Small time heat kernel asymptotics at the Riemannian and sub-Riemannian cut locus

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[1] D. Barilari, U. B. R. Neel, Small time heat kernel asymptotics at the sub-Riemannian cut-locus, to appear on JDG.

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an old question:

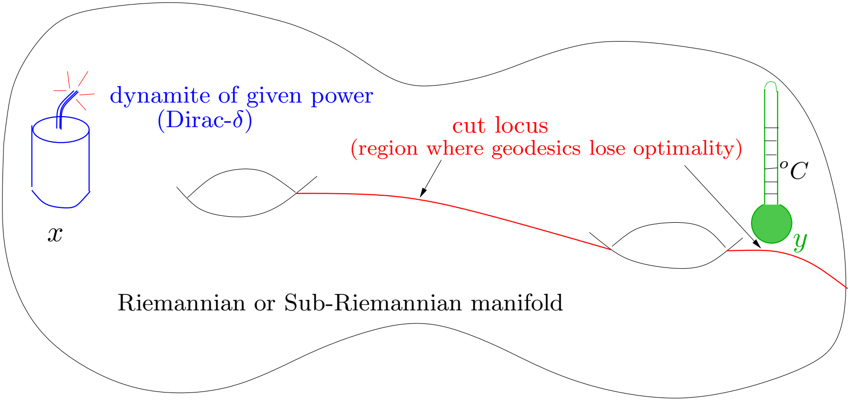
What is the relation between:

(sub)-Riemannian distance

the structure of optimal geodesics

$\longleftrightarrow$ small-time heat-kernel asymptotics

In particular, we are interested in what happens at the cut locus



Can we recognize that we are at the cut locus by measuring the heat?

Definition of sub-Riemannian structure

Definition

A sub-Riemannian manifold is a pair $(M, \{X_1, \dots, X_m\})$ such that $\{X_1, \dots, X_m\}$ satisfies the Hörmander condition.

$$\forall\ q \in M,\ \mathrm{Lie}_q\{X_1, \dots, X_m\} = T_q M$$

In general $\underbrace{dim(M)}_{n} \geq \underbrace{\dim(Span(X_1(q), \dots X_m(q)))}_{k(q)} \leq \underbrace{\#\text{of vector fields}}_{m}$

This definition includes:

structure		example
Riemannian struct. with M parallelizable	$n = k = m$	flat torus
Riemannian struct. with M non-parallelizable	$n = k < m$	2-sphere in $\mathbf{R}^3$
Carnot groups	$n > k = m$	Heisenberg
equiregular sub-Riemannian struct.	$n > k \leq m$	contact struct.
non-equiregular sub-Riemannian struct.	$n > k \leq m$	Martinet
rank-varying sub-Riemannian struct.	$n \geq k(q) \leq m$	Grushin
others		

Define $\blacktriangle^1(q) := Span(X_1(q), \dots X_m(q))$, $\blacktriangle^{i+1} := \blacktriangle^i + [\blacktriangle^i, \blacktriangle]$. If $\dim(\blacktriangle^i)$, $i = 1, \dots, m$ do not depend on the point, it is called **equiregular**

Horizontal Curves and Carnot-Caratheodory distance

Definition

A Lipschitz continuous curve $\gamma : [0,T] \to M$ is said to be *horizontal* if $\exists$ $u_1(.), \ldots, u_m(.) \in L^\infty([0,T], \mathbf{R})$, s.t.

$$\dot\gamma(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t)) \qquad \text{for a.e. } t \in [0,T].$$

Definition

The Carnot-Caratheodory distance is

$$d(q_0,q_1) = \inf\{\int_0^T \sqrt{u_1^2(t) + \ldots + u_m^2(t)}\, dt \ \mid \ \text{the corresponding trajectory joins } q_0 \text{ to } q_1\}$$

→thanks to the Hörmander condition, this distance gives to M a structure of metric space (compatible with its topology) of Hausdorff dimension

- $Q = n$ in the Riemannian case
- $Q = const > n$ for equiregular sub-Riemannian structures
- $Q = Q(q) \geq n$ in the general case

Laplace operator

Definition

the sub-Riemannian Laplacian is $\qquad \Delta\phi := \mathrm{div}\big(\mathrm{grad}_H(\phi)\big)$

where

- $\mathrm{grad}_H\phi = \sum_i^m X_i(\phi)X_i$
- div is the classical divergence computed with respect to a given smooth volume μ.

Remarks

- Second order terms do not depends on μ since

$$\Delta\phi = \sum_{i=1}^{m} X_i^2(\phi) + (div(X_i))X_i(\phi)$$

- If the structure is equiregular there is a regular intrinsic volume (Popp's volume) and the corresponding Laplacian is called “intrinsic”.
- In the Riemannian case Popp's volume is the Riemannian volume.

Existence of the heat kernel

Theorem (Hörmander, Strichartz)

Consider a sub-Riemannian manifold $(M, \{X_1, \ldots, X_k\})$ which is complete as metric space. Then

- *the sub-Riemannian Laplacian Δ w.r.t. a regular volume μ is hypoelliptic ($\Leftarrow$ Hörmander condition)*
- *The sub-Riemannian heat equation $\partial_t \phi(t,q) = \Delta\phi(t,q)$ admits a smooth kernel $p_t(x,y)$ ($\Leftarrow$ completeness)*

Computation of Minimizers:

Candidates minimizers are computed via the Pontryagin Maximum Principle

- normal extremals: projection on q of solutions of

$$H(q,\lambda) = \frac{1}{2}\sum_{1}^{m}\langle\lambda, X_i(q)\rangle^2$$

  lying on the level set $H = 1/2$

- abnormal extremals satisfy $\langle\lambda, X_i(q)\rangle \equiv 0$.

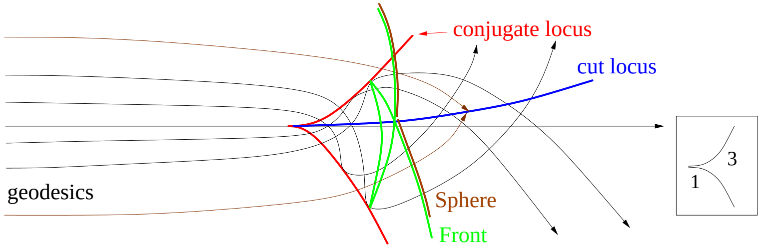
→Normal extremals are geodesics:

Definition

a geodesic is a curve $\gamma : [0,T] \to M$, parametrized by constant velocity, s.t. for every suff. small interval $[t_1,t_2] \subset [0,T]$, $\gamma|_{[t_1,t_2]}$ is optimal between $\gamma(t_1)$ and $\gamma(t_2)$.

→abnormal extremals can be geodesics or not. In this talk I will assume that there are no abnormal minimizers

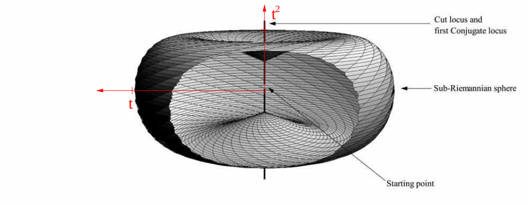
In any case one expects that candidate optimal trajectories loose optimality after some time.



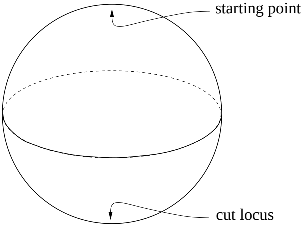
- conjugate locus: where local optimality is lost
  (the differential of the exponential map is degenerate)
- cut locus: where global optimality is lost
- sphere(ε): set of points at distance ε from a given point (level sets of the value function)
- front(ε): end point of geodesics at time ε from a given point

Recall that:

- if $\dim(M) > \dim(\text{Span}\{X_1, \ldots X_m\})$ then the cut locus and the conjugate locus are adiacent to the starting point



- If $\dim(M) = \dim(\text{Span}\{X_1, \ldots, X_m\})$ then they are far from the starting point



The relation between the distance and the kernel

Can we relate $p_t(x, y)$ with $d(x, y)$?

What is known in SRG?

Assume that there are no abnormal extremals

- **On the diagonal**.

$$p_t(x,x) = \frac{C + o(1)}{t^{Q/2}} \text{ (Ben Arous and Leandre, '91)} \tag{1}$$

  Here Q is the Hausdorff dimension

  $\rightarrow p_t(x,x) = \frac{1}{(4\pi t)^{n/2}}(1 + \frac{K(x)}{6}t + o(t))$
  (Riemannian, Minakshisundaram-Pleijel, 1949)

  $\rightarrow p_t(x,x) = \frac{1}{t^2}(1 + \frac{k(x)}{3}t + o(t))$ (3D contact, Barilari, 2012)

- **Off diagonal and off cut locus**. Fix $x \neq y$. If y is not in the cut locus of x

$$p_t(x,y) = \frac{C + o(1)}{t^{n/2}} e^{-d^2(x,y)/4t} \text{ (Ben Arous, '88)}$$

- **In any point of the space including the cut locus**.

$$\lim_{t \to 0} 4t \log p_t(x,y) = -d^2(x,y) \text{ (Leandre, '87)} \tag{2}$$

the gap: what happens on the cut locus?

Specific examples shows that on the cut locus

$$p_t(x,y) = \frac{C + o(1)}{t^r} e^{-d^2(x,y)/4t} \text{ with } r \geq n/2$$

in Riemannian

- on S^1 we have $r = 1/2 = n/2$
- on the cylinder we have that $r = 1 = n/2$
- on S^2 we have that $r = 3/2 > 1 = n/2$
(Fischer, Jungster, and Williams, 1984)

→why this difference?

in sub-Riemannian

- on the Heisenberg group on the z axis we have $r = 2 > 3/2 = n/2$
(by Gaveau 1977)

But the problem was open for 20 years.

reasons ???

- absence of results in the Riemannian case
  (however there was a pioneering ideas of Molcanov in '75 that was overlooked)
- no information on the cut locus in sub-Riemannian geometry besides those on the Heisenberg group and symmetric nilpotent $(n, n+1)$ groups

Now we have a better understanding of the cut locus in sub-Riemannian geometry (at least in STEP 2)

Complete results on:

- local structure in 3D contact (Agrachev, Gauthier and Kupka, '96)
- SU(2), SO(3), Sl(2) with the metric induced by the Killing form (Francesco Rossi and U.B., 2009)
- SE(2) by Yuri Sachkov (2010-2011)
- non-symmetric nilpotent (4,5) case (Barilari, U.B. 2013)

Partial results on

- nilpotent (4,10) by Brockett (????)
- nilpotent (3,6) by Myasnichenko (2002)
- nilpotent (2,3,4) (2,3,5) Yuri Sachkov (2004)

Heat-kernel asymptotic at the cut locus

Announcement

Geometry, Analysis and Dynamics on Sub-Riemannian Manifolds
IHP, Paris, Sep-Dec 2014

Organizing Committee: A. Agrachev, U.B, Y. Chitour, F. Jean, M. Sigalotti, L. Rifford

Scientific Commitee: A. Agrachev, L. Ambrosio, U. Boscain, Y. Chitour, R. Bryant, E. Falbel, A. Figalli, B. Franchi, J.P. Gauthier, N. Garofalo, F. Jean, I. Kupka, A. Malchiodi, R. Montgomery, P. Pansu, J. Petitot, L. Rifford, A. Sarychev, F. Serra Cassano, M. Sigalotti, E. Trélat, I. Zelenko.

- 4 courses at M2 level
- 4 workshops
- several thematic days
- many seminars

→there will be the possibility of financing students

→we have money for several invitations. We are looking for more ...

→www.cmap.polytechnique.fr/subriemannian

The Molcanov technique

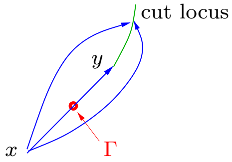
(how to get information on the heat kernel asymptotic at the cut locus)

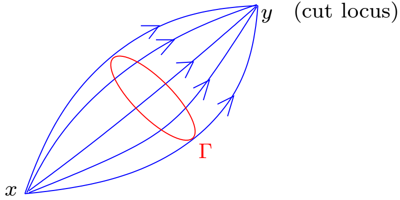
$\rightarrow$Assume that there are no abnormal minimizers.

By the semi-group property (or Chapman-Kolmogorov equation, for probabilists), we have

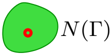
$$p_t(x,y) = \int_M p_{t/2}(x,z)p_{t/2}(z,y)\,\mu(dz)$$

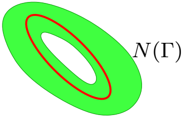
Let Γ the set of midpoints of the geodesics going from x to y.





$$p_t(x,y) = \int_{N(\Gamma)} p_{t/2}(x,z)p_{t/2}(z,y)\,\mu(dz) + \int_{M\setminus N(\Gamma)} p_{t/2}(x,z)p_{t/2}(z,y)\,\mu(dz)$$





First term $I_{N(\Gamma)} = \int_{N(\Gamma)} p_{t/2}(x,z)p_{t/2}(z,y)\,\mu(dz)$

On $N(\gamma)$ there are no cut points neither from x neither from $y \Rightarrow$ we can use the Ben Arous expansion

$$p_t(x,z) = \frac{1}{t^{n/2}} e^{-d^2(x,z)/4t}(C_1(x,z) + O(t)), \quad p_t(z,y) = \frac{1}{t^{n/2}} e^{-d^2(z,y)/4t}(C_2(z,y) + O(t))$$

Then

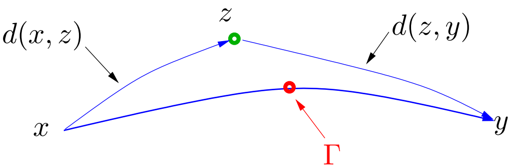
$$\begin{aligned} I_{N(\Gamma)} &= \int_{N(\Gamma)} \frac{1}{t^n} e^{-\frac{d^2(x,z)+d^2(z,y)}{4t}}(C(x,y,z) + O(t))\mu(dz) \\ &= \int_{N(\Gamma)} \frac{1}{t^n} e^{\frac{-h_{x,y}(z)}{t}}(C(x,y,z) + O(t))\mu(dz) \end{aligned}$$

Where $h_{x,y}(z) = \frac{d^2(x,z)+d^2(z,y)}{4}$ is called the **Hinged energy function**. Now

- For t small only the behaviour of $h_{x,y}(z)$ around its minimum is important (Laplace integral).
- For the same reason $\int_{M \setminus N(\Gamma)}$ is small

the hinged energy function and its minimum

$$h_{x,y}(z) = \frac{d^2(x,z) + d^2(z,y)}{4}$$



Lemma

$h_{x,y}(z)$ obtains its minimum exactly on Γ and it is smooth in a neighborhood of Γ.

The analysis of the asymptotic $I_{N(\Gamma)}$ permits to obtain

Theorem (Barilari, U.B., Neel)

Assume that there is only one optimal geodesic from x to y. If there exists a coordinate system around z_0 such that

$$h_{x,y}(z) = \frac{1}{4} d^2(x,y) + z_1^{2m_1} + \ldots + z_n^{2m_n} + o(|z_1|^{2m_1} + \ldots + |z_n|^{2m_n}) \quad (3)$$

for some integers $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n$ then

$$p_t(x,y) = \frac{C + o(1)}{t^{n - \sum_i \frac{1}{2m_i}}} \exp\left(-\frac{d^2(x,y)}{4t}\right). \quad (4)$$

- If the minimum is not degenerate then by Morse Lemma $h_{x,y}(z) = \frac{1}{4} d^2(x,y) + z_1^2 + \ldots + z_n^2$.
  In this case one gets $t^{n - n\frac{1}{2}} = t^{n/2}$
- If the number of minimal geodesics connecting x to y is not one but finite one gets sever contributions of the kind above
- If there exists a one (or more) parameter family of optimal geodesics joining x to y and coordinates such that $h_{x,y}$ does not depend on certain variables. Then some $m_i = +\infty$.

What is the relation among the expansion of $h_{x,y}(z)$ and the properties of optimal geodesics joining x to y?

Recall that geodesics are projections of solution to the Hamiltonian system defined by $H(q,\lambda) = \sum \langle \lambda, X_i(q) \rangle^2$ corresponding to the level set 1/2.

Define the exponential map $\mathcal{E}_x$ map as follows:

$$(\lambda_0, t) \in T_x^* M \cap \{H = 1/2\} \times \mathbf{R}^+ \to \{\text{projection of the solution starting from } (x, \lambda_0)\}$$

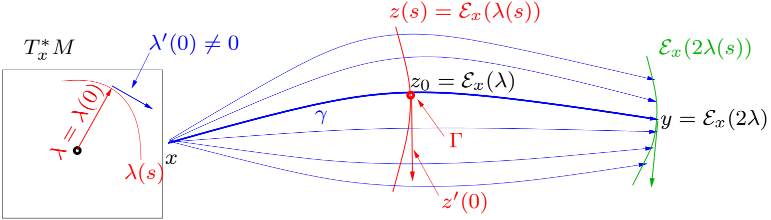
Properties:

- For every λ_0, $\gamma(t) = \mathcal{E}_x(\lambda_0, t)$ is a geodesic parameterized by arclength.
- $\mathcal{E}_x(\lambda_0, t)$ depends only on the product $\lambda_0 \, t$ i.e. We can consider it as a map from $T_x^* M$ to M.
- The first conjugate time of is $t_{con}(\gamma) = \min\{t > 0, \ (\lambda_0, t) \text{ is a critical point of } \mathcal{E}_x\}$.

Conjugacy of $\mathcal{E}_x$ and Degeneracy of $Hess_{z_0} h_{x,y}(z)$

Theorem (Barilari, U.B., Neel)

- *x and y are conjugate along γ if and only if the Hessian of $h_{x,y}$ at z_0 is degenerate.*
- *In particular γ is conjugate in the direction $\lambda'(0)$ (i.e. $\frac{d}{ds}\mathcal{E}_x(2\lambda(s))|_{s=0} = 0$) if and only if the Hessian of $h_{x,y}$ at z_0 is degenerate in the corresponding direction $z'(0)$ (i.e. $z'(0)$ if $\frac{d^2}{ds^2}h_{x,y}(z(s))|_{s=0} = 0$).*
- *The dimension of the space of perturbations for which γ is conjugate is equal to the dimension of the kernel of the Hessian of $h_{x,y}$ at z_0.*



The main result

Theorem (Barilari, U.B., Neel)

- *(less degenerate case): when x and y are not conjugate:*

$$h_{x,y}(z) = \frac{1}{4}d^2(x,y) + z_1^2 + \ldots + z_n^2 + o(|z_1|^2 + \ldots + |z_n|^2), \text{ and}$$
$$p_t(x,y) = \frac{C + O(t)}{t^{n/2}} e^{-d^2(x,y)/4t},$$

- *(most degenerate case): when the only non degenerate direction is t:*

$$h_{x,y}(z) = \frac{1}{4}d^2(x,y) + z_1^2 + o(|z_1|^2) \text{ and}$$
$$p_t(x,y) = \frac{C + O(t)}{t^{n-(1/2)}} e^{-d^2(x,y)/4t}$$

- *when the degeneration is only in one direction and it is "minimal":*

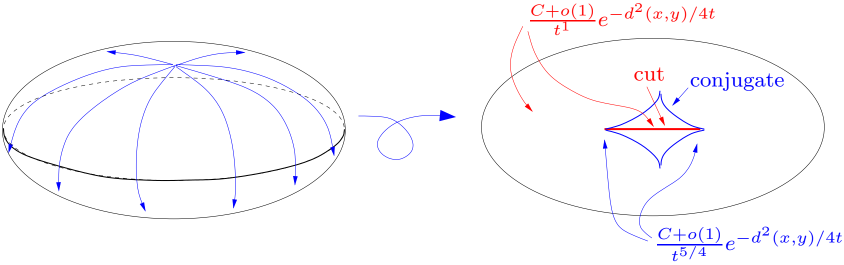
$$h_{x,y}(z) = \frac{1}{4}d^2(x,y) + z_1^2 + \ldots + z_{n-1}^2 + z_n^4 + o(|z_1|^2 + \ldots + |z_n|^4), \text{ and}$$
$$p_t(x,y) = \frac{C + O(t)}{t^{(n/2)+(1/4)}} e^{-d^2(x,y)/4t}, \tag{5}$$

the case of a Riemannian surface (with G. Charlot)

For a generic conjugate point on a surface we get

$$P_t(x,y) = \frac{C+O(t)}{t^{2-(1/2+1/4)}} e^{-d^2(x,y)/4t} = \frac{C+O(t)}{t^{5/4}} e^{-d^2(x,y)/4t}$$

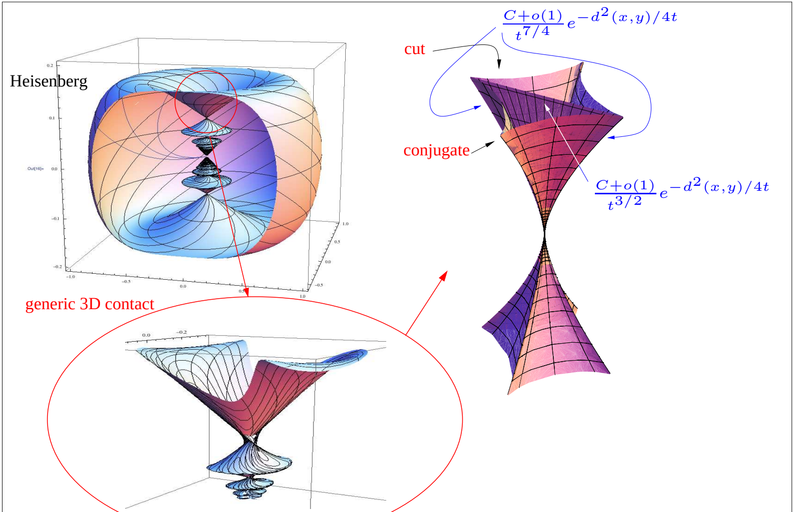
This was not known even for the Ellipsoid (see Barilari-Jendrej, 2012)



the local 3D contact case (with G. Charlot)

For a generic conjugate point in 3D contact we get

$$P_t(x,y) = \frac{C + O(t)}{t^{3-(1/2+1/2+1/4)}} e^{-d^2(x,y)/4t} = \frac{C + O(t)}{t^{7/4}} e^{-d^2(x,y)/4t}$$



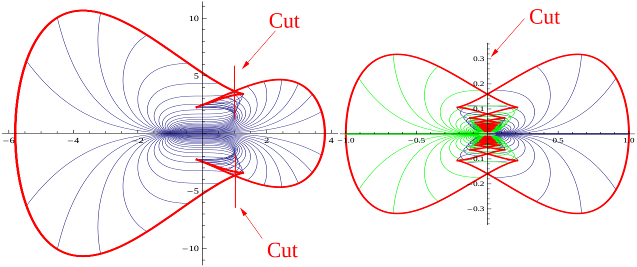
Remarks

- In general h is not "diagonalizable" and there are mixed terms.
- case with abnormals ???

Grushin-Baouendi: $X_1 = (1, 0)$, $X_2 = (0, x)$

$$\mu = dx\,dy$$

$$\Delta = \partial_x^2 + x^2 \partial_y^2$$



	$p_t(q, q')$ q Riemannian point	$p_t(q, q')$ q degenerate point
diagonal (Leandre Ben Arous)	$\sim \frac{C}{t}$	$\sim \frac{C}{t^{3/2}}$
off diagonal off cut (Ben Arous)	$\sim \frac{C}{t} e^{-d^2(q,q')/(4t)}$	$\sim \frac{C}{t} e^{-d^2(q,q')/(4t)}$
off diagonal cut (non-conjugate)	$\sim \frac{C}{t} e^{-d^2(q,q')/(4t)}$	$\sim \frac{C}{t} e^{-d^2(q,q')/(4t)}$
off diagonal cut conjugate	$\sim \frac{C}{t^{5/4}} e^{-d^2(q,q')/(4t)}$	— (no cut conjugate)

Recall that if μ is the Riemannian volume $\frac{1}{|x|}dx\,dy$ then the Laplace-Beltrami operator for the Grushin metric is

$$\Delta = \partial_x^2 + x^2\partial_y^2 - \frac{1}{x}\partial_x$$

which is essentially self-adjoint on the half plane. Hence no heat is passing through the Grushin set. [C. Laurent, U.B. Annales Institut Fourier, to appear]

Thanks