A Continuous Time Approach for the Asymptotic Value in Two-Person Zero-Sum Repeated Games

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- All sets under consideration are finite.

Inductively, at stage $t = 1, ...,$ knowing the past history $h_t = (\omega_1, i_1, j_1, ..., i_{t-1}, j_{t-1}, \omega_t)$, player I chooses $i_t \in I$, player J chooses $j_t \in J$.

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The payoff at stage t is $g_t = g(i_t, j_t, \omega_t)$ and the total payoff is the discounted sum $\sum_t \lambda(1-\lambda)^{t-1} g_t$, where $\lambda \in]0, 1]$.

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This discounted game has a value v_λ.

The Shapley Operator

The Shapley operator $T(\lambda, \cdot)$ associates to a real function f on Ω (a point in $\mathbb{R}^\Omega$) the function:

$$T(\lambda, f)(\omega) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} [\lambda g(x, y, \omega) + (1-\lambda) \sum_{\tilde{\omega}} Q(x, y, \omega)(\tilde{\omega}) f(\tilde{\omega})]$$

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Lemma

The Shapley operator $T(\lambda, \cdot)$ is well defined from $\mathbb{R}^\Omega$ to itself. Its unique fixed point is v_λ.

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From stage 1 on, the parameter is fixed and the information of the players after stage n is $a_{n+1} = b_{n+1} = \{i_n, j_n\}$.

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$\mathbf{X} = \Delta(I)^K$ and $\mathbf{Y} = \Delta(J)^L$ are the type-dependent mixed action sets of the players; g is extended on $\overline{M} \times \mathbf{X} \times \mathbf{Y}$ by:

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Given (p, q, x, y), let $x(i) = \sum_k x_i^k p^k$ be the (total) probability of action i and $p(i)$ be the conditional probability on K given the action i, hence $p^k(i) = \frac{p^k x_i^k}{x(i)}$ (and similarly for y and q).

The resulting form of the Shapley operator is:

$$
\begin{aligned}
T(\lambda, f)(p,q) &= \sup_{x \in \mathbf{X}} \inf_{y \in \mathbf{Y}} \{\lambda \sum_{k,\ell} p^k q^\ell g(k,\ell,x^k,y^\ell) \\
&\quad + (1-\lambda) \sum_{i,j} x(i) y(j) f(p(i), q(j))\} \qquad (1)
\end{aligned}
$$

acting on the set of concave/convex continuous functions defined on the product of simplexes $\Delta(K) \times \Delta(L)$.
These equations are due to Aumann and Maschler (1966) and Mertens and Zamir (1971).

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and the recursive formula for the n stage value is obtained similarly

$$v_n = \mathbf{T}[\frac{1}{n}, v_{n-1}] \tag{3}$$

with obviously $v_0 = 0$.

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$$v_\Pi = \mathtt{val}\{t_1 g_1 + (1 - t_1) \mathsf{E} v_{\Pi_{t_1}}\}$$

where Π_{t_1} is the normalization on $[0,1]$ of the trace of the partition Π on the interval $[t_1, 1]$.

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$$V_\Pi(t_k) = \mathtt{val}\{(t_{k+1} - t_k) g_{k+1} + \mathsf{E} V_\Pi(t_{k+1})\} \tag{4}$$

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By taking the linear extension we define this way for every finite partition Π, a function $V_\Pi(t)$ on $[0, 1]$.

Lemma

Assume $n \to \mu(n)$ decreasing. Then V_Π is C-Lipschitz in t, where C is a bound on the payoffs.

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Here $u(p) = \mathtt{val}_{\Delta(I)\times\Delta(J)} \sum_k p^k g(k,x,y)$ is the value of the one shot non revealing game, where the informed player does not use his information and $\mathtt{Cav}_C$ is the concavification operator: given ϕ, a real bounded function defined on a convex set C, $\mathtt{Cav}_C(\phi)$ is the smallest function greater than ϕ and concave, on C.

Extensions of these results to games with lack of information on both sides were achieved by Mertens and Zamir (1971). In addition they identified the limit as the only solution of the system of implicit functional equations with unknown ϕ:

$$\phi(p,q) = \mathtt{Cav}_{p \in \Delta(K)} \min\{\phi, u\}(p,q), \tag{5}$$

$$\phi(p,q) = \mathtt{Vex}_{q \in \Delta(L)} \max\{\phi, u\}(p,q) \tag{6}$$

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Here again u stands for the value of the non revealing game: $u(p,q) = \mathtt{val}_{X \times Y} \sum_{k,\ell} p^k q^\ell g(k,\ell,x,y)$ and we will write **MZ** for the corresponding operator

$$\phi = \mathbf{MZ}(u). \tag{7}$$

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the Shapley equation can be written as a finite set of polynomial equalities and inequalities involving $\{x^k_\lambda, y^k_\lambda, v_\lambda(k), \lambda\}$ thus it defines a semi-algebraic set in some euclidean space $\mathbb{R}^N$, hence by projection v_λ has an expansion in Puiseux series.

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The existence of $\lim_{n \to \infty} v_n$ is obtained by an algebraic comparison argument, Bewley and Kohlberg (1976).

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We extend here an approach that was initially introduced by Laraki (2002) for the discounted case.

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We consider a sequence of partitions Π_n and we define W_n as V_{Π_n}.

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It satisfies $W_n(1,p,q) = 0$ and for $t_n^m = \sum_{k \leq m} \mu_n^k$

$$W_n(t_n^m, p, q) = \max_{x \in \mathbf{X}} \min_{y \in \mathbf{Y}} [\mu_n^{m+1} g(x, y, p, q) + \sum_{i,j} \overline{x}(i) \overline{y}(j) W_n(t_n^{m+1}, p(i), q(j))] \quad (8)$$

Extend $W_n(\cdot, p, q)$ to $[0,1]$ by linear interpolation.
$W_n(\cdot,\cdot,\cdot)$ is a C Lipschitz function, concave in p, convex in q.

Moreover if W is an accumulation point of the equi-continuous family $\{W_n\}$ then for all (t, p, q), W is a fixed point of the operator S

$$SU(t,p,q) = \max_{x \in \mathbf{X}} \min_{y \in \mathbf{Y}} \left[\sum_{i,j} \overline{x}(i)\overline{y}(j) U(t, p(i), q(j)) \right] \tag{9}$$

Let $\mathbf{X}(t, p, q, W) \subseteq \Delta(I)^K$ be the set of strategies for player I that are optimal for the above game.
$x \in \mathbf{X}$ is non-revealing at p if $\bar{x}(i) > 0$ implies $p(i) = p$ and similarly for a subset of strategies.
Recall that $u(p, q)$ is the value of the game played on non-revealing strategies.

The Variational Inequalities

We introduce the following properties:
For any $(p, q) \in \Delta(K) \times \Delta(L)$ and any C^1 test function $\phi : [0, 1] \to \mathbf{R}$:
(P1) If, for some $t \in [0, 1)$, $\mathbf{X}(t, p, q, W)$ is non-revealing at p and $W(\cdot, p, q) - \phi(\cdot)$ has a global maximum at t, then $u(p, q) + \phi'(t) \geq 0$.
(P2) If, for some $t \in [0, 1)$, $\mathbf{Y}(t, p, q, W)$ is non-revealing at q and $W(\cdot, p, q) - \phi(\cdot)$ has a global minimum at t then $u(p, q) + \phi'(t) \leq 0$.

Theorem

*Any accumulation point W of the family $\{W_n\}$ satisfies **(P1)** and **(P2)**.*

proof

- Let t, p and q such that $\mathbf{X}(t, p, q, W)$ is non-revealing and $W(\cdot, p, q) - \phi(\cdot)$ admits a global maximum at t.

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- Let W_n converge to W and define t^{θ_n} to be a global maximum of $W_n(\cdot,p,q)-\phi(\cdot)$ on $\{t_n^k\}$. Then $t^{\theta_n}\to t$.

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- Let W_n converge to W and define t^{θ_n} to be a global maximum of $W_n(\cdot,p,q)-\phi(\cdot)$ on $\{t_n^k\}$. Then $t^{\theta_n}\to t$.
- One has

$$W_n\left(t^{\theta_n},p,q\right) = \max_x \min_y [\mu_n^{\theta_n} g(x,y,p,q) + \sum_{i,j} \overline{x}(i)\overline{y}(j) W_n(t^{\theta(n)+1},p(i),q(j))]$$

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$$W_n\left(t^{\theta_n}, p, q\right) \leq \mu_n^{\theta_n} g(x_n, y, p, q) + \sum_i \overline{x}_n(i) W_n\left(t^{\theta_n+1}, p_n(i), q\right)$$

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By concavity of W_n with respect to p

$$\sum_i \overline{x}_n(i) W_n\left(t^{\theta_n+1}, p_n(i), q\right) \leq W_n\left(t^{\theta_n+1}, p, q\right)$$

proof

Hence:

$$0 \leq \mu_n^{\theta_n} g(x_n, y, p, q) + [W_n\left(t^{\theta_n+1}, p, q\right) - W_n\left(t^{\theta_n}, p, q\right)]$$

Since t^{θ_n} is a global maximum of $W_n(\cdot, p, q) - \phi(\cdot)$:

$$\phi\left(t^{\theta_n+1}\right) - \phi\left(t^{\theta_n}\right) \geq W_n\left(t^{\theta_n+1}, p, q\right) - W_n\left(t^{\theta_n}, p, q\right)$$

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Hence:

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Assume $\{x_n\}$ converges to some x (hence non-revealing at p by upper sami continuity).

Passing to the limit:

$$g(x, y, p, q) + \phi'(t) \geq 0 \ .$$

Since this inequality holds true for every y non revealing, one obtains :

$$u(p, q) + \phi'(t) \geq 0 \ .$$

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- ***(P3)** $W_1(1,p,q) \leq W_2(1,p,q)$ for any $(p,q) \in \Delta(K) \times \Delta(L)$.*

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Then $W_1 \leq W_2$ on $[0, 1] \times \Delta(K) \times \Delta(L)$.

The comparison principle

We argue by contradiction, assuming that

$$\max_{t\in[0,1],p\in P,q\in Q}[W_1(t,p,q)-W_2(t,p,q)]=\delta>0\,.$$

Then, for $\varepsilon>0$ sufficiently small,

$$\delta(\varepsilon):=\max_{t\in[0,1],s\in[0,1],p\in P,q\in Q}[W_1(t,p,q)-W_2(s,p,q)-\frac{(t-s)^2}{2\varepsilon}+\varepsilon s]\;>0 \tag{10}$$

Moreover $\delta(\varepsilon)\to\delta$ as $\varepsilon\to0$.

The comparison principle

We claim that there is $(t_\varepsilon, s_\varepsilon, p_\varepsilon, q_\varepsilon)$, point of maximum above, such that $\mathbf{X}(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player 1 and $\mathbf{Y}(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player 2.

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Take an extreme point of the (convex hull of the) compact set where the difference $W_1(t_\varepsilon, ., .) - W_2(s_\varepsilon, ., .)$ is maximal and use the fact that both functions are fixed points of the operator S.
Finally we note that $t_\varepsilon < 1$ and $s_\varepsilon < 1$ for ε sufficiently small, because $\delta(\varepsilon) > 0$ and $W_1(1, p, q) \leq W_2(1, p, q)$ for any (p, q) by **P3**.

The comparison principle

Since the map $t \mapsto W_1(t, p_\varepsilon, q_\varepsilon) - \frac{(t - s_\varepsilon)^2}{2\varepsilon}$ has a global maximum at t_ε and since $X(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player I, condition **P1** implies that

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} \geq 0\ . \qquad (11)$$

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In the same way, since the map $s \mapsto W_2(s, p_\varepsilon, q_\varepsilon) + \frac{(t_\varepsilon - s)^2}{2\varepsilon} - \varepsilon s$ has a global minimum at s_ε and since $Y(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player J, we have by condition **P2** that

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \varepsilon \leq 0 \, .$$

The comparison principle

Since the map $t \mapsto W_1(t, p_\varepsilon, q_\varepsilon) - \frac{(t-s_\varepsilon)^2}{2\varepsilon}$ has a global maximum at t_ε and since $X(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player I, condition **P1** implies that

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} \geq 0 \, . \tag{11}$$

In the same way, since the map $s \mapsto W_2(s, p_\varepsilon, q_\varepsilon) + \frac{(t_\varepsilon - s)^2}{2\varepsilon} - \varepsilon s$ has a global minimum at s_ε and since $Y(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player J, we have by condition **P2** that

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \varepsilon \leq 0 \, .$$

This latter inequality contradicts (11).

Corollary

The family V_Π has only one accumulation point V hence converges.

V satisfies $V(t,p,q) = (1-t)V(0,p,q)$. Write $v(p,q) = V(0,p,q)$, hence one can take $\phi(t) = (1-t)v(p,q)$ and v is characterized by the property:
p_0 extreme point of the epigraph of $p \mapsto v(p,q_0)$ implies $v(p_0,q_0) \leq u(p_0,q_0)$ and one recovers the Mertens-Zamir system:

$$v(p,q) = \mathtt{Cav}_{p\in\Delta(K)} \min\{v,u\}(p,q), \tag{12}$$

$$v(p,q) = \mathtt{Vex}_{q\in\Delta(L)} \max\{v,u\}(p,q) \tag{13}$$

Contents

The same tols extend to the study of absorbing games and can be applied to the "splitting game".
Sketch of the approach :
The family of value functions is relatively compact
Consider two accumulation points w_1 and w_2 and a point (t, ω) where the difference $w_1 - w_2$ is maximal.
Deduce a variational inequality at (t, ω) for any majorant of w_1 and a dual property
Prove a comparison principle.

Reference:

Cardaliaguet P., R. Laraki and S. Sorin (2012) A continuous time approach for the asymptotic value in two-person zero-sum repeated games, *SIAM J. on Control and Optimization*, **50**, 1573-1596.