Observability of gravity water-waves

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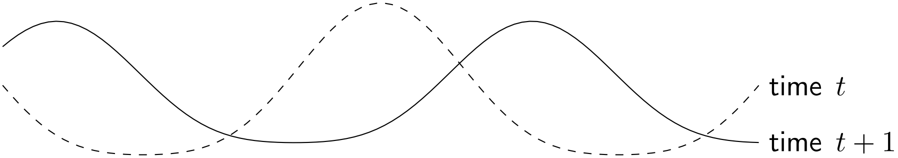


Water waves are, in general, generated by the motion of a solid component of the boundary or by impulsive pressures (blowing) applied on the free surface.

Question: which waves can be generated ?

Question: which 2d or 3d water waves can be generated in a tank?

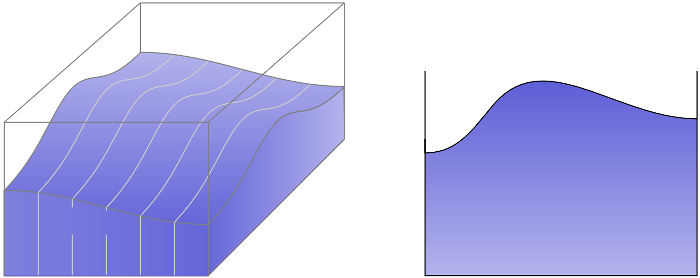


Figure: 3d and 2d waves in a rectangular tank

Main results

1) With Baldi and Han-Kwan: **controllability** of gravity-capillary waves

2) **Observability** of gravity water waves

There are many results for equations describing water waves : Benjamin-Ono, KdV, Saint-Venant; see works by Cerpa, Crépeau, Coron, Dubois, Glass, Guerrero, Laurent, Linares, Ortega, Petit, Rosier, Rouchon, Russell, Zhang....

Here we consider the dynamics of an incompressible, irrotational liquid flow

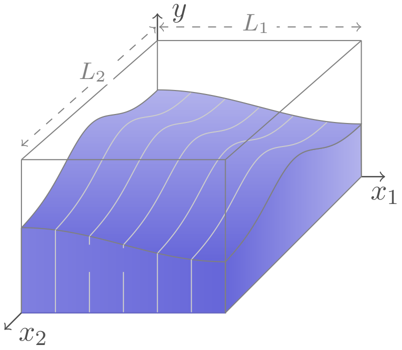
- moving under the force of gravitation and/or surface tension,
- in a time-dependent domain Ω with a free boundary.

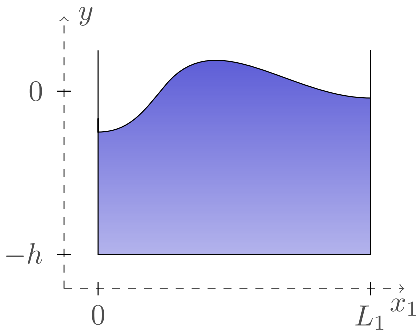
This problem is **quasi linear** and it is not a partial differential equation but instead a pseudo-differential equation, involving the Dirichlet-Neumann operator which is **nonlocal** (Ψ DO).

The fluid domain Ω has a **free surface**. At time $t \geq 0$,

$$\Omega(t) = \{ (x,y) \in Q \times \mathbb{R} \, : \, -h < y < \eta(t,x) \},$$

where η is an unknown, $Q = [0, L_1] \times [0, L_2]$ or $Q = [0, L_1]$.





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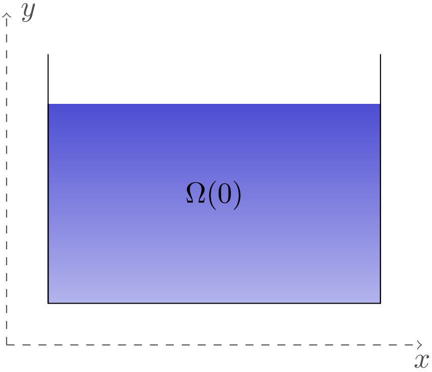
$$
\begin{aligned}
&\partial_t v + v \cdot \nabla v + \nabla(P + gy) = 0 && \text{in } \Omega \\
&\operatorname{div} v = 0 && \text{in } \Omega \\
&v \cdot n = 0 && \text{on the bottom and walls} \\
&\partial_t \eta = \sqrt{1 + |\nabla \eta|^2}\, v \cdot n && \text{on the free surface} \\
&P - P_{ext} = \kappa \operatorname{div} \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) && \text{on the free surface}
\end{aligned}
$$

g gravity, P pressure, P_{ext} external pressure, κ surface tension.

Moreover $\operatorname{curl} v = 0$ so that $v = \nabla \phi$.

One problem



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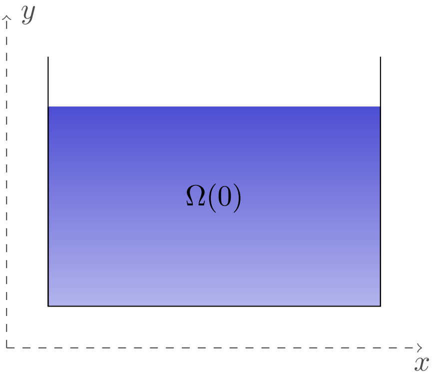

Figure: Initial and desired state at time T

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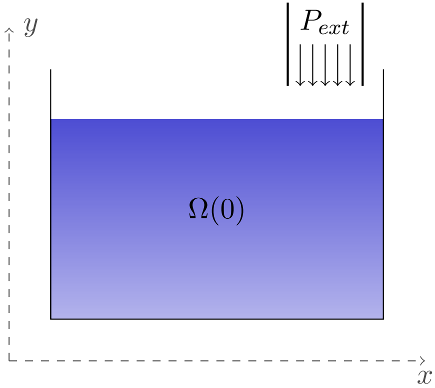

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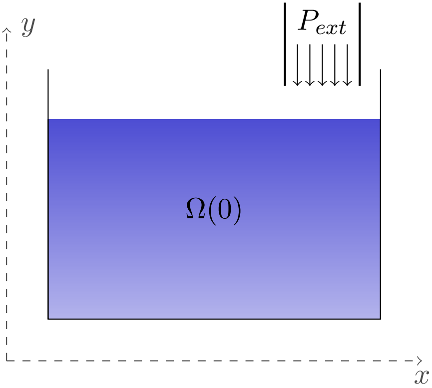

Figure: Initial and desired state at time T

Then $\operatorname{curl} v = 0$ and the assumption $v = \nabla\phi$ holds.

Reduction to the case

$$\Omega(t) = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \,:\, -h < y < \eta(t,x) \}.$$

Periodization

Justification by ABZ / Thibault de Poyferré.

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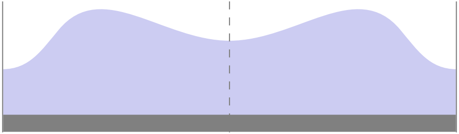

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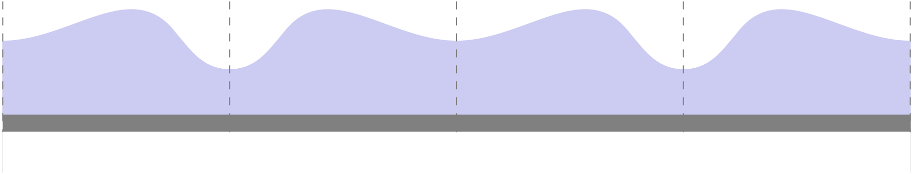

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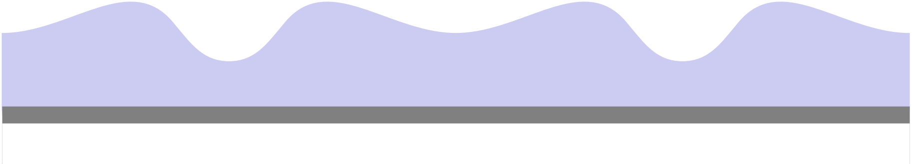

Justification by ABZ / Thibault de Poyferré.

Case 1: with surface tension
Local controllability of 2D gravity-capillary water waves

Notations: $d = 1$, $\psi(t,x) = \phi(t,x,\eta(t,x))$

Theorem (T.A., Baldi, Han-Kwan)

Let $T > 0$ *and consider* $\omega \subset \mathbb{T}$. *There exist* $s > 0$ *(large) and* $M_0 > 0$ *(small) s.t. for any* (η_{in}, ψ_{in}), $(\eta_{final}, \psi_{final})$ *in* $H_0^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$

$$\|\eta_{in}\|_{H^{s+\frac{1}{2}}} + \|\psi_{in}\|_{H^s} < M_0, \quad \|\eta_{final}\|_{H^{s+\frac{1}{2}}} + \|\psi_{final}\|_{H^s} < M_0,$$

there exists P_{ext} *in* $C^0([0,T]; H^s(\mathbb{T}))$ *supported in* $[0,T] \times \omega$, *such that the Cauchy problem with data* $(\eta|_{t=0}, \psi|_{t=0}) = (\eta_{in}, \psi_{in})$ *has a unique solution*

$$(\eta, \psi) \in C^0([0,T]; H_0^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T}))$$

satisfying

$$(\eta|_{t=T}, \psi|_{t=T}) = (\eta_{final}, \psi_{final}).$$

Linearized equation (neglect gravity):

$$\begin{cases} \partial_t \eta = |D_x| \psi \\ \partial_t \psi - \partial_x^2 \eta = P_{ext} \end{cases}$$

Then $u = \psi - i|D_x|^{\frac{1}{2}}\eta$ satisfies the **dispersive** equation

$$\partial_t u + i\, |D_x|^{\frac{3}{2}}\, u = P_{ext}.$$

The dispersive analysis of the nonlinear water waves is based on

- study in Eulerian coordinates (Zakharov, Craig-Sulem, Lannes)
- complete paralinearization of the equations (A-Métivier)
- Symmetrization (A-Burq-Zuily)
- normal forms (A-Delort; A-Baldi)

(**Oversymplifying**) One can rewrite the WW system as:

$$\frac{\partial u}{\partial t} + V(u)\partial_x u + i\, |D_x|^{\frac{3}{4}} \left(c(u)\, |D_x|^{\frac{3}{4}}\, u \right) = P_{ext}$$

where V, c are real-valued functions.

The linearized system at the origin has constant coefficient and can be controlled by means of Fourier analysis, Reid (1995) or multipliers Biccari (2015). But this is not enough since the problem is quasi-linear.

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$$P = \partial_t + V\partial_x + i\,|D_x|^{\frac{3}{4}}\left(c\,|D_x|^{\frac{3}{4}}\,\cdot\,\right)$$

where $V = V(\underline{u})$ and $c = c(\underline{u})$ are real-valued and $c-1$ is small enough.

Using 3 change of variables (preserving the L^2-norm in x)

$$(1+\partial_x\kappa(t,x))^{\frac{1}{2}}\,h(t,x+\kappa(t,x))$$

$$h(a(t),x)$$

$$h(t,x-b(t))$$

we replace P by

$$Q = \partial_t + W\partial_x + i\,|D_x|^{\frac{3}{2}} + R$$

where $W = W(t,x)$ satisfies $\int_{\mathbb{T}} W(t,x)\,dx = 0$ and R is of order zero.

- Nontrivial since the equation is nonlocal and cancellation of the term of order $1/2$.

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To study $\partial_t + W\partial_x + i\,|D_x|^{\frac{3}{2}} + R'$, we seek an operator A such that

$$i\big[A, |D_x|^{\frac{3}{2}}\big] + W\partial_x A \quad \text{is a zero order operator}$$

We find (study of the standing wave problem A.-Baldi) an operator of the form

$$A = \mathrm{Op}\left(q(t,x,\xi)e^{i\beta(t,x)|\xi|^{\frac{1}{2}}}\right)$$

with

$$\beta = \beta_0(t) + \frac{2}{3\sqrt{\kappa}}\partial_x^{-1}W.$$

Then

$$\big(\partial_t + W\partial_x + i\,|D_x|^{\frac{3}{2}}\big)A = A\big(\partial_t + i\,|D_x|^{\frac{3}{2}} + R''\big)$$

with R'' of order 0.

Notice that $A \in \mathrm{Op}\, S^0_{\rho,\rho}$ with $\rho = 1/2$ (quasi-linear). For Benjamin-Ono one has a similar conjugation but with $A \in \mathrm{Op}\, S^0_{1,0}$ (semi-linear).

Ingham type inequality

For some given real-valued function $\beta \in C^3(\mathbb{R})$, set

$$\mu_n(t) = \mathrm{sign}(n) \left[|n|^{\frac{3}{2}} t + \beta(t) |n|^{\frac{1}{2}} \right].$$

For any $T \in (0,1]$ there are $C(T)$ and $\delta(T)$ such that, if

$$\left\| (\partial_t \beta, \partial_t^2 \beta, \partial_t^3 \beta) \right\|_{L^\infty} \leq \delta(T)$$

then

$$C(T) \sum_{n \in \mathbb{Z}} |w_n|^2 \leq \int_0^T \left| \sum_{n \in \mathbb{Z}} w_n e^{i\mu_n(t)} \right|^2 dt.$$

For $\beta = 0$: Ingham, Kahane, Ball-Slemrod, Haraux.
$\beta(t)|n|^{\frac{1}{2}}$ is sub-principal but not perturbative.

Then Observability (of the real part only), HUM -> L^2 control, H^s control, quasi-linear scheme...

To obtain an Ingham inequality we used in an essential way the fact the equation is dispersive and the infinite speed of propagation ($3/2 > 1$):

$$\mu_n(t) = \text{sign}(n)\left[|n|^{\frac{3}{2}}t + \beta(t)|n|^{\frac{1}{2}}\right].$$

Now we consider the case **without surface tension :**

By contrast, for gravity waves (without surface tension) one has an exponent $1/2 < 1$ and one does not expect the same result (high frequencies travel at a low speed).

Microlocal (x, ξ) –> **Global** $\int f(x)dx$

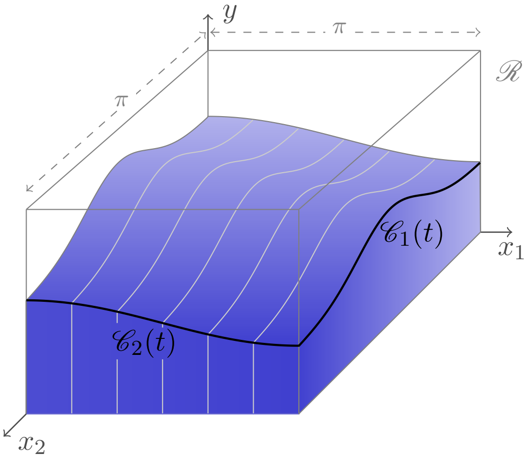
Some computations inspired by the works by **Benjamin–Olver** of the conservation laws for WW.

BOUNDARY OBSERVABILITY

Banksy

The only <u>known coercive quantity</u> is the **energy** :

$$\mathcal{H} = \frac{g}{2} \int \eta^2(t,x)\, dx + \frac{1}{2} \iint_{\Omega(t)} |\nabla_{x,y}\phi(t,x,y)|^2 \; dxdy.$$



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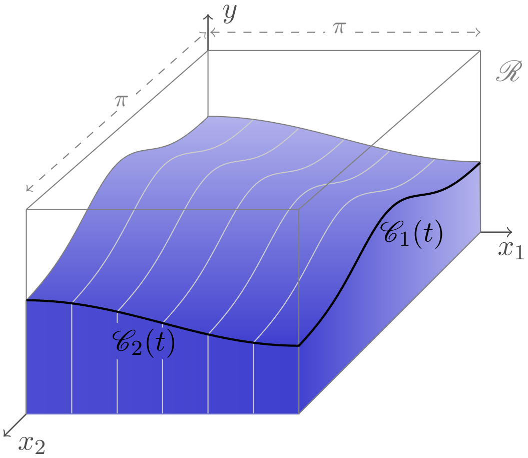


Estimate $\mathcal{H}$ by looking only at the motion of some of the curves of contact between the free surface and the vertical walls?

Theorem (A. 2015)

Recall $\psi(t,x) = \phi(t,x,\eta(t,x))$. *Set*

$$\Theta := -\eta \partial_t \psi - \frac{g}{2}\eta^2.$$

Let $\chi \in C_0^\infty((0,\pi)^2)$. *There exist* K_0, κ, c *s.t., for any* N *in* $\mathbb{N}$, *if*

$$\eta_0, \psi_0 = \chi(x) \sum_{|n|+|m|\leq N} a_{nm} \cos(nx_1)\cos(mx_2) \quad \textit{with} \quad |a_{nm}| \leq cN^{-\kappa},$$

then the solution exists on $[0, T_N]$ *with* $T_N = K_0 + K_0 N^{\frac{1}{2}+\varepsilon}$ *and*

$$\int_0^{T_N} \left[\int_0^\pi \Theta(t,\pi,x_2)\, dx_2 + \int_0^\pi \Theta(t,x_1,\pi)\, dx_1 \right] dt \geq \mathcal{H}.$$

Sharp: a wave-packet travels at a speed $1/\sqrt{N}$ and might take a time $\sqrt{N}$ to reach the boundary.
Consider a 1D wave-packet solving the linearized equation :

$$\partial_t u + \omega(D_x)u = 0, \quad \omega(\xi) = \sqrt{|\xi|}, \quad u(0,x) = h(x)\exp\left(iNx\right).$$

Then

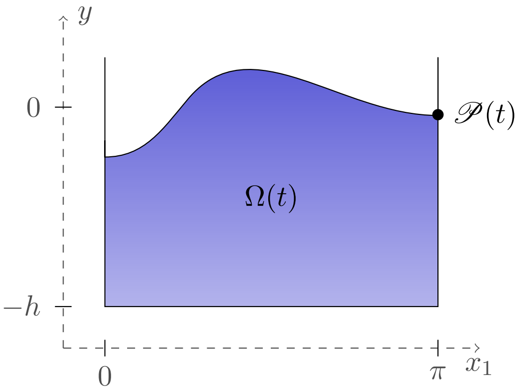
$$u(t,x) = a(t,x)\exp\left(i\left[Nx - \omega\left(N\right)t\right]\right)$$

where

$$a(t,x) = h\left(x - \omega'\left(N\right)t\right) + R$$

with

$$\omega'\left(N\right) = \frac{1}{2\sqrt{N}}, \quad \left\|R(t)\right\|_{H^s} \leq CN^{-\frac{3}{2}}\left(1+t\right)\left\|h\right\|_{H^s}.$$



Consider a smooth enough solution of the water waves equations which is independent of x_2. Introduce

$$m(t) = \eta(t, \pi).$$

Then

$$\Theta(t,\pi) := \Big(-\eta\partial_t\psi - \frac{g}{2}\eta^2 \Big)(t,\pi) = \frac{1}{2}\big[g m(t)^2 - m(t) m'(t)^2 \big].$$

Example: consider the 1D wave eq with Dirichlet boundary condition:

$$\partial_t^2 u - \partial_x^2 u = 0, \quad u(t,0) = u(t,1) = 0.$$

Multiply the equation by $x\partial_x u$ and integrate by parts

$$\int_0^T (\partial_x u(t,1))^2 \, dt = 2 \int_0^1 (\partial_t u)(x\partial_x u) \, dx \Big|_0^T + \iint_S \left[(\partial_t u)^2 + (\partial_x u)^2\right] dxdt$$

where $S = (0,T) \times (0,1)$. Since

$$\left| \int_0^1 (\partial_t u)(x\partial_x u) \, dx \right| \le \mathcal{E} := \frac{1}{2} \int_0^1 \left[(\partial_t u)^2 + (\partial_x u)^2\right] dx,$$

and $d\mathcal{E}/dt = 0$, one has

$$\int_0^T (\partial_x u(t,1))^2 \, dt \ge (T-2) \int_0^1 \left[(\partial_t u)^2 + (\partial_x u)^2\right](0,x) \, dx.$$

Lemma

For smooth enough solutions, the following property holds:

With $m(t) = \eta(t,\pi)$ *and* $\Theta(t,\pi) = \frac{1}{2}\big[gm(t)^2 - m(t)m'(t)^2\big]$ *one has*

$$\pi \int_0^T \Theta(t,\pi)\,dt = \frac{T}{2}\mathcal{H} \qquad \textit{energy}$$

$$+ \frac{\pi}{2}\int_0^T \int_{-h}^{m(t)} (\partial_y \phi)^2(t,\pi,y)\,dydt \qquad \textit{positive}$$

$$+ \frac{1}{2}\int_0^T \int_0^\pi \left(h + \frac{7}{4}\eta\right)(\partial_x \phi)^2(t,x,-h)\,dxdt \qquad \textit{positive}$$

$$- \frac{1}{4}\int_0^\pi \eta\psi\,dx\,\Big|_{t=0}^{t=T} - \int_0^\pi x\eta\partial_x\psi\,dx\,\Big|_{t=0}^{t=T} \qquad \textit{boundary}$$

$$- \frac{7}{4}\int_0^T \iint_{\Omega(t)} (\partial_x \eta)(\partial_x \phi)(\partial_y \phi)\,dxdydt \qquad \textit{remainder.}$$

Remark: The identity holds more generally with parameters. Consequently, if

$$\eta(t,x) = \varepsilon\eta'(\sqrt{\mu}t, \sqrt{\mu}x), \quad \phi(t,x,y) = \frac{\varepsilon}{\sqrt{\mu}}\phi'(\sqrt{\mu}t, \sqrt{\mu}x, y),$$

then one recovers the identity for $\partial_t^2 u - \partial_x^2 u = 0$ as $\varepsilon \to 0$ and $\mu \to 0$.

Cauchy problem

Balance between:

- rough initial data since the reflected/periodized domain will not be smooth. Only Lipschitz continuous in general but here the angle between the free surface and the vertical boundary of the canal is a right angle so the reflected/periodized domain enjoy additional smoothness.
- On the other hand, we need to consider smooth enough solutions to justify the computations.

Example: One can consider initial data (η_0, ψ_0) with

$$\psi_0 = 0 \quad \text{and } \eta_0 \in H^s(Q) \quad \text{with } s \in (d/2 + 2, 7/2).$$

Ros-Oton and Serra (see also Biccari). Pohozaev identity with Dirichlet boundary condition: $s \in (0,1)$ and $u \in H^s(\mathbb{R}^d)$ vanishes in $\mathbb{R}^d \setminus Q$, then

$$\int_Q (x\cdot\nabla u)(-\Delta)^{\frac{1}{2}} u\,dx = \frac{2s-d}{2}\int_Q u(-\Delta)^s u\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial Q}\left(\frac{u}{\operatorname{dist}(x,\partial Q)^s}\right)^2 (x\cdot\nu)\,dS.$$

Here

$$\begin{aligned}\int_Q (x\cdot\nabla\psi)G(\eta)\psi\,dx &= \frac{1}{2}\int_R \left|\nabla_{x,y}\phi\right|^2 \begin{pmatrix} x \\ y\end{pmatrix}\cdot n\,dS - \frac{d-1}{2}\iint_\Omega \left|\nabla_{x,y}\phi\right|^2\,dxdy \\ &+ \frac{1}{2}\int_Q (\eta - x\cdot\nabla\eta)\big[(\partial_x\phi)^2 + (\partial_y\phi)^2 - 2(\partial_y\phi)(\partial_n\phi)\big]\,\big|_{y=\eta}\,dx,\end{aligned}$$

Multiplier method used in this way: Compute, by two different methods,

$$I := \iint_{[0,T]\times[0,\pi]} \left[(\partial_t \eta)(x\partial_x \psi) - (\partial_t \psi)(x\partial_x \eta)\right] dxdt$$

Method 1: Integration by parts / Method 2 : write

$$\text{(Craig-Sulem-Zakharov)} \qquad \partial_t \eta = \frac{\delta \mathcal{H}}{\delta \psi}, \quad \partial_t \psi = -\frac{\delta \mathcal{H}}{\delta \eta},$$

to observe that

$$I = \iint_{[0,T]\times[0,\pi]} \left[x\psi_x \frac{\delta \mathcal{H}}{\delta \psi} + x\eta_x \frac{\delta \mathcal{H}}{\delta \eta}\right] dxdt.$$

Then use Lannes' derivative formula for the DN to obtain

$$\begin{aligned}\frac{1}{2}\int \left[m(t)^2 - m(t)m'(t)^2\right] dt = {} & \frac{1}{4}\iint \psi G(\eta)\psi \, dxdt + \frac{1}{4}\iint \eta^2 \, dxdt \\ & - \frac{1}{4}\int \eta\psi \, dx \Big|_{t=0}^{t=T} - \int x\eta\partial_x \psi \, dx \Big|_{t=0}^{t=T} \\ & + R\end{aligned}$$

with

$$R := \iint \left[\frac{3}{8}\eta\left(V^2 + 2BV\partial_x\eta - B^2\right) + \frac{1}{2}(G(\eta)\psi)xV + \frac{1}{2}B(x\psi_x)\right] dxdt.$$

Guided by Benjamin–Olver, use

$$\int u(x,\eta(x))\,dx + \int f(x,\eta(x))\partial_x\eta\,dx = \iint (\partial_y u - \partial_x f)\,dydx + \int u(x,-h)\,dx + \int f\,dy\Big|_{x=0}^{x=\pi}.$$

Then

$$R = \frac{1}{2}\int_0^T \int_{-h}^{m(t)} (\partial_y\phi)^2(t,1,y)\,dydt + \frac{1}{2}\int_0^T\int_0^1 \left(h + \frac{7}{4}\eta\right)(\partial_x\phi)^2(t,x,-h)\,dxdt - \frac{7}{4}\int_0^T\iint_{\Omega(t)} (\partial_x\eta)(\partial_x\phi)(\partial_y\phi)\,dxdydt.$$

$$\int_0^\pi \left| |D_x|^{\frac{1}{2}} \psi \right|^2 dx \lesssim \frac{1}{2} \iint_{\Omega(t)} |\nabla_{x,y}\phi(t,x,y)|^2 \, dxdy \leq \mathcal{H}.$$

Corollary

Assume that

$$|\partial_x \eta(t,x)| \leq \frac{1}{7}, \quad T \geq 4 + \frac{20\pi}{\sqrt{g}} A, \quad \|\partial_x \psi\|_{L^2} \leq \sqrt{2} A \left\| |D_x|^{\frac{1}{2}} \psi \right\|_{L^2}.$$

Then

$$\pi \int_0^T \Theta(t,\pi) \, dt \geq \mathcal{H}.$$

The assumption for $\|\partial_x \psi\|_{L^2}$ holds at $t = 0$ with $A = K\sqrt{N}$ if the Fourier transform of $\psi(0)$ is supported in $[-N, N]$. Moreover, for small data, one can propagate the estimate $\|\partial_x \psi(0)\|_{L^2} \lesssim K\sqrt{N} \|\psi(0)\|_{\dot{H}^{1/2}}$ on large time intervals of size $\sqrt{N}$.

Thank you!