Controllability and Feedback Stabilization of a 1d simplified fluid-structure system

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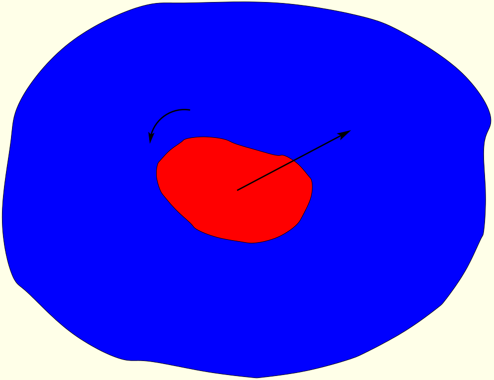
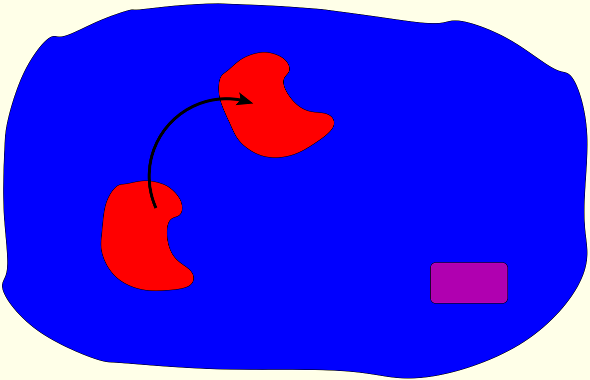
Presentation of the problem



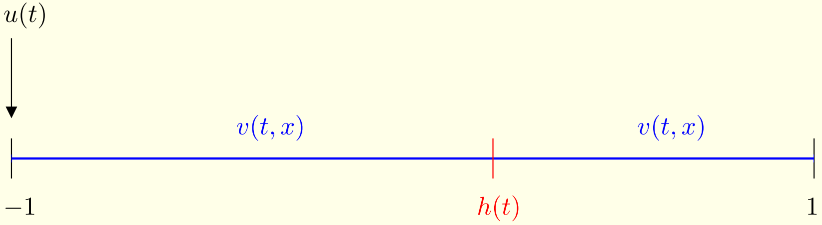
- Fluid: viscous, incompressible (Navier-Stokes)
- Structure: rigid body (Newton's laws)

Controllability results



- 1d: Doubova and Fernandez-Cara, 2005
- 2d: Imanuvilov and T., 2007, case of rigid ball
- 2d: Boulakia and Osses, 2008, rigid body with geometrical properties
- 3d: Boulakia and Guerrero, 2011

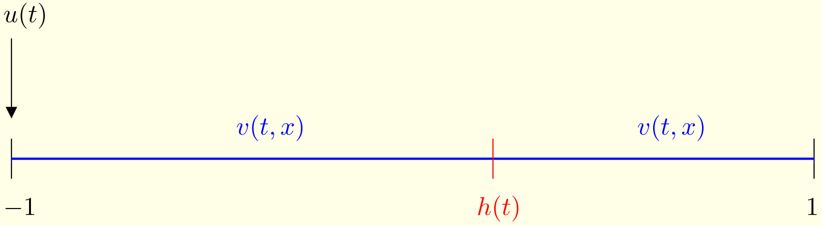
Notations



- $h = h(t)$ trajectory of the particle
- $v = v(t, x)$ fluid velocity, defined in the fluid domain $[-1, 1] \setminus \{h(t)\}$.
- $u = u(t)$ control of the system, velocity of the fluid at $x = -1$.

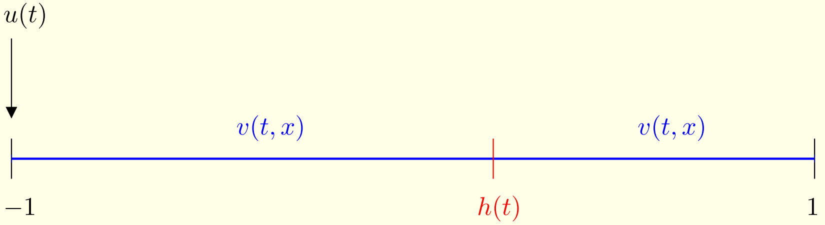
$$[f](x) = f(x^+) - f(x^-).$$

The 1d simplified model



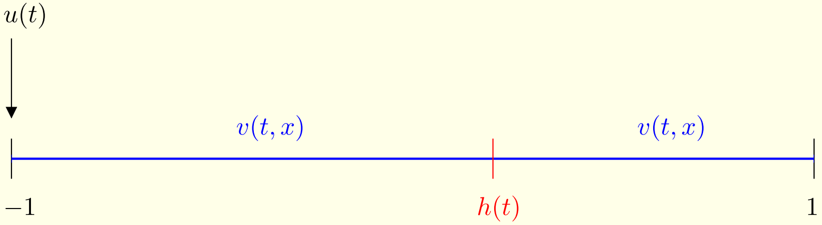
$$
\begin{cases}
v_t - v_{xx} + v v_x = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\[2em]
v(t,-1) = u(t), \quad v(t,1) = 0 \qquad (t \geqslant 0),
\end{cases}
$$

The 1d simplified model



$$\begin{cases} v_t - v_{xx} + v v_x = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t, h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ \\ v(t,-1) = u(t), \quad v(t,1) = 0 \qquad (t \geqslant 0), \end{cases}$$

The 1d simplified model



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\begin{cases}
v_t - v_{xx} + v v_x = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\
v(t, h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\
m\ddot{h}(t) = [v_x](t, h(t)) \qquad (t \geqslant 0), \\
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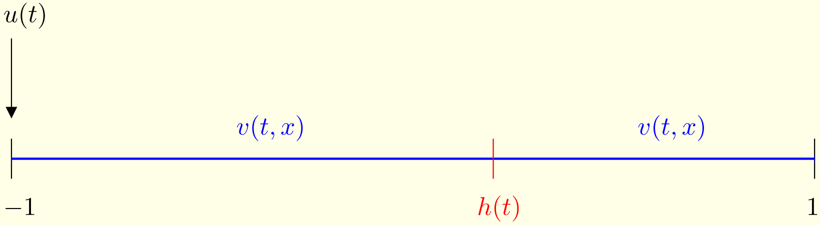
The 1d simplified model



$$\begin{cases} v_t - v_{xx} + vv_x = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ m\ddot{h}(t) = [v_x](t,h(t)) \qquad (t \geqslant 0), \\ v(t,-1) = u(t), \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{h_0\}. \end{cases} \tag{1}$$

The controllability result

Theorem

Let $\tau > 0$. For initial data (position and velocities)

$$h_0 \in (-1,1), \quad h_1 \in \mathbb{R}, \quad v_0 \in H_0^1(-1,1), \quad v_0(h_0) = h_1,$$

small enough

$$|h_0| + |h_1| + \|v_0\|_{H_0^1(-1,1)} \leqslant r,$$

there exists $u \in C([0,\tau])$ such that

$$v \in L^2([0,\tau], H^2((-1,1)\backslash\{h(t)\})) \cap C([0,\tau], H^1(-1,1)), \quad h \in C^1[0,\tau],$$

and

$$\dot{h}(\tau) = 0, \quad v(\tau,\cdot) = 0 \quad \textit{and} \quad h(\tau) = 0.$$

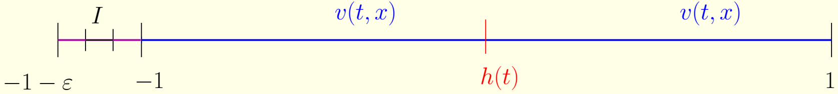
Comments

- Well-posedness: Vázquez and Zuazua
- Previous controllability results: Doubova and Fernández-Cara, use of two controls, one at each boundary.
- Classical method for parabolic systems: we first prove the internal controllability and deduce from it the boundary controllability.

Difficulties

- Nonlinearity from the Burgers equation;
- Coupling between the Burgers equation and the ODE;
- We want to control both the velocity and the position of the particle;
- The domain of the Burgers equation is moving in time and depends on the position of the particle.

From interior null-controllability to boundary null-controllability:



We extend v_0 by 0 outside $(-2, -1)$ and consider the control problem

$$\begin{cases} v_t - v_{xx} + v v_x = \hat{u}(t,x) \qquad (t \geqslant 0, \quad x \in [-2,1] \setminus \{h(t)\}), \\ v(t, h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ m\ddot{h}(t) = [v_x](t, h(t)) \qquad (t \geqslant 0), \\ v(t,-2) = 0, \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-2,1] \setminus \{h_0\}. \end{cases}$$

Change of variables

$$y = Y(t,x) = \begin{cases} \dfrac{(x-h(t))}{1-h(t)}, x > h(t), \\[2ex] \dfrac{2(x-h(t))}{1+h(t)}, x < h(t). \end{cases}$$

We define a new function V by setting

$$V(t, Y(t,x)) = v(t,x) \qquad (t \in [0,\tau],\ x \in (-2, h(t)) \cup (h(t), 1)).$$

System after change of variables

$$\begin{cases} V_t - A_h V_{yy} + B_h V_y V + C_h \dot{h} V_y = u(t,y), \\ \qquad\qquad\qquad\qquad (t \geqslant 0, \quad y \in [-2,1] \setminus \{0\}), \\ m\ddot{h}(t) = [D_h V_y](t,0), \qquad (t \geqslant 0), \\ V(t,0) = \dot{h}(t), \qquad (t \geqslant 0), \\ V(t,-2) = 0, \quad V(t,1) = 0, \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ V(0,y) = V_0(y) \quad x \in [-2,1] \setminus \{0\}, \end{cases}$$

System after change of variables

$$
\begin{cases}
V_t = V_{yy} + f_1 + u \qquad (t \geqslant 0, \quad y \in [-2,1] \setminus \{0\}), \\
m\dot{g}(t) = [V_y](t,0) + f_2(t) \qquad (t \geqslant 0), \\
V(0,t) = g(t) \qquad (t \geqslant 0), \\
\dot{h}(t) = g(t), \\
V(-2,t) = 0, \quad V(t,1) = 0 \qquad (t \geqslant 0), \\
h(0) = h_0, \quad g(0) = h_1, \\
V(y,0) = V_0(y) \qquad (y \in [-2,1] \setminus \{0\}),
\end{cases}
$$

where

$$
\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} (A_h - I)V_{yy} - B_h V_y V - C_h g V_y \\ [(D_h - I)V_y](t,0) \end{bmatrix}.
$$

Null-controllability of a linear system

$$\begin{cases} V_t = V_{yy} + f_1 + u \qquad (t \geqslant 0, \quad y \in [-2,1] \setminus \{0\}), \\ m\dot{g}(t) = [V_y](t,0) + f_2(t) \qquad (t \geqslant 0), \\ V(0,t) = g(t) \qquad (t \geqslant 0), \\ V(-2,t) = 0, \quad V(t,1) = 0 \qquad (t \geqslant 0), \\ g(0) = h_1, \\ V(y,0) = V_0(y) \qquad (y \in [-2,1] \setminus \{0\}), \end{cases}$$

where $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is *given*.

Null-controllability of a linear system

$$X = L^2(-2,1) \times \mathbb{R}, \quad U = L^2(I).$$

$$\mathcal{D}(A) = \left\{ z = \begin{bmatrix} V \\ g \end{bmatrix} \in H_0^1(-2,1) \times \mathbb{R} \,\middle|\, V(0) = g, \right.$$

$$\left. V_{|_{(-2,0)}} \in H^2(-2,0),\ V_{|_{(0,1)}} \in H^2(0,1) \right\},$$

$$A \begin{bmatrix} V \\ g \end{bmatrix} = \begin{bmatrix} V_{xx} \\ m^{-1}[V_x](0) \end{bmatrix}.$$

The previous system can be written as

$$\begin{cases} \dot{z} = Az + Bu + f, \\ z(0) = z_0. \end{cases}$$

General framework

- Assume X Hilbert.
- Assume $A : D(A) \to X$, $A = A^*$ and A is a negative operator.
- Assume $B : U \to X$ is a bounded operator.

We consider the system

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0 \in X. \end{cases}$$

Definition

The pair (A, B) is null-controllable in time τ if

$$\forall z_0 \in X, \quad \exists u \in L^2([0,\tau], U) \quad z(\tau) = 0.$$

Null-controllability with source term

Assume the pair (A, B) is null-controllable. Is it possible to obtain the null-controllability of

$$\begin{cases} \dot{z} = Az + Bu + f, \\ z(0) = z_0. \end{cases} \qquad ((A, B, f))$$

where $f : [0, \infty) \to X$?
More precisely, we would like to prove

$$\forall z_0 \in X, \quad \forall f \in \mathcal{F} \subsetneq L^2([0, \tau], X), \quad \exists u \in L^2([0, \tau], U) \quad z(\tau) = 0.$$

Functional spaces

$$\mathcal{F} = \left\{ f \in L^2([0,\tau], X) \;\middle|\; \frac{f}{\rho_{\mathcal{F}}} \in L^2([0,\tau], X) \right\},$$

$$\mathcal{U} = \left\{ u \in L^2([0,\tau], U) \;\middle|\; \frac{u}{\rho_0} \in L^2([0,\tau], U) \right\}.$$

$$\rho_{\mathcal{F}}, \rho_0 : [0,\tau] \to \mathbb{R}_+$$

continuous with

$$\rho_{\mathcal{F}}, \rho_0 \;\text{ decreasing and }\; \rho_{\mathcal{F}}(\tau) = \rho_0(\tau) = 0.$$

The Key Proposition

Proposition

Assume

1. *the pair* (A, B) *is null-controllable in any time* $t > 0$*;*
2. $\tau > 0$ *and* $\rho_{\mathcal{F}}$ *and* ρ_0 *are as above with some extra assumptions.*

Then

$$\forall z_0 \in X, \quad \forall f \in \mathcal{F}, \quad \exists u \in \mathcal{U} \quad \textit{such that} \quad \frac{z}{\rho_0} \in C([0,\tau], X).$$

Remark

Since ρ_0 *is a continuous function satisfying* $\rho_0(\tau) = 0$,

$$\frac{z}{\rho_0} \in C([0,\tau], X) \Longrightarrow z(\tau) = 0.$$

Consequence of the proposition

It is sufficient to have the null-controllability of (A, B):

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0 \in X. \end{cases}$$

By duality, it is equivalent to prove the observability inequality

$$C \int_0^\tau \|B^* e^{tA^*} z_0\|_U^2 \mathrm{d}t \geqslant \|e^{\tau A^*} z_0\|^2 \qquad (z_0 \in X).$$

Classical method for parabolic systems: global Carleman estimates, local Carleman estimates, spectral methods.

Spectral method

Result due to Fattorini and Russell:

Proposition

- *Assume that A admits an orthonormal basis of eigenvectors $(V_k)_{k\geqslant 1}$ with the corresponding decreasing sequence of eigenvalues $(-\lambda_k)_{k\geqslant 1}$.*
- *Assume that*

$$\inf_{k\geqslant 0}(\lambda_{k+1}-\lambda_k)>0,$$

$$\lambda_k = rk^2 + O(1) \qquad (k\to\infty),$$

$$\|B^*V_k\|_U \geqslant m \qquad (k\geqslant 1).$$

- *Assume that U is a separable Hilbert space.*

Then the pair (A,B) is null-controllable in any time $\tau>0$.

Consequence 1: without source term

Take

$$I = (-2, a), \quad \text{for some } a \in (-2, 1).$$

Then the system

$$\begin{cases} V_t - V_{xx} = \mathbf{1}_I u \qquad (t \geqslant 0, \quad x \in [-2,1] \setminus \{0\}), \\ V(t,0) = g(t) \qquad (t \geqslant 0), \\ m\dot{g}(t) = [V_x](t,0) \qquad (t \geqslant 0), \\ V(t,-2) = 0, \quad V(t,1) = 0 \qquad (t \geqslant 0), \\ g(0) = g_0, \\ V(0,x) = V_0(x) \quad x \in [-2,1] \setminus \{0\} \end{cases}$$

is null-controllable for any time $\tau > 0$.
For all (V_0, g_0), there exists $u \in L^2([0,\tau]; L^2(I))$ such that

$$V(\tau, \cdot) = 0, \quad g(\tau) = 0.$$

Consequence 2: with source term

The system

$$\begin{cases} V_t - V_{xx} = \mathbf{1}_I u + f_1 & (t \geqslant 0, \quad x \in [-2,1] \setminus \{0\}), \\ V(t,0) = g(t) & (t \geqslant 0), \\ m\dot{g}(t) = [V_x](t,0) + f_2 & (t \geqslant 0), \\ V(t,-2) = 0, \quad V(t,1) = 0 & (t \geqslant 0), \\ g(0) = g_0, \\ V(0,x) = V_0(x) \quad x \in [-2,1] \setminus \{0\} \end{cases}$$

is null-controllable for any time $\tau > 0$.
For all (V_0, g_0), for all $(f_1, f_2)/\rho_{\mathcal{F}} \in L^2(0,\tau; L^2(-2,1) \times \mathbb{R})$,
there exists $u/\rho_0 \in L^2([0,\tau]; L^2(I))$ such that

$$\frac{(V,g)}{\rho_0} \in C([0,\tau]; L^2(-2,1) \times \mathbb{R})$$

and in particular

$$V(\tau, \cdot) = 0, \quad g(\tau) = 0.$$

Consequence 3: with source terms and control of the position

The system

$$\begin{cases} V_t - V_{xx} = \mathbf{1}_I u + f_1 \qquad (t \geqslant 0, \quad x \in [-2,1] \setminus \{0\}), \\ V(t,0) = g(t) \qquad (t \geqslant 0), \\ m\dot{g}(t) = [V_x](t,0) + f_2 \qquad (t \geqslant 0), \\ V(t,-2) = 0, \quad V(t,1) = 0 \qquad (t \geqslant 0), \\ \dot{h}(t) = g(t) \qquad (t \geqslant 0), \\ h(0) = h_0, \quad g(0) = g_0, \\ V(0,x) = V_0(x) \quad x \in [-2,1] \setminus \{0\} \end{cases}$$

is null-controllable for any time $\tau > 0$.
For all (V_0, g_0, h_0), for all $(f_1, f_2)/\rho_{\mathcal{F}} \in L^2(0,\tau; L^2(-2,1) \times \mathbb{R})$, there exists $u/\rho_0 \in L^2([0,\tau]; L^2(I))$ such that

$$\frac{(V,g)}{\rho_0} \in C([0,\tau]; L^2(-2,1) \times \mathbb{R}) \quad \text{and} \quad h(\tau) = 0.$$

Resolution of our fluid-structure problem

$$\begin{cases} V_t = V_{yy} + f_1 + u \qquad (t \geqslant 0, \quad y \in [-2,1] \setminus \{0\}), \\ m\dot{g}(t) = [V_y](t,0) + f_2(t) \qquad (t \geqslant 0), \\ V(0,t) = g(t) \qquad (t \geqslant 0), \\ \dot{h}(t) = g(t), \\ V(-2,t) = 0, \quad V(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad g(0) = h_1, \\ V(y,0) = V_0(y) \qquad (y \in [-2,1] \setminus \{0\}), \end{cases}$$

where

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} (A_h - I)V_{yy} - B_h V_y V - C_h g V_y \\ [(D_h - I)V_y](t,0) \end{bmatrix}.$$

Fixed point procedure

$$\mathcal{N} \ : \ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \ \mapsto \ \begin{bmatrix} V \\ g \\ h \end{bmatrix} \ \mapsto \ \begin{bmatrix} (A_h - I)V_{yy} - B_h V_y V - C_h g V_y \\ [(D_h - I)V_y](t,0) \end{bmatrix}.$$

For r small enough, $\mathcal{N}$ is a contractive mapping from

$$\left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{F}, \quad \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{\mathcal{F}} \leqslant r. \right\}$$

into itself.

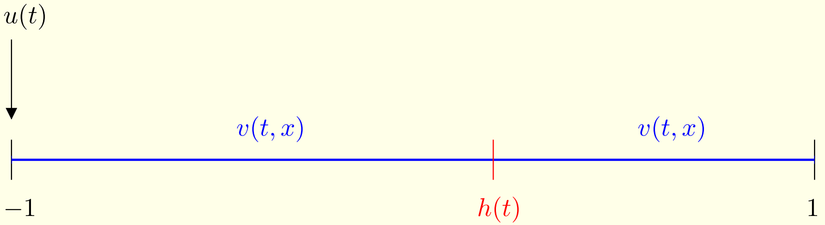
Conclusion

- General method to control problem of the form

$$\begin{cases} \dot{z} = Az + Bu + f, \\ z(0) = z_0, \end{cases}$$

  with $A = A^* < 0$.
- We obtain the controllability of a fluid-structure problem in 1d, with only one control.
- This general method can be applied for other parabolic systems, the important hypothesis is the null-controllability of (A, B).

Feedback stabilization



$$\begin{cases} v_t - v_{xx} + v v_x = f^S & (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) & (t \geqslant 0), \\ m\ddot{h}(t) = [v_x](t,h(t)) + m\ell^S & (t \geqslant 0), \\ v(t,-1) = a^S + u(t), \quad v(t,1) = b^S & (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{h_0\}. \end{cases} \tag{3}$$

Stationary states

For f^S, ℓ^S, a^S and b^S given, find V^S and H^S such that

$$\begin{cases} -V^S_{yy} + V^S V^S_y = f^S \qquad (y \in (-1,1) \setminus \{H^s\}), \\ 0 = [V^S_y](H^S) + m\ell^S, \\ V^S(-1) = a^S, \quad V^S(1) = b^S, \\ V^S(H^S) = 0. \end{cases} \tag{4}$$

Stationary states

If $f^S \equiv 0$, $\ell^S = 0$, the stationary state (V^S, H^S) satisfies

$$\left\{\begin{array}{c} -V^S_{yy} + V^S V^S_y = 0 \qquad (y \in (-1,1) \setminus \{H^s\}), \\ 0 = [V^S_y](H^S), \\ V^S(-1) = a^S, \quad V^S(1) = b^S, \\ V^S(H^S) = 0. \end{array}\right. \tag{5}$$

Stationary states

If $f^S \equiv 0$, $\ell^S = 0$, the stationary state (V^S, H^S) satisfies

$$\begin{cases} -V^S_{yy} + V^S V^S_y = 0 \qquad (y \in (-1,1) \setminus \{H^s\}), \\ 0 = [V^S_y](H^S), \\ V^S(-1) = a^S, \quad V^S(1) = b^S, \\ V^S(H^S) = 0. \end{cases} \tag{5}$$

One find that V^S is one of these solutions ($C > 0$):

- $V^S(y) = 2C \tan(C(y - H^S))$,
- $V^S(y) = -2C \tanh(C(y - H^S))$,
- $V^S \equiv 0$.

Feedback Stabilization result

Theorem

Assume (V^S, H^S) *is a stationary state. There exist* $\mu > 0$ *and* $(\widehat{\varphi}, \widehat{g}, \widehat{k}) \in L^2(-1,1) \times \mathbb{R}^2$*, such that, if*

$$\|v_0 - V^S\|_{L^2((-1,1))} + |h_1| + |h_0 - H^S| \leqslant \mu$$

then there exists a solution

$$v \in L^2_{loc}(\mathbb{R}_+, H^1(-1,1)) \cap C(\mathbb{R}_+, L^2(-1,1)), \quad h \in H^1_{loc}(\mathbb{R}_+),$$

with

$$v(t,-1) = a^S + \int_{-1}^{1} \left(v(t,X) - V^S(y)\right) \widehat{\varphi} \mathrm{dy} + m\dot{h}(t)\widehat{g} + h(t)\widehat{k}.$$

and satisfying

$$\|v(t) - V^S\|_{L^2(-1,1)} + |\dot{h}(t)| + |h(t)| \leqslant Ce^{-\sigma t} \left(\|v_0 - V^S\|_{L^2((-1,1))} + |h_1| + |h_0|\right).$$

Change of variables

$$X(t,y) := y + \eta(y)(h(t) - H^S),$$

with $\eta \in C^\infty((-1,1))$ satisfying

- $\eta(H^S) = 1$, $\eta_y(H^S) = 0$,
- $\eta(1) = \eta(-1) = 0$.

Assume that the particle remains close to H^S:

$$\left\|\eta'\right\|_{L^\infty(-1,1)} \left|h(t) - H^S\right| < 1 \quad \text{for all } t$$

then, $X(t,\cdot)$ is a bijection

$$X(t,\cdot) : (-1,1) \to (-1,1), \quad with \quad X(t,H^s) = h(t).$$

We denote by $Y(t,\cdot)$ the inverse of $X(t,\cdot)$.

Change of variables

We set

$$V(t,y) := v(t, X(t,y)).$$

To simplify we assume

$$H^S = 0.$$

We also set

$$V = W + V^S$$

System after change of variables

$$
\begin{cases}
W_t - W_{yy} + h\left(\eta' V_y^S - \eta f^S\right)_y + (V^S W)_y - \eta \ell V_y^S = F(\mathbf{X}) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t \geqslant 0, \quad y \in (-1,1) \setminus \{0\}), \\
W(t,0) = \ell(t) \qquad (t \geqslant 0), \\
m\dot{\ell}(t) = [W_y](t,0) \qquad (t \geqslant 0), \\
\dot{h}(t) = \ell(t) \qquad (t \geqslant 0), \\
W(t,-1) = u(t), \quad W(t,1) = 0 \qquad (t \geqslant 0), \\
h(0) = h^0, \quad \ell(0) = \ell^0, \\
W(0,y) = V^0(y) - V^S(y) \quad y \in (-1,1) \setminus \{0\}.
\end{cases}
$$

Feedback stabilization of parabolic systems

$$\mathbf{X}' = A\mathbf{X} + Bu \quad \text{in} \ [\mathcal{D}(A^*)]', \quad \mathbf{X}(0) = \mathbf{X}^0, \qquad (6)$$

where

- $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ compact resolvent in the real Hilbert space $\mathcal{H}$ and generator of an analytic semigroup in $\mathcal{H}$;
- $B : \mathbb{R} \to [\mathcal{D}(A^*)]'$ is strictly relatively bounded.

There is a finite number of "unstable" modes: for any prescribed $\sigma > 0$, there are only N eigenvalues of A with real part strictly greater than $-\sigma$: λ_k, $k = 1, \ldots, N$ (N depending on σ).

Feedback stabilization of parabolic systems

Theorem

Assume the following unique continuation property

$$\forall \varepsilon \in \mathcal{D}(A^*), \quad \forall k \in \{1, \dots, N\}$$
$$A^*\varepsilon = \bar{\lambda}_k \varepsilon \quad \textit{and} \quad B^*\varepsilon = 0 \implies \varepsilon = 0.$$

Then there exists $K \in \mathcal{L}(\mathcal{H}; \mathbb{R})$ *such that* $A + BK$ *is exponentially stable of order* $-\sigma$*: the solution* $\mathbf{X}$ *of* (6) *with* $u = K\mathbf{X}$ *satisfies*

$$\|\mathbf{X}(t)\| \leqslant \|\mathbf{X}^0\| e^{-\sigma t}.$$

Application to our case

$$\mathcal{H} := L^2((-1,1)) \times \mathbb{R}^2,$$

$$\mathcal{D}(A) := \left\{ \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} \in \mathcal{H} \ ; \right.$$
$$\left. W \in H^2((-1,1) \setminus \{0\}) \cap H_0^1((-1,1)), \quad W(0) = \ell \right\}.$$

$$A \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} := \begin{bmatrix} W_{yy} - h\left(\eta' V_y^S - \eta f^S\right)_y - (V^S W)_y + \eta \ell V_y^S \\ \dfrac{1}{m}[W_y](0) \\ \ell \end{bmatrix}$$

Application to our case

Assume

$$\lambda \in \mathbb{C}, \quad \Re\lambda \geqslant -\sigma.$$

Assume

$$\begin{cases} \lambda\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (-1,1)\setminus\{0\}, \\ \varphi(0) = g, \\ m\lambda g = [\varphi_y](0) + k + \int_{-1}^{1} \eta V_y^S \varphi \mathrm{dy}, \\ \lambda k = \int_{-1}^{1} \left(-\eta' V_y^S + \eta f^S\right)_y \varphi \mathrm{dy}, \\ \varphi(-1) = \varphi(1) = 0, \end{cases} \quad \text{and} \quad \varphi_y(-1) = 0$$

do we have $g = k = 0$ and $\varphi \equiv 0$ in $(-1,1)$?

Application to our case

$$\begin{cases} \lambda\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (0,1), \\ \varphi(0) = \varphi(1) = 0, \\ \lambda\varphi_y(0) = \int_0^1 \left(\eta' V_y^S - \eta f^S\right)_y \varphi \mathrm{dy} - \int_0^1 \lambda\eta V_y^S \varphi \mathrm{dy} \end{cases}$$

Application to our case

$$\begin{cases} \lambda\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (0,1), \\ \varphi(0) = \varphi(1) = 0, \\ \lambda\varphi_y(0) = \int_0^1 \left(\eta' V_y^S - \eta f^S\right)_y \varphi \mathrm{dy} - \int_0^1 \lambda\eta V_y^S \varphi \mathrm{dy} \end{cases}$$

Mutliplying by ηV_y^S and integrating by parts, we obtain

$$0 = (\lambda - V_y^S(0))\varphi_y(0). \tag{7}$$

Application to our case

Is it possible to find a non null solution to the following problem?

$$\begin{cases} V_y^S(0)\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (0,1), \\ \varphi(0) = \varphi(1) = 0. \end{cases} \qquad (\star)$$

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If we multiply by φ, we obtain

$$V_y^S(0)\int_0^1 |\varphi|^2 \mathrm{dy} + \int_0^1 |\varphi_y|^2 \mathrm{dy} + \frac{1}{2}\int_0^1 V_y^S |\varphi|^2 \mathrm{dy} = 0.$$

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Using Poincaré inequality, we deduce

$$\lambda = V_y^S(0) \leqslant \frac{\|[F^S]^+\|_{L^\infty} - 2\pi^2}{3}, \quad F^S(y) = \int_0^y f^S(s)\mathrm{ds}.$$

Conclusion for the linear case

Proposition

There exists $(\widehat{\varphi}, \widehat{g}, \widehat{k}) \in L^2(-1,1) \times \mathbb{R}^2$,

$$\begin{cases} W_t - W_{yy} + h\left(\eta' V_y^S - \eta f^S\right)_y + (V^S W)_y - \eta \ell V_y^S = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (t \geqslant 0, \quad y \in (-1,1) \setminus \{0\}), \\ W(t,0) = \ell(t) \qquad (t \geqslant 0), \\ m\dot{\ell}(t) = [W_y](t,0) \qquad (t \geqslant 0), \\ \dot{h}(t) = \ell(t) \qquad (t \geqslant 0), \\ W(t,-1) = \int_{-1}^{1} W(t)\widehat{\varphi}\,\mathrm{dy} + m\dot{h}(t)\widehat{g} + h(t)\widehat{k} \qquad (t \geqslant 0). \end{cases}$$

admits a solution

$$W \in L^2(\mathbb{R}_+, H^1(-1,1)) \cap C(\mathbb{R}_+, L^2(-1,1)), \quad h \in C^1(\mathbb{R}_+),$$

with

$$\|W(t)\|_{L^2(-1,1)} + |\dot{h}(t)| + |h(t)| \leqslant C e^{-\sigma t}\left(\|W_0\|_{L^2((-1,1))} + |h_1| + |h_0|\right).$$

Fixed Point Procedure

We set

$$\mathcal{V} := \left\{ \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} \in \mathcal{H} \ ; \ W \in H_0^1((-1,1)), \quad W(0) = \ell \right\},$$

$$\mathcal{V}_K := [\mathcal{D}(A+BK), \mathcal{H}]_{1/2}.$$

One can show that $N : \mathcal{V}_K \to \mathcal{V}'$ satisfies the following relations

$$\|N(\mathbf{X})\|_{\mathcal{V}'} \leqslant C \|\mathbf{X}\|_{\mathcal{H}} \|\mathbf{X}\|_{\mathcal{V}_K}, \quad (\mathbf{X} \in \mathcal{V}_K \cap B_{\mathcal{H}}(0,R)),$$

$$\|N(\mathbf{X}_1) - N(\mathbf{X}_2)\|_{\mathcal{V}'} \leqslant C \Big(\|\mathbf{X}_1 - \mathbf{X}_2\|_{\mathcal{H}} \|\mathbf{X}_1\|_{\mathcal{V}_\mathcal{K}} + \|\mathbf{X}_2\|_{\mathcal{H}} \|\mathbf{X}_1 - \mathbf{X}_2\|_{\mathcal{V}_\mathcal{K}} \Big) \quad (\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{V}_K \cap B_{\mathcal{H}}(0,R)).$$

Fixed Point Procedure

Assume $\rho > 0$ and let us consider

$$\mathbf{Z} \in \mathcal{W} := L^2(\mathcal{V}_K) \cap H^1(\mathcal{V}') \cap C(\mathcal{H})$$

with

$$\|\mathbf{Z}\|_{\mathcal{W}} \leqslant \rho \|\mathbf{X}_0\|_{\mathcal{H}}.$$

Then for ρ small enough,

$$\| N\left(\mathbf{Z}\right) \|_{L^2(\mathcal{V}')} \leqslant C \, \|\mathbf{Z}\|_{\mathcal{W}}^2$$

Fixed Point Procedure

We define

$$\mathbf{X}' + (A + BK)\mathbf{X} + N(\mathbf{Z}) = 0, \quad \mathbf{X}(0) = \mathbf{X}_0.$$

Then

$$\|\mathbf{X}_{\mathbf{Z}}\|_{\mathcal{W}} \leqslant C(\|\mathbf{X}_0\|_{\mathcal{H}} + \|\mathbf{Z}\|_{\mathcal{W}}^2),$$

and

$$\|\mathbf{X}_{\mathbf{Z}^1} - \mathbf{X}_{\mathbf{Z}^2}\|_{\mathcal{W}} \leqslant C(\left\|\mathbf{Z}^1\right\|_{\mathcal{W}} + \left\|\mathbf{Z}^2\right\|_{\mathcal{W}}) \left\|\mathbf{Z}^1 - \mathbf{Z}^2\right\|_{\mathcal{W}},$$

For ρ small enough, the ball $B_\rho := \{\|\mathbf{Z}\|_{\mathcal{W}} \leqslant \rho\|\mathbf{X}_0\|_{\mathcal{H}}\}$ for the mapping $\mathbf{Z} \mapsto \mathbf{X}$. It is a contraction on this ball, and thus we can apply the Banach fixed point.

Lyapunov function

- Natural Lyapunov function:

$$L \stackrel{\text{def}}{=} \int_0^\infty e^{2\sigma t} A_K^{*\frac{1}{2}} e^{-A_K^* t} e^{-A_K t} A_K^{\frac{1}{2}} dt$$

- Norm on $\mathcal{H}$:

$$\|\xi\|_{\mathcal{H}}^2 \sim (\xi | L\xi)_{\mathcal{H}}$$

- Accretivity of A_K and a priori estimate:

$$LA_K + A_K^* L = 2\sigma L + A_K^{*\frac{1}{2}} A_K^{\frac{1}{2}}$$

and

$$2(A_K \xi | L\xi) = 2\sigma \|\xi\|_{\mathcal{H}}^2 + \|A_K^{1/2}\xi\|_{\mathcal{H}}^2$$

$$\frac{d}{dt}\|\mathbf{X}(t)\|^2 + 2\sigma\|\mathbf{X}(t)\|^2 + \|A_K^{1/2}\mathbf{X}(t)\|^2 = (N(\mathbf{X}(t))|L\mathbf{X}(t))$$

Conclusion

- We obtain the feedback stabilizability of a fluid-structure problem in 1d, with only one boundary control, provided σ is not too large or there does not exist a solution to $(\star)$.
- We are working on the same problem in several dimensions.