

# Models of Fluid - Solid Coupling and Exact Controllability

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## Coupling between Navier-Stokes Equations and Lamé System

$$\Omega_F(t) \subset \mathbb{R}^3$$

$$\Omega_S(t) = \Omega \setminus \overline{\Omega_F(t)}$$

$$v_t + (v \cdot \nabla_x)v - \operatorname{div}_x \sigma(v, p) = 0 \text{ in } \Omega_F(t), 0 < t < T$$

$$\operatorname{div}_x v = 0 \text{ in } \Omega_F(t), 0 < t < T$$

$$v(x, 0) = v_0(x) \text{ in } \Omega_F(0)$$

$$\sigma(v, p) = \nu(\nabla_x v + \nabla_x v^T) - pI$$

### Lagrangian trajectories

$$\frac{\partial X}{\partial t}(y, t) = v(X(y, t), t)$$

$$X(y, 0) = y, \quad y \in \Omega_F(0)$$

## Boundary Conditions

$$\begin{aligned}v &= 0 \text{ on } \Gamma_e \times (0, T) \\v(X(y, t), t) &= \frac{\partial w}{\partial t}(y, t) \text{ on } \Gamma_S(0) \times (0, T)\end{aligned}$$

## Lamé System

$$\begin{aligned}w_{tt} - \operatorname{div}_y \sigma(w) &= 0 \text{ in } \Omega_S(0) \times (0, T) \\w(y, 0) = w_0(y), \frac{\partial w}{\partial t}(y, 0) &= w_1(y) \text{ in } \Omega_S(0) \\ \sigma(w) &= \lambda \operatorname{trace} \varepsilon(w) + 2\mu \varepsilon(w) \\ \varepsilon(w) &= \frac{1}{2}(\nabla_y w + \nabla_y w^T)\end{aligned}$$

## Boundary Conditions

$$\sigma(w)n = (\sigma(v, p) \circ X) \operatorname{Cof}(\nabla_y X)n \quad \text{on } \Gamma_S(0) \times (0, T)$$

## Features.

Evolution problem, non-stationary problem posed on a non-cylindrical domain in space-time.

Free Interface problem.

Mixed Nature (Hyperbolic-Parabolic)

Hyperbolic metric and Parabolic metric

Hyperbolic scaling and Parabolic scaling

Characteristic speeds do not match at the interface

Natural spaces do not match at the interface.

**Idea.** Solid deformations may have only finitely many degrees of freedom.  
Hyperbolicity is negligible.  
Parabolic dominates.  
Solid is treated as a perturbation of the fluid system.

## Motion of Rigid Bodies

$x_B$  - position of centre of mass

$\omega$  - angular velocity

$Q$  - Orientation of solid (orthogonal matrix)

$$S(t) = Q(t)S(0) + x_B(t)$$

$$x = Q(t)y + x_B(t)$$

Velocity of Solid

$$v(x, t) = \omega(t) \wedge (x - x_B(t)) + x'_B(t) \text{ (Interface condition)}$$

$$Q'(t)y = \omega(t) \wedge Q(t)y$$

Newton's Laws:

$$m''_B(t) = - \int_{\partial S(t)} \sigma(v, p)n$$

$$I\omega'(t) = - \int_{\partial S(t)} (x - x_B(t)) \wedge \sigma(v, p)n$$

## STOKES MODEL

$$y' - \Delta y + \nabla p = u1_U \quad \text{in } Q = \Omega \times (0, T)$$

$$\operatorname{div} y = 0 \quad \text{in } Q$$

$$y = s' \quad \text{on } \Sigma_i = \Gamma_i \times (0, T)$$

$$y = 0 \quad \text{on } \Sigma_e = \Gamma_e \times (0, T)$$

$$y(0) = y_0 \quad \text{in } \Omega$$

$$s'' + s = - \int_{\Gamma_i} \sigma(y, p) n \quad \text{in } (0, T)$$

$$s(0) = s_0, s'(0) = s_1$$

$$\sigma = \nu(\nabla y + \nabla y^T) - pI$$

## Heat -Solid Model

$$y' - \Delta y = u1_U \quad \text{in } Q$$

$$y = 0 \quad \text{on } \Sigma_e$$

$$y = s' \cdot n \quad \text{on } \Sigma_i$$

$$y(0) = y^0 \quad \text{in } \Omega$$

$$s'' + s = - \int_{\Gamma_i} (\partial_n y) n \quad \text{in } (0, T)$$

$$s(0) = s_0, s'(0) = s_1$$

## Helmholtz Model

$$y'' - \Delta y = u1_U \quad \text{in } Q = \Omega \times (0, T)$$

$$y = 0 \quad \text{on } \Sigma_e = \Gamma_e \times (0, T)$$

$$\partial_n y = s' \cdot n \quad \text{on } \Sigma_i = \Gamma_i \times (0, T)$$

$$y(0) = y_0, y'(0) = y_1 \quad \text{in } \Omega$$

$$s'' + s = - \int_{\Gamma_i} y' n \quad \text{in } (0, T)$$

$$s(0) = s_0, s'(0) = s_1$$

## Resume

NSE + Lamé  $\longrightarrow$  NSE + Rigid Body

Eliminates infinite degrees of freedom for solid deformation. Infact, no solid deformation at all.

NSE + Rigid body  $\longrightarrow$  Stokes Model.

Allows some finite degrees of freedom for solid deformation; contains viscosity effects.

Stokes Model  $\longrightarrow$  Heat-Solid Model.

Eliminates pressure effects,

contains viscosity effects

contains some solid deformation.

Heat -Solid Model  $\longrightarrow$  Helmholtz Model.

Eliminates viscous effects

Adds weak compressibility effects

contains some solid deformation.

Linearization principle holds for control problems

Nonlinear equation  $\longrightarrow$  Linear equation

Viscous effects are treated via Carleman observability estimates.

# Finite Energy Solutions (Helmholtz Model)

Energy Space

$$Z = V \times L^2(\Omega) \times \mathbb{R}^2 \times \mathbb{R}^2$$

$$V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_e\}$$

Skew adjoint operator,

Energy conservation

Dynamics is well defined.

Stone's Theorem .

## Exact observability

Consider free dynamics (no control )

$$\begin{aligned}\phi'' - \Delta\phi &= 0 && \text{in } Q \\ \phi &= 0 && \text{on } \Sigma_e \\ \partial_n\phi &= w'n && \text{on } \Sigma_i \\ w'' + w &= - \int_{\Gamma_i} \phi'n && \text{in } (0, T)\end{aligned}$$

Initial conditions

$$\phi(0) = \phi_0, \quad \phi'(0) = \phi, \quad w(0) = w_0, \quad w'(0) = w_1$$

Observation is  $\phi'$  on  $U$  during  $0 < t < T$

$$\|(\phi_0, \phi_1, w_0, w_1)\|_{Egy}^2 \leq c \int_0^T \int_U |\phi'|^2$$

## **Exact observability estimate**

Infinite dimensional Kalman Rank Condition.

## Consequence of observability estimate

### Unique continuous principle (UCP)

observation  $\longrightarrow$  solution is injective

observation  $= 0 \Rightarrow$  solution  $= 0$ .

This is not obvious.

Its validity depends on  $(U, T)$

**Example.**  $U = \Omega$ .

UCP is not enough; we require observability estimate for controllability.

## Exact Controllability

**Theorem.** (Observability  $\Rightarrow$  Controllability)

Assume  $T > 0$ .

Assume Exact observability estimate for  $(U, T)$ .

Then for any  $(y_0, y_1, s_0, s_1) \in Z$ , there is a control  $u \in L^2(U \times (0, T))$  such that

$$(y(T), y'(T), s(T), s'(T)) = (0, 0, 0, 0).$$

Besides,  $u$  can be obtained as follows.

Define  $J : Z \longrightarrow \mathbb{R}$  by

$$J(\phi_0, \phi_1, w_0, w_1) = \frac{1}{2} \int_0^T \int_U |\phi'|^2 + \int_{\Omega} \nabla \phi_0 \cdot \nabla y_0 + \int_{\Omega} \phi_1 y_1 \\ + w_0 \cdot s_0 + w_1 \cdot s_1$$

Then  $J$  has a unique minimizer  $(\tilde{\phi}_0, \tilde{\phi}_1, \tilde{w}_0, \tilde{w}_1)$ .

Let  $(\tilde{\phi}, \tilde{w})$  be the corresponding solution .

Take  $u = \tilde{\phi} \mathbf{1}_U$ . Moreover this has minimal norm in  $L^2(U \times (0, T))$  among all possible controls.

## **Remark.**

Exact observability estimate ensures the coercivity of the functional  $J$  and thereby guarantees the stability of minimizing sequences.

## Proving Observability estimate (I)

- Multiplier Method
- Multiplier  $\rho\phi, \rho s$  with  $\rho(t) = t^2(T - t)^2$

$$\|(\phi_0, \phi_1, w_0, w_1)\|_Z \leq c \left\{ \int_0^T \int_{\Omega} |\phi'|^2 + \int_0^T |w|^2 \right\}$$

**Next step.** Compactness arguments

$$\int_0^T |w|^2 \leq c \int_0^T \int_{\Omega} |\phi'|^2$$

This method works if  $U = \Omega, T > 0$  arbitrary.

Control acts on entire fluid domain.

Observation is made on entire fluid domain.

## Proving observability (II)

### Frequency variable method

### Localized Control for Helmholtz Model

$$y'' - \Delta y = u 1_U \quad \text{in } Q$$

$$y = 0 \quad \text{on } \Sigma_e, \quad \partial_n y = s' \cdot n \quad \text{on } \Sigma_i$$

$$s'' + s = - \int_{\Gamma_i} y' n \quad \text{in } (0, T)$$

Initial Conditions.

**Theorem.** Assume  $U$  is a neighbourhood of  $\Gamma_e \cup \Gamma_i$ . Then exists time  $T > 0$  such that we have exact controllability and observability in the energy space. No information on time of controllability. Control, observations only in a neighbourhood of  $\Gamma_e$  and  $\Gamma_i$ .

## HELMHOLTZ Model as I order System

$$z'(t) = Az(t) + Bu(t)$$

$$z(t) = (y(t), y'(t), s(t), s'(t))$$

$$A(f, g, k, l) = (g, \Delta f, l, -k - \int_{\Gamma_i} gn)$$

$$D(A) = \{(f, g, k, l); f \in H^2(\Omega), g \in H^1(\Omega) \\ f = g = 0 \quad \text{on } \Gamma_e \\ \partial_n f = l \cdot n \quad \text{on } \Gamma_i \\ k, l \in \mathcal{C}^2\}$$

$$Bv = (0, v1_U, 0, 0)$$

$A$  is skew - adjoint  $A^* = -A$

$$B^*(f, g, k, l) = g1_U$$

Observability Inequality/Estimate

Passage : Time  $t \rightarrow$  Frequency  $\omega$

Infinite dimensional **Hautus test** (cf. TUCSNAK-WEISS)

**Proposition**  $z'(t) = Az(t) + Bu(t)$

$A$  skew - adjoint, with compact resolvent

$\{\Phi_k\}$  on basis of eigenvectors  $\{i\mu_k\}$  eigenvalues. Define

$$E_\lambda = \overline{\langle \Phi_k; |\mu_k| > \lambda \rangle}$$

(1) High Frequency Condition:

$\exists \delta > 0, \alpha > 0$  such that  $\forall \omega \in \mathbb{R}, |\omega| > \alpha$

$$\|(i\omega I - A)\Phi\|^2 + \|B^*\Phi\|^2 \geq \delta^2 \|\Phi\|^2$$

$$\forall \Phi \in E_\alpha \cap D(A).$$

(2) Low Frequency Condition:

$$B^* \Phi \neq 0 \quad \forall \text{ eigenvector } \Phi \text{ of } A$$

Then we have exact observability.

No information on time  $T$  of observability

(High & Low Frequency Conditions)

## Verification (2)

$$A\Phi = i\omega\Phi$$

. This is equivalent to

$$\left\{ \begin{array}{l} g = i\omega f, \quad \Delta f = i\omega g \\ l = i\omega k, \quad -k - \int_{\Gamma_i} gn = i\omega l \\ \text{Boundary Conditions} \end{array} \right.$$

- zero is not an eigenvalue of  $A$

- If  $B^*\Phi = 0$  then  $g = 0$  on  $U$ .

$$\Delta f + \omega^2 f = 0, f = 0 \text{ on } U. \text{ UCP} \Rightarrow f \equiv 0.$$

Hence  $\Phi = 0$

## Verification (1)

Proof by Contradiction, technical.

Roughly, in high frequency part, we can neglect solid

Helmholtz model  $\approx$  uncoupled model

$$y'' - \Delta y = 0 \text{ in } Q, y = 0 \text{ on } \Sigma_e,$$

$$\partial_n y = 0 \text{ on } \Sigma_i$$

choice of  $U \Rightarrow (1)$

## Proving Observability (III)

- Method of lateral propagation of energy for for the wave equation in one space variable.

**Theorem.** Assume  $U$  is a neighbourhood of  $\Gamma_e$  only. (No control near the solid boundary)

(H) Assume solid is a ball /disc.

Then observability estimate holds with an optimal time.

(H) ensures that no energy is trapped near the solid boundary.

Here, we need control/observation only in neighbourhood of  $\Gamma_e$  only. No control is required near the solid.

Introduce Polar coordinates  $(r, \theta)$ . Decomposition of the problem into spherical harmonics.

**Surprise.** Only the first harmonic mode is coupled to the solid. Other modes are free. Thus we are led to consider the wave equation in one -space variable coupled with the solid.

$$\psi'' - \frac{1}{r} \partial_r (r \partial_r \psi) + \frac{1}{r} \psi = 0$$

$$r_0 < r < R, 0 < t < T.$$

$$\psi(t, R) = 0$$

$$\partial_r \psi(t, r_0) = \pi w'(t)$$

$$w''(t) + w(t) = r_0 \psi'(t, r_0)$$

Initial conditions

Observability estimate

$$\begin{aligned} & \int_{r_0}^R |\psi'(0, r)|^2 r dr + \int_{r_0}^R |\partial_r \psi(0, r)|^2 r dr + \\ & + \int_{r_0}^R \frac{1}{r} |\psi(0, r)|^2 dr + |w'(0)|^2 + |w(0)|^2 \\ & \leq c \int_0^T \int_{R_0}^R |\psi'(t, r)|^2 r dr dt \end{aligned}$$

**UCP.** If  $T > 2(R_0 - r_0)$  and

$$\int_0^T \int_{R_0}^R |\psi'(t, r)|^2 r dr = 0$$

then  $(\psi, w) \equiv (0, 0)$ .

Lateral propagation of waves.

Interchange space and time.

Integrate energy density with respect to time  
at a fixed space point.

**Conjecture.** When solid region is convex, there is controllability/observability with control/observation made only on  $\Gamma_e$  with optimal time.

## References

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