Vectorial Ingham–Beurling type estimates

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Working group in control theory

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We consider the coupled string–beam system

\[
\begin{align*}
  u_{tt} - u_{xx} + au + bw &= 0, \\
  w_{tt} + w_{xxxx} + cu + dw &= 0
\end{align*}
\]

with usual initial conditions and with Dirichlet–hinged boundary conditions on a bounded interval \((0, \ell)\), where \(a, b, c, d\) are given coupling constants.

Given \(T > 0\), we investigate the validity of the estimates

\[
c_1 E(0) \leq \int_0^T |u_x(t, 0)|^2 + |w_x(t, 0)|^2 \, dt \leq c_2 E(0)
\]

with suitable positive constants \(c_1, c_2\) where \(E(0)\) denotes the usual initial energy \((\mathcal{H} = H_0^1 \times L^2 \times H_0^1 \times H^{-1})\).
Following K.–Loreti, we may write these estimates in the abstract form

\[ c_1 \sum_{k \in \mathbb{Z}} |x_k|^2 \leq \int_0^T |x(t)|^2 \, dt \leq c_2 \sum_{k \in \mathbb{Z}} |x_k|^2 \]

where

\[ x(t) = \sum_{k \in \mathbb{Z}} x_k U_k e^{i \omega_k t} \]

with square-summable complex coefficients \( x_k \). Here \( (U_k) \) is a given sequence of unit vectors in \( \mathbb{C}^4 \) and \( (\omega_k) \) is a given sequence of real numbers, depending on the parameters of the problem (eigenvector traces and eigenvalues).
ASSUMPTIONS AND NOTATIONS

- Let $\Omega := (\omega_k)_{k \in \mathbb{Z}}$ be a family of real numbers satisfying the \textit{gap condition}
  \[ \gamma := \inf_{k \neq n} |\omega_k - \omega_n| > 0. \]

- Let $(U_k)_{k \in \mathbb{Z}}$ be a corresponding family of unit vectors in some finite-dimensional complex Hilbert space $H$ and consider the sums
  \[ x(t) = \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \]
  with square summable complex coefficients $x_k$.

- By the gap condition $\Omega$ has a finite upper density defined by
  \[ D^+ = D^+(\Omega) := \lim_{r \to \infty} n^+(r)/r \]
  where $n^+(r)$ denotes the maximum number of terms $\omega_k$ contained in an interval of length $r$. We have $D^+ \leq 1/\gamma$. 

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Statement of the results

MAIN RESULT

We recall that

\[ x(t) = \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t}, \quad x_k \in \mathbb{C}. \]

Theorem

(Barhoumi, K., Mehrenberger)

(a) If \( T > 2\pi D^+ \), then the estimates

\[ c_1 \sum_{k \in \mathbb{Z}} |x_k|^2 \leq \int_0^T \|x(t)\|_H^2 \, dt \leq c_2 \sum_{k \in \mathbb{Z}} |x_k|^2 \]

hold with suitable \( c_1, c_2 > 0 \).

(b) Conversely, if the above estimates hold true and \( \dim H = d \), then

\( T \geq 2\pi D^+/d. \)
ONE-DIMENSIONAL EXAMPLES

In the scalar case $d = 1$ the critical length is $T = 2\pi D^+$ by Beurling’s original theorem.

- For $\omega_k = k$ we have $D^+ = 1$ and the critical length is $2\pi$ in correspondence with Parseval’s equality:
  \[
  \int_0^{2\pi} \left| \sum_{k \in \mathbb{Z}} x_k e^{ikt} \right|^2 dt = 2\pi \sum_{k \in \mathbb{Z}} |x_k|^2.
  \]

- For $\omega_k = k^3$ we have $D^+ = 0$, so that
  \[
  c_1 \sum_{k \in \mathbb{Z}} |x_k|^2 \leq \int_0^T |x(t)|^2 dt \leq c_2 \sum_{k \in \mathbb{Z}} |x_k|^2
  \]
  for any $T > 0$ (the constants $c_1, c_2 > 0$ depend on $T$).

- Ingham’s earlier sufficient condition ensured the preceding estimates for $T > 2\pi / \gamma = 2\pi$. (We recall that $D^+ \leq 1 / \gamma$.)
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- Ingham’s earlier sufficient condition ensured the preceding estimates for $T > 2\pi/\gamma = 2\pi$. (We recall that $D^+ \leq 1/\gamma$.)
If \( d > 1 \), \((U_k)\) is \( d \)-periodical and \( U_1, \ldots, U_d \) is an orthonormal basis of \( H \), then the critical length is \( T = 2\pi D^+ / d \). Indeed,

\[
\int_0^T \left| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \right|^2_H dt = \sum_{j=1}^d \int_0^T \left| \sum_{k \in \mathbb{Z}} x_{kd+j} e^{i\omega_{kd+j} t} \right|^2 dt
\]

and we may apply the scalar case to each sum on the right side.

We show later that the critical length can be anything between \( 2\pi D^+ / d \) and \( 2\pi D^+ \).
If \( d > 1 \), \((U_k)\) is \(d\)-periodical and \(U_1, \ldots, U_d\) is an orthonormal basis of \(H\), then the critical length is \(T = \frac{2\pi D^+}{d}\). Indeed,

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We show later that the critical length can be anything between \(\frac{2\pi D^+}{d}\) and \(2\pi D^+\).
Theorem

If

$$\gamma := \inf_{k \neq n} |\omega_k - \omega_n| > 0$$

and $T > 2\pi / \gamma$, then we have

$$c_1 \sum_{k \in \mathbb{Z}} |x_k|^2 \leq \int_0^T \left| \sum_{k \in \mathbb{Z}} x_k e^{i \omega_k t} \right|^2 dt \leq c_2 \sum_{k \in \mathbb{Z}} |x_k|^2.$$

Proof by introducing suitable orthogonalizing weight functions and imitating the proof of Parseval’s equality.
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Theorem

If $\Omega_1 \cup \cdots \cup \Omega_M$ be a finite partition of $\Omega = \{\omega_k\}$ and

$$T > \frac{2\pi}{\gamma(\Omega_1)} + \cdots + \frac{2\pi}{\gamma(\Omega_M)}$$

then we have

$$c_1 \sum_{k \in \mathbb{Z}} |x_k|^2 \leq \int_0^T \left| \sum_{k \in \mathbb{Z}} x_k e^{i\omega_k t} \right|^2 dt \leq c_2 \sum_{k \in \mathbb{Z}} |x_k|^2.$$

For $M = 1$ this reduces to Ingham’s theorem.
Proof by using a Fourier transform method of Kahane, made constructive by using a method of Haraux.

Example. For \( \omega_k = k^3 \) and \( \Omega_j := \{ \omega_{kM+j} : k \in \mathbb{Z} \} \), \( j = 1, \ldots, M \) we have \( \gamma = 1 \) but

\[
\frac{2\pi}{\gamma(\Omega_1)} + \cdots + \frac{2\pi}{\gamma(\Omega_M)} \leq M \frac{2\pi}{M^3/4} \to 0, \quad M \to \infty.
\]
WEAKENING OF $T > 2\pi/\gamma$

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**Upper Density and Partitions**

**Proposition**

(Baiocchi, K., Loreti) For every $T > 2\pi D^+$ there exists a finite partition of $\Omega$ such that

$$\frac{2\pi}{\gamma(\Omega_1)} + \cdots + \frac{2\pi}{\gamma(\Omega_M)} < T.$$  

**Proof.**

We choose $\gamma' > 0$ such that $T > \frac{2\pi}{\gamma'} > 2\pi D^+$, and then a large integer $M$ such that $\frac{2\pi}{\gamma'} > 2\pi \frac{n^+(M\gamma')}{M\gamma'}$, i.e., $n^+(M\gamma') < M$. Arranging the exponents into an increasing sequence $(\omega_k)_{k \in K}$ we have $\omega_{k+M} - \omega_k > M\gamma'$ for all $k$, so that the sets $\Omega_j := \{\omega_{Mk+j} : k \in K\}$ satisfy

$$\sum_{j=1}^M \frac{2\pi}{\gamma(\Omega_j)} \leq \sum_{j=1}^M \frac{2\pi}{M\gamma'} = \frac{2\pi}{\gamma'} < T.$$ 

UPPER DENSITY AND PARTITIONS

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\]
THE VECTORIAL CASE

We fix an orthonormal basis \((E_n)_{n \in \mathbb{N}}\) of \(H\) and we develop each \(U_k\) into Fourier series:

\[
U_k = \sum_{n \in \mathbb{N}} u_{kn} E_n.
\]

If \(T > 2\pi D^+\), then using the scalar case we have

\[
\int_0^T \left\| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \right\|_H^2 \, dt = \sum_{n \in \mathbb{N}} \int_0^T \left| \sum_{k \in \mathbb{Z}} x_k u_{kn} e^{i\omega_k t} \right|^2 \, dt
\]

\[
\approx \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} |x_k u_{kn}|^2
\]

\[
= \sum_{k \in \mathbb{Z}} |x_k|^2
\]

with \(\approx\) meaning equivalence.
NECESSITY OF $T \geq 2\pi D^+/d$. NOTATIONS

Assume by scaling that

$$\int_0^{2\pi} \left| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \right|^2 dt \asymp \sum_{k \in \mathbb{Z}} |x_k|^2.$$

We need to show that $D^+ \leq d$. We adapt a method of Gröchenig and Razafinjatovo.

Fix $R > 0$, $y \in \mathbb{R}$, $r > 0$ and set

$$V = V_{y,r} := \text{Vect} \{ U_k e^{i\omega_k t} : |\omega_k - y| < r \},$$

$$W = W_{R,y,r} := \text{Vect} \{ U e^{ik t} : U \in H, |k - y| < r + R \}.$$

Note that

$$n^+(2r) = \sup_y \dim V,$$

$$\dim W \leq (2r + 2R)d.$$
NECESSITY OF $T \geq 2\pi D^+ / d$. NOTATIONS

Assume by scaling that
\[
\int_0^{2\pi} \left| \sum_{k \in \mathbb{Z}} x_k U_k e^{i\omega_k t} \right|^2 dt \preceq \sum_{k \in \mathbb{Z}} |x_k|^2.
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Note that
\[ n^+(2r) = \sup_y \dim \ V, \]
\[ \dim \ W \leq (2r + 2R)d. \]
PROOF OF $D^+ \leq d$

We prove that

$$\dim V \leq (1 + o_R(1)) \dim W \quad \text{as} \quad R \to \infty.$$  

This will imply that

$$n^+(2r) = \sup_y \dim V \leq (2r + 2R)d(1 + o_R(1))$$

and hence that

$$D^+ = \lim_{r \to \infty} \frac{n^+(2r)}{2r} \leq d(1 + o_R(1))$$

for all $R > 0$. Letting $R \to \infty$ this yields $D^+ \leq d$. 

PROOF OF $\dim V \leq (1 + o_R(1)) \dim W$

Let $P$, $Q$ be the orthogonal projections of $L^2(0, 2\pi; H)$ onto $V$ and $W$. Then

$$S := P \circ Q|_V \in L(V, V)$$

has norm $\leq 1$ and rank $\leq \dim W$, so that

$$\text{Tr } S \leq \dim W.$$ 

It remains to show that

$$\text{Tr } S \geq (1 - o_R(1)) \dim V.$$
PROOF OF $\text{Tr } S \geq (1 - o_R(1)) \dim V$

Let $(f_k)$ be a bounded biorthogonal sequence to $e_k := U_k e^{i\omega_k t}$ in $L^2(0, 2\pi; H)$. Since

$$\text{Tr } S = \sum_{|\omega_k - y| < r} (Se_k, f_k)_{L^2(0, 2\pi; H)} = \sum_{|\omega_k - y| < r} (Qe_k, Pf_k)_{L^2(0, 2\pi; H)}$$

we have

$$\text{Tr } S - \dim V = \sum_{|\omega_k - y| < r} ((Q - I)e_k, Pf_k)_{L^2(0, 2\pi; H)}$$

$$\geq -\left(\sup \|f_k\|\right)(\dim V) \sup_{|\omega_k - y| < r} \| (Q - I)e_k \|_{L^2(0, 2\pi; H)}$$

$$= -o_R(1) \dim V$$

by a direct computation.
ESTIMATE OF $\|(Q - I) e_k\|_{L^2(0, 2\pi; H)}$

Using the Fourier expansion

$$e_k = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sum_{j=1}^{d} (e_k, E_j e^{int}) E_j e^{int}$$

where $(E_j)$ is an orthonormal basis of $H$, we have

$$\|(Q - I) e_k\|_{L^2(0, 2\pi; H)}^2 = \frac{1}{2\pi} \sum_{|n-y| \geq r+R} \sum_{j=1}^{d} |(e_k, E_j e^{int})|^2$$

$$= \frac{1}{2\pi} \sum_{|n-y| \geq r+R} \sum_{j=1}^{d} |(U_k, E_j)|^2 \left| \int_0^{2\pi} e^{i(\omega_k - n)t} \, dt \right|^2$$

$$\leq \frac{2d}{\pi} \sum_{|n-y| \geq r+R} \frac{1}{|\omega_k - n|^2}.$$
Since $|n - y| \geq r + R$ and $|\omega_k - y| < r$ imply $|n - \omega_k| > R$, it follows that

$$\| (Q - I)e_k \|_{L^2(0, 2\pi; H)}^2 \leq \frac{2d}{\pi} \sum_{|n - y| \geq r + R} \frac{1}{|\omega_k - n|^2} \leq \frac{4d}{\pi} \sum_{n=0}^{\infty} \frac{1}{(R + n)^2} \approx \int_{R}^{\infty} \frac{1}{x^2} \, dx = \frac{4d}{\pi R}.$$
PARTITIONS AND UPPER DENSITY

In order to show that the critical value of $T$ may be anything between $2\pi D^+/d$ and $2\pi D^+$, we use the following result:

**Theorem**

Let $\Omega$ be a set of real numbers with a finite upper density $D^+$ and let $\alpha_1, \alpha_2, \ldots$ be a finite or infinite sequence of numbers in $[0, 1]$ satisfying

$$\alpha_1 + \alpha_2 + \cdots \geq 1.$$  

Then there exists a partition

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots$$

such that the upper density of $\Omega_j$ is equal to $\alpha_j D^+$ for every $j$. 

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OPTIMALITY OF THE MAIN THEOREM

Given $1/d \leq \alpha \leq 1$ arbitrarily we choose $\alpha_1, \ldots, \alpha_d \geq 0$ such that

$$\alpha_1 + \cdots + \alpha_d = 1 \quad \text{and} \quad \max\{\alpha_1, \ldots, \alpha_d\} = \alpha.$$

Applying the above theorem we obtain a partition $\Omega = \Omega_1 \cup \cdots \cup \Omega_d$ such that $D^+(\Omega_j) = \alpha_j D^+$ for all $j$. Fix an orthonormal basis $E_1, \ldots, E_d$ of $H$ and set $U_k = E_j$ if $\omega_k \in \Omega_j$. Then using the identity

$$\int_0^T \left\| \sum_{k \in \mathbb{Z}} x_k U_k e^{i \omega_k t} \right\|^2_H \, dt = \sum_{j=1}^d \int_0^T \left\| \sum_{\omega_k \in \Omega_j} x_k e^{i \omega_k t} \right\|^2 \, dt$$

and applying the scalar case of the theorem we conclude that the required estimates hold if $T > 2\pi \alpha D^+$, and they fail if $T < 2\pi \alpha D^+$. 