Minimal time issues for the observability of Grushin-type equations

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GT Contrôle
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Null-controllability problem at \( T \) for the heat equation through \( \Gamma \)

Let \( \Omega \subset \mathbb{R}^d \), \( \Gamma \subset \partial \Omega \).

The heat equation with control function \( v \in L^2(0, T; L^2(\Gamma)) \):

\[
\begin{align*}
\frac{\partial}{\partial t} u - \Delta_x u &= 0, \quad \text{in } (0, T) \times \Omega, \\
u(t, x) &= v(t, x)1_{\Gamma}(x), \quad \text{in } (0, T) \times \partial \Omega, \\
u(0, x) &= u_0(x) \quad \text{in } \Omega.
\end{align*}
\]

Given \( u_0 \in L^2(\Omega) \), can we find a control function \( v \in L^2(0, T; L^2(\Gamma)) \) s.t. \( u(T, \cdot) = 0 \)?

\( \rightarrow \) YES [Fursikov Imanuvilov ’96, Lebeau Robbiano ’95]
For any time \( T > 0 \) and any non-empty open subset \( \Gamma \subset \partial \Omega \).
By duality [Fattorini-Russell ’71],

Null-controllability $\Leftrightarrow$ Observability.

Observability of the heat equation through $(0, T) \times \Gamma$

Let $\Omega \subset \mathbb{R}^d$, $\Gamma \subset \partial \Omega$.
We consider the following heat equation:

$$
\begin{aligned}
\partial_t z - \Delta_x z &= 0, \quad \text{in } (0, T) \times \Omega, \\
z(t, x) &= 0, \quad \text{in } (0, T) \times \partial \Omega, \\
z(0, x) &= z_0(x) \quad \text{in } \Omega.
\end{aligned}
$$

Does there exist a constant $C > 0$ such that for all $z_0 \in H^1_0(\Omega)$

$$
\|z(T, \cdot)\|_{L^2(\Omega)} \leq C \|\partial_\nu z\|_{L^2((0, T) \times \Gamma)}
$$

YES Proof by Carleman estimates [Fursikov Imanuvilov ’96].
General motivation

Understand what happens for degenerate parabolic equations.

Typical examples:

- **Degeneracies on the boundary** of the domain:

\[
\begin{cases}
\partial_t z - \partial_x (x^{2\alpha} \partial_x z) = 0, & (t, x) \in (0, T) \times (0, L), \\
z(t, 0) = z(t, L) = 0, & t \in (0, T).
\end{cases}
\]

See [Cannarsa Martinez Vancostenoble ’16] for latest developments.

- **Degeneracies inside** the domain \( \Omega = (-L, L) \times (0, \pi) \):

\[
\begin{cases}
\partial_t z - \partial_{xx} z - |x|^{2\alpha} \partial_{yy} z = 0, & (t, x, y) \in (0, T) \times \Omega, \\
z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial \Omega.
\end{cases}
\]
Focus on the case of interior degeneracies

In $\Omega = (-L, L) \times (0, \pi)$, Grushin type operators:

$$\begin{cases} 
\partial_t z - \partial_{xx} z - |x|^{2\alpha} \partial_{yy} z = 0, & (t, x, y) \in (0, T) \times \Omega, \\
z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial \Omega.
\end{cases}$$

- Boundary observation through $\Gamma \subset \partial \Omega$:

$$\|z(T)\|_{L^2(\Omega)} \leq C \|\partial_\nu z\|_{L^2((0,T) \times \Gamma)}.$$

- Internal/distributed Observation through $\omega \subset \Omega$:

$$\|z(T)\|_{L^2(\Omega)} \leq C \|z\|_{L^2((0,T) \times \omega)}.$$
**Known results**

\[ \Omega = (-L, L) \times (0, \pi), \text{ Grushin type operators:} \]

\[
\begin{cases}
\partial_t z - \partial_{xx} z - |x|^{2\alpha} \partial_{yy} z = 0, & (t, x, y) \in (0, T) \times \Omega, \\
z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial \Omega.
\end{cases}
\]

[Beauchard Cannarsa Guglielmi ’14]

- **\( \alpha < 1 \):** Observable in any time \( T > 0 \) from any open subset \( \omega \subset \Omega \) or \( \Gamma \subset \partial \Omega \).
  
  \[ \Rightarrow \text{ Like the usual heat equation.} \]

- **\( \alpha > 1 \):** Not observable from \( \omega \), whatever \( T > 0 \), when \( \overline{\omega} \cap \{x = 0\} = \emptyset \).
  
  \[ \Rightarrow \text{ The strong degeneracy prevents from any observability result.} \]
The case $\alpha = 1$

$\Omega = (-L, L) \times (0, \pi)$, Grushin type operators:

$$\begin{cases} 
\partial_t z - \partial_{xx} z - |x|^2 \partial_{yy} z = 0, \\
 z(t, x, y) = 0, 
\end{cases} \quad (t, x, y) \in (0, T) \times \Omega, \quad (t, x, y) \in (0, T) \times \partial \Omega.$$

If $\omega = \omega_x \times (0, \pi)$, and $\overline{\omega_x} \cap \{0\} \times (0, \pi) = \emptyset$, there exists a critical time $T_\omega^* > 0$ such that

- The equation is not observable in any time $T < T_\omega^*$.
- For any $T > T_\omega^*$, the equation is observable:

$$\|z(T)\|_{L^2(\Omega)} \leq C \|z\|_{L^2((0, T) \times \omega)}.$$

cf [Beauchard Cannarsa Guglielmi ’14].

If there is an horizontal strip which does not meet $\omega$, there is never observability whatever $T > 0$ is [Koenig ’17].
The case $\alpha = 1$

$\Omega = (-L, L) \times (0, \pi)$, Grushin type operators:

$$\begin{cases}
\partial_t z - \partial_{xx} z - |x|^2 \partial_{yy} z = 0, & (t, x, y) \in (0, T) \times \Omega, \\
z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial \Omega.
\end{cases}$$

If $\Gamma = \Gamma_x \times (0, \pi)$, with $\Gamma_x = \{L\}, \{-L\}$ or $\{-L, L\}$, there exists a critical time $T^*_\Gamma > 0$ such that

- The equations are not observable in any time $T < T^*_\Gamma$.
- For any $T > T^*_\Gamma$, the equations are observable:

$$\|z(T)\|_{L^2(\Omega)} \leq C \|\partial_x z\|_{L^2((0, T) \times \Gamma)}.$$

cf [Beauchard Cannarsa Guglielmi ’14]: $T^*_\Gamma \geq L^2 / 2$. 

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Observations from two ends

\[ \Omega = (-L, L) \times (0, \pi) : \]

\[
\begin{align*}
\partial_t z - \partial_{xx} z - x^2 \partial_{yy} z &= 0 & \text{in} & (0, T) \times \Omega, \\
z(t, x, y) &= 0 & \text{in} & (0, T) \times \partial \Omega, \\
z(0, x, y) &= z_0(x, y) & \text{in} & \Omega,
\end{align*}
\]

**Theorem [K. Beauchard, J. Dardé & S.E. 2018]**

Let \( T > L^2/2 \) and \( \Gamma = \{-L, L\} \times (0, \pi) \). Then there exists \( C > 0 \) such that for all smooth solutions \( z \),

\[
\| z(T) \|_{L^2(\Omega)} \leq C \| \partial_x z \|_{L^2((0, T) \times \Gamma)} .
\]

**Sharp time:** No observability if \( T < L^2/2 \).

\( \rightsquigarrow \) See [Beauchard Cannarsa Guglielmi 2014].

**Same as in [Beauchard Miller Morancey 2015], but \( \neq \) proof.**
Observation from one end

\[ \Omega = (-L, L) \times (0, \pi) : \]

\[
\begin{cases}
\partial_t z - \partial_{xx} z - x^2 \partial_{yy} z = 0 & \text{in } (0, T) \times \Omega, \\
z(t, x, y) = 0 & \text{in } (0, T) \times \partial \Omega, \\
z(0, x, y) = z_0(x, y) & \text{in } \Omega,
\end{cases}
\]

Theorem [K. Beauchard, J. Dardé & S.E. 2018]

Let \( T > L^2/2 \) and \( \Gamma = \{ L \} \times (0, \pi) \). Then there exists \( C > 0 \) such that for all smooth solutions \( z \),

\[
\| z(T) \|_{L^2(\Omega)} \leq C \| \partial_x z \|_{L^2((0, T) \times \Gamma)}.
\]

\[\Rightarrow\] Same time as for \( \Gamma = \{-L, L\} \times (0, \pi) \).

Sharp time: No controllability if \( T < L^2/2 \).

\[\Rightarrow\] See [Beauchard Cannarsa Guglielmi 2014].
Grushin equations in non-symmetric domains

\[ \Omega = (-L_-, \, L_+) \times (0, \pi) : \]

\[
\begin{aligned}
\partial_t z - \partial_{xx} z - x^2 \partial_{yy} z = 0 & \quad \text{in } (0, \, T) \times \Omega, \\
z(t, x, y) = 0, & \quad (t, x, y) \in (0, \, T) \times \partial \Omega.
\end{aligned}
\]

**Theorem** [K. Beauchard, J. Dardé & S.E. 2018]

Let \( T > L^2_+ / 2 \) and \( \Gamma = \{ L_+ \} \times (0, \pi) \). Then there exists \( C > 0 \) such that for all smooth solutions \( z \),

\[
\| z(T) \|_{L^2(\Omega)} \leq C \| \partial_x z \|_{L^2((0, T) \times \Gamma)}.
\]

**Sharp time** \( \rightsquigarrow \) Agmon estimates.
The role of boundary conditions

\[ \Omega = (0, L) \times (0, \pi) : \]

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t z - \partial_{xx} z - x^2 \partial_{yy} z = 0 \\
z(t, x, y) = 0,
\end{array} \right. & \quad \text{in } (0, T) \times \Omega, \\
(t, x, y) \in (0, T) \times (\partial\Omega \setminus \{0\} \times (0, \pi)).
\end{aligned}
\]

+ Dirichlet conditions \( z(t, 0, y) = 0 \) in \((0, T) \times (0, \pi)\)

or Neumann conditions \( \partial_x z(t, 0, y) = 0 \) in \((0, T) \times (0, \pi)\).

**Theorem** [K. Beauchard, J. Dardé & S.E. 2018]

When \( \Gamma = \{L\} \times (0, 1) \), the critical time for observability is

- \( T^* = L^2 / 6 \) in the Dirichlet case;
- \( T^* = L^2 / 2 \) in the Neumann case.

\[ \rightarrow \text{The BC at } x = 0 \text{ plays an important role.} \]
Our results also apply to

- **Heisenberg equations** in tensorized domains:

\[ \partial_t - \partial_x^2 - (x \partial_y + \partial_z)^2. \]

see [Beauchard Cannarsa ’17] for previous results.

- Some **inverse problems** similar to the ones of [Beauchard Cannarsa Yamamoto ’14].

- Slightly more general settings for Grushin equations:
  - \( \Omega = \Omega_x \times \Omega_y \), with \( \Omega_y \) of any dimension,
  - Operators of the form \( \partial_t - \partial_x^2 - q(x)^2 \partial_y^2 \) with

\[ q \in C^3(-L_-, L_+), \quad q(0) = 0, \quad \inf \partial_x q > 0. \]

\[ \Rightarrow \text{Critical time} \ T^* = \frac{1}{q'(0)} \int_{0}^{L_+} q(x) \, dx \text{ when observed from } L_+. \]
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3. Further comments
Each result concerns observability properties for the equation:

$$\begin{cases}
(\partial_t - \partial_x^2 - x^2 \partial_y^2)z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \Omega, \\
z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial\Omega, \\
z(0, ., .) = z_0 \in L^2(\Omega).
\end{cases}$$

We shall take advantage of the tensorized form of the problem:

$$z(t, x, y) = \sum_n z_n(t, x) \sin(ny).$$

$$\begin{cases}
(\partial_t - \partial_x^2 + n^2 x^2)z_n(t, x) = 0, & (t, x) \in (0, T) \times (-L, L), \\
z_n(t, -L) = z_n(t, L) = 0, & t \in (0, T), \\
z_n(0, .) = z_{0,n} \in H^1_0(-L, L),
\end{cases}$$
Use Fourier series.

\[
\begin{cases}
(\partial_t - \partial_x^2 + n^2x^2)z_n(t, x) = 0, & (t, x) \in (0, T) \times (-L, L), \\
z_n(t, -L) = z_n(t, L) = 0, & t \in (0, T), \\
z_n(0, .) = z_0, n \in L^2(-L, L),
\end{cases}
\]

? Uniform ? observability problems

\[\rightsquigarrow\] If observed from \(-L\) and \(L\):

\[
\int_{-L}^{L} |z_n(T, x)|^2 dx \leq C \int_0^T (|\partial_x z_n(t, -L)|^2 + |\partial_x z_n(t, L)|^2) dt.
\]

\[\rightsquigarrow\] If observed from \(L\):

\[
\int_{-L}^{L} |z_n(T, x)|^2 dx \leq C \int_0^T |\partial_x z_n(t, L)|^2 dt.
\]
To prove a uniform observability result in time $T$, we use

- A careful analysis of the cost of observability of a family of 1-d heat equations as $n \to \infty$, for a given $T_0 > 0$ (small):

$$\|z_n(T_0)\|_{L^2(-L,L)} \leq Ce^{An} \|\text{observation}\|_{L^2(0,T_0)},$$

with $C$ and $A$ independent of $n \in \mathbb{N}$.

- The dissipation of each semigroup:

$$\|z_n(T)\|_{L^2(-L,L)} \leq e^{-\mu(n(T-T_0))} \|z_n(T_0)\|_{L^2(-L,L)},$$

with $\mu$ independent of $n$.

$\Rightarrow$ Uniform observability provided

$$T \geq T_0 + \frac{A}{\mu}.$$ 

As $T_0$ is arbitrarily small, this gives $T > A/\mu$. 

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Minimal times for Observability of Grushin Eq.
Follows the same strategy as in [Beauchard Cannarsa Guglielmi ’14].

The dissipation rates of each semi-group are known ($\mu = 1$ in the case $x \in (-L, L)$)

→ It mainly remains to analyze the cost of observability of a family of 1-d heat equations, in the asymptotics $n \to \infty$ and for a given $T_0 > 0$ (small):

$$\| z_n(T_0) \|_{L^2(-L, L)} \leq C e^{A n} \| \text{observation} \|_{L^2(0, T_0)}.$$ 

**Remark**

To get a sharp result, we should obtain sharp estimates on $A$. 

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Observation from both boundaries $\pm L$

\[
\begin{cases}
(\partial_t - \partial_x^2 + n^2 x^2)z_n(t, x) = 0, & (t, x) \in (0, T) \times (-L, L), \\
z_n(t, -L) = z_n(t, L) = 0, & t \in (0, T), \\
z_n(0, \cdot) = z_0,n \in H_0^1(-L, L),
\end{cases}
\]

Then

\[
w_n(t, x) = z_n(t, x) \exp \left( -\frac{n \coth(2nt)}{2} (L^2 - x^2) \right)
\]

satisfies

\[
\begin{cases}
(\partial_t - \partial_x^2 + 2\theta_n(t)x\partial_x + \theta_n'(t) \frac{L^2}{2} + \theta_n(t)) w_n(t, x) = 0 \\
w_n(t, -L) = w_n(t, L) = 0,
\end{cases}
\]

where we set $\theta_n(t) = n \coth(2nt)$. No terms in $x^2$ anymore!

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Minimal times for Observability of Grushin Eq.
(Carleman type) Energy estimates:

\[
\int_{-L}^{L} \left( |\partial_x w_n(T_0)|^2 - \frac{n^2 L^2}{\sinh(2nT_0)^2} \frac{L^2}{2} |w_n(T_0)|^2 \right) \, dx
\leq 2L \int_0^{T_0} \left( |\partial_x w_n(t, -L)|^2 + |\partial_x w_n(t, L)|^2 \right) dt.
\]

⇒ Given \( T_0 > 0 \), \( \exists n_0 \in \mathbb{N} \), s.t. for all \( n \geq n_0 \),

\[
\int_{-L}^{L} |w_n(T_0)|^2 = \int_{-L}^{L} |z_n(T_0, x)|^2 \exp \left( -n \coth(2nT_0)(L^2 - x^2) \right)
\leq C \int_0^{T_0} \left( |\partial_x z_n(t, -L)|^2 + |\partial_x z_n(t, L)|^2 \right) dt
\]

\( \sim \) Observability cost in \( \exp(nL^2/2) \) at time \( T_0 \).

To be combined with the dissipation in \( \exp(-n(T - T_0)) \).
Why this choice? see [Dardé Ervedoza, ’16, ’17].

The fundamental solution of

$$ (\partial_t - \partial_{xx} + x^2)K(t, x, y) = \delta_{t=0} \delta_{x=y} $$

is given by the Mehler kernel

$$ K(t, x, y) = \frac{1}{(2\pi \sinh(2t))^{1/2}} e^{-\coth(2t) \left( \frac{|x|^2 + |y|^2}{2} \right)} e^{-\frac{2x \cdot y}{\sinh(2t)}}. $$

Scaling \((t, x, y) \rightarrow (nt, \sqrt{n}x, \sqrt{n}y)\) and \(y \rightsquigarrow -iL\), gives

$$ |K_n(t, x, iL)| = \frac{1}{(2\pi \sinh(2nt))^{1/2}} e^{-n\coth(2nt) \left( \frac{|x|^2 - L^2}{2} \right)}. $$

\(\rightsquigarrow\) The weight function is the exponential envelop of \(1/K_n(t, x, iL)\).
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The role of the boundary condition at $x = 0$

Corresponding to Dirichlet boundary conditions:

\[
\begin{align*}
(\partial_t - \partial_x^2 + n^2 x^2)z_n(t, x) &= 0, & (t, x) &\in (0, T) \times (0, L), \\
z_n(t, 0) = z_n(t, L) &= 0, & t &\in (0, T),
\end{align*}
\]

Corresponding to Neumann boundary condition

\[
\begin{align*}
(\partial_t - \partial_x^2 + n^2 x^2)z_n(t, x) &= 0, & (t, x) &\in (0, T) \times (0, L), \\
\partial_x z_n(t, 0) = z_n(t, L) &= 0, & t &\in (0, T),
\end{align*}
\]

- Cost of observability in $\exp(nL^2/2)$ by symmetry arguments.
- Dissipation in $\exp(-3nt)$ for Dirichlet BC, $\exp(-nt)$ for Neumann BC.

$\Rightarrow T^* = L^2/6$ for Dirichlet BC, $T^* = L^2/2$ for Neumann BC.
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Observation at one end

\[
\begin{cases}
(\partial_t - \partial_x^2 + n^2 x^2) z_n(t, x) = 0, & (t, x) \in (0, T) \times (-L, L), \\
z_n(t, -L) = z_n(t, L) = 0, & t \in (0, T), \\
z_n(0, .) = z_0, n \in H^1_0(-L, L),
\end{cases}
\]

Goal

Get a \textbf{uniform} observability inequality

\[
\|z_n(T)\|_{L^2(-L, L)} \leq C \exp(A n) \|\partial_x z_n(t, L)\|_{L^2(0, T)},
\]

with a precise knowledge of $A$. 

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Minimal times for Observability of Grushin Eq.
Proof done in several steps:

- Passing the information from the right $x = L$ to the left of the singularity $x = -\varepsilon$, $\varepsilon > 0$ small.

- Passing the information from the singularity $x = 0$ to the extreme left of the domain $-L$.

- A gluing argument.

The first two steps are done by Carleman estimates.
From $x = L$ to $x = 0$ (or $-\varepsilon$)

$$
\varphi_{R,n}(t, x) = n\theta(t)\psi_R(x) + \theta(t), \quad (t, x) \in (0, T) \times (-\varepsilon, L)
$$

with $\theta$ and $\psi_R$ as follows

$$
\theta \in C^\infty(0, T), \quad \theta(t) = \begin{cases} 
1/t & \text{for } t < T/4, \\
1 & \text{for } t \in (T/3, 2T/3), \\
1/(T - t) & \text{for } t > 3T/4, \\
\geq 1 & \text{for } t \in (0, T), 
\end{cases}
$$

$$
\psi_R(x) = \frac{L^2 - x^2}{2} + 2\varepsilon(L - x), \quad x \in (-\varepsilon, L).
$$

Remark

$$
\varphi_{R,n}(t, x) \sim n \coth(2nt) \frac{L^2 - x^2}{2}.
$$
Then \( \exists n_0 > 0 \) and \( C > 0 \) s.t. for all \( n \geq n_0 \), for all \( u_n \) satisfying

\[
\begin{cases}
(\partial_t - \partial_x^2 + n^2 x^2)u_n = 0, & (t, x) \in (0, T) \times (-\varepsilon, L), \\
u_n(t, -\varepsilon) = u_n(t, L) = 0, & t \in (0, T), \\
u_n(0, \cdot) = u_{0,n} \in H_0^1(-\varepsilon, L),
\end{cases}
\]

we have

\[
n^{3/2} \left\| \theta^{3/2} u_n e^{-\varphi_{R,n}} \right\|_{L^2((0,T) \times (-\varepsilon,L))} \leq Cn^{1/2} \left\| \theta^{1/2} \partial_x u_n(t, L^+) e^{-\theta(t)} \right\|_{L^2(0,T)}
\]

A typical Carleman estimate:

- **Weight function adapted to the potential** \( x^2 \), inspired by the Mehlker kernel.
- **The Carleman parameter is chosen** = \( n \).
- **Sources terms can be handled easily.**
From $x = 0$ to $x = -L$

There, the potential $n^2 x^2$ improves the observability property.

We choose a weight function:

$$\varphi_{L,n}(t, x) = n\theta(t)A - \sqrt{n}\theta(t) \left( \frac{x^2}{2} + 2Lx \right), \quad (t, x) \in (0, T) \times (-L, 0),$$

where $A$ is a suitable positive constant.
\[ \exists n_0 > 0 \text{ and } C > 0 \text{ s.t. for all } n \geq n_0, \text{ for all } u_n \text{ satisfying } \]
\[ \left\{ \begin{array}{l}
(\partial_t - \partial_x^2 + n^2 x^2)u_n = 0, \quad (t, x) \in (0, T) \times (-L, 0), \\
u_n(t, -L) = u_n(t, 0) = 0, \quad t \in (0, T), \\
u_n(0, .) = u_{0,n} \in H^1_0(-L, 0). 
\end{array} \right. \]

we have
\[ n^{3/4} \left\| \theta^{3/2} u_n e^{-\varphi_{L,n}} \right\|_{L^2(0, T) \times (-L, 0)} \leq C n^{1/4} \left\| \theta^{1/2} \partial_x u_n(t, 0) e^{-n\theta(t)A} \right\|_{L^2(0, T) \times (-L, 0)}. \]

The weight function
\[ \varphi_{L,n}(t, x) = n\theta(t)A - \sqrt{n\theta(t)} \left( \frac{x^2}{2} + 2Lx \right), \]

- \( \varphi_{L,n} \) is essentially constant in space \( \simeq n\theta(t)A \).
- Variations (in \( x \)) of the weight function are of lower order.
- \( A \) will be chosen to match \( \psi_{R,n}(t, 0) \) for the gluing argument.
1. Introduction
   - Context
   - Results

2. Proofs
   - Strategy
   - Control on the two lateral boundaries
   - Boundary condition at $x = 0$
   - Observation at one end

3. Further comments
Our results also apply to

- Non-symmetric geometric settings.
- **Heisenberg equations** in tensorized domains:

\[
\partial_t - \partial_x^2 - (x \partial_y + \partial_z)^2.
\]

see [Beauchard Cannarsa ’17] for previous results.

- Some **inverse problems** similar to the ones of [Beauchard Cannarsa Yamamoto ’14].

- Slightly more general settings for Grushin equations:
  
  - \(\Omega = \Omega_x \times \Omega_y\), with \(\Omega_y\) of any dimension,
  
  - Operators of the form \(\partial_t - \partial_x^2 - q(x)^2 \partial_y^2\) with

\[
q \in C^3(-L_-, L_+), \quad q(0) = 0, \quad \inf \partial_x q > 0.
\]

\(\leadsto\) **Critical time** \(T^* = \frac{1}{q'(0)} \int_0^{L_+} q(x) \, dx\) when observed from \(L_+\).
Open problems

- Determine the correct **geometric condition in non-tensorized settings** for which observability holds for Grushin operators.

- **Determine the time** required for observability for Kolmogorov like operators

\[ \partial_t - \partial_{vv} + v^2 \partial_x, \quad \text{or} \quad \partial_t - \partial_{vv} + v^2(-\Delta_x)^{1/2}. \]

See [Beauchard Helffer Henry Robbiano ’15].

- Precisely describe the **reachable sets** in each of the above situations, and **how it evolves in time**. See [Dardé Ervedoza ’16].
Merci pour votre attention !

Ref: Minimal time issues for the observability of Grushin-type equations, Karine Beauchard, Jérémie Dardé, Sylvain Ervedoza, 2018. Available on HaL.