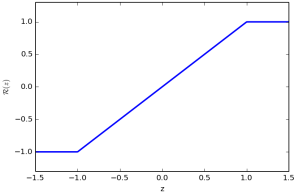
Notation

- For $u \in L^1(-1,1)$, $p \in (-1,1)$ $\delta > 0$ small we consider:

$$\mathcal{OBS}(u) = \frac{1}{2\delta}\int_{p-\delta}^{p+\delta} u(x)dx.$$

- The function $\mathcal{R}$ is defined by:

$$\forall z \in \mathbb{R}, \quad \mathcal{R}(z) = \begin{cases} 1 & \text{if } z \geq 1, \\ z & \text{if } 1 \geq z \geq -1, \\ -1 & \text{if } -1 \geq z. \end{cases}$$



Asymptotic Stabilization

Theorem

For any choice of $p \in (-1,1)$ and $\delta > 0$ small and $\bar{u} > 0$ we can find $\epsilon, C > 0$ such that the system:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, & t > 0,\ x \in (-1,1), \\ u(t,1^-) \in \mathcal{A}_r\left(-\bar{u}\right), & dt\ a.e. \\ u(t,-1^+) \in \mathcal{A}_l\left(\bar{u}\left(1 - \epsilon\mathcal{R}\left(\frac{\mathcal{OBS}(u(t,.))}{\bar{u}(1+\epsilon)}\right)\right)\right) \end{cases}$$

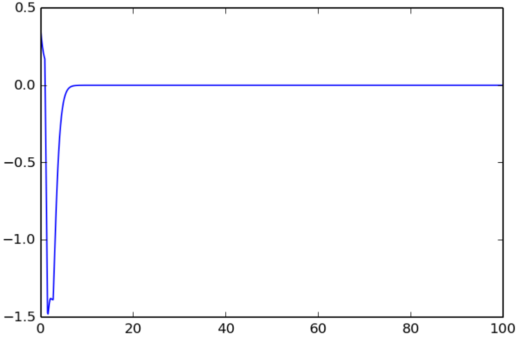
has a unique entropy weak solution u. It satisfies:

$$||u(t,.) - S^2_{p,\bar{u}}||_{L^1(-1,1)} \leq \frac{e^{-Ct}}{C}||u_0 - S^2_{p,\bar{u}}||_{L^1(-1,1)}.$$

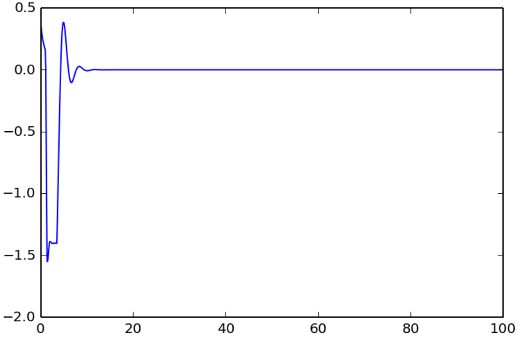
Steps of the proof

1. "Relaxation of the initial data".
2. Shock wave with singularity in observation zone.
3. DDE on the observation.
4. Bound on observation $\Rightarrow$ bound on solution.

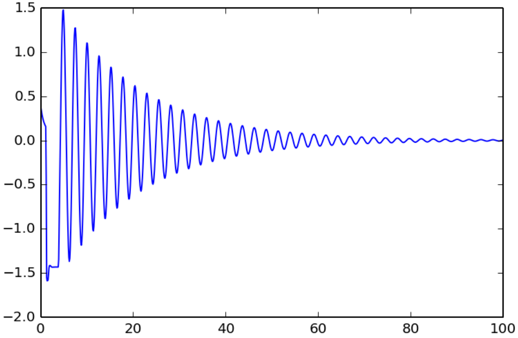
Observation Examples I



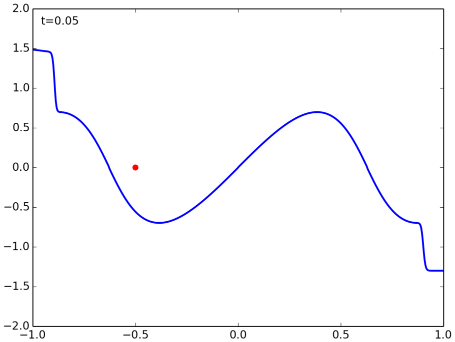
Observation Examples II



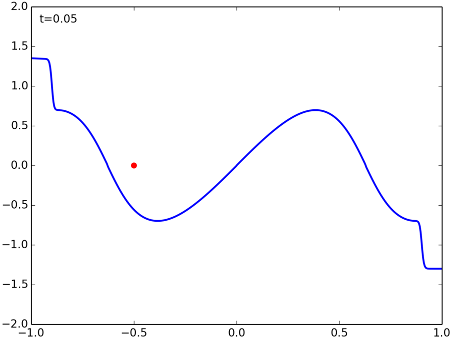
Observation Examples III



Example with bad parameters



Example with good parameters



Further Problems

- Sharp condition on ϵ.
- Robustness with respect to viscosity perturbation.
- Finite time stabilization for stationary shock.
- Control independent of observation zone.
- Local stabilization around inflexion point with non convex flux.
- Extension to systems.