Moving pointwise control of a simplified fluid-structure system

Sorin Micu

Department of Mathematics, University of Craiova, Romania

E-mail: sd_micu@yahoo.com

(joint work with **N. Cindea, I. Roventa, M. Tucsnak**)

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Outline

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- The model

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- Statement of the main result: stabilizability and controllability

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The model

- We consider a simplified 1D model for a swimmer which is able to undergo self-propelled motions in a viscous fluid.
- More precisely we consider a point mass moving in a long and thin pipe filled with a fluid. The fluid is modelled by the viscous Burgers equation whereas the point mass obeys Newton's second law.
- The control variable is a force acting on the swimmer.

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- More precisely we consider a point mass moving in a long and thin pipe filled with a fluid. The fluid is modelled by the viscous Burgers equation whereas the point mass obeys Newton's second law.
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The main result asserts that for any initial data there exists a control such that, at the end of the control process,

- the point reaches approximately its destination
- the velocities of the fluid and of the point mass simultaneously vanish ("silent swimmer").

The model

$$\begin{cases} \dot{v}(t,y) - v_{yy}(t,y) + v(t,y)v_y(t,y) = 0 & y \in (-1,1) \setminus \{h(t)\}, \\ v(t,-1) = v(t,1) = 0 & t \in (0,T), \\ \dot{h}(t) = v(t,h(t)) & t \in (0,T), \\ \ddot{h}(t) = [v_y](t,h(t)) + u(t) & t \in (0,T), \\ v(0,y) = v_0(y) & y \in (-1,1), \\ h(0) = h_0, \qquad \dot{h}(0) = g_0. \end{cases} \tag{1}$$

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\tag{1}
$$

- $v = v(t,y)$ denotes the velocity field of the fluid
- $h = h(t)$ indicates the position of the point mass
- the force exerted by the fluid on the mass is given by the jump of the derivative of v when crossing the mass, denoted by $[v_y](t,h(t))$
- $u = u(t)$ is the control acting on the point mass.

Short bibliography

- J.L. Vasquez and E. Zuazua *(CPDE 2003, M3AS 2006)* introduce a simplified model of a 1-D fluid on the real line containing one or several immersed rigid bodies and study the existence of solutions and their asymptotic behavior as time goes to infinity.
- A. Doubova and E. Fernandez-Cara *(M3AS-2005)* proves a null-controllability result based on Carleman estimates and fixed point techniques. Controls acting on both ends points.
- Y. Liu, T. Takahashi and M. Tucsnak *(ESAIM COCV 2013)* proves a null-controllability result by using spectral methods and a new fixed point procedure. The control acts at one end only.
- V. Starovoitov, J. San Martin, M. Tucsnak *(ARMA 2002)*, E. Feireisl *(JEE, 2003)*, T. Takahashi *(Adv. Diff. Eq. 2003)*, J. Ortega, L. Rosier, T. Takahashi *(Ann. Inst. H. Poincaré 2007)*, O. Imanuvilov, T. Takahashi *(JMPA, 2007)*, M. Boulakia, A. Osses *(ESAIM COCV 2008)*, O. Glass, F. Sueur *(Proc. AMS 2012)*

In our problem we consider the 1-D simplified model in an interval and with the control active only on the moving point.

Main result

Theorem

Let $v_0 \in L^2[-1,1]$, $g_0 \in \mathbb{R}$ and $h_0 \in (-1,1)$. Then for every $\tilde{h}_1 \in (-1,1)$ and $\eta > 0$ there exist $T > 0$ and $u \in L^\infty[0,T]$ such that the solution of (1) satisfies

$$v(T,\cdot) = 0, \qquad |h(T) - \tilde{h}_1| \leq \eta, \qquad \dot{h}(T) = 0. \tag{2}$$

The strategy to prove Theorem 1 consists in two main steps.

- Stabilizability: Choose u in a feedback form for which an appropriate Lyapounov functions is non increasing along the trajectories of the closed loop system. Prove the global existence of solutions and that the proposed feedback law steers the system arbitrarily close to a desired final state as time tends to infinity.
- Controllability: Prove the exact controllability to a well-chosen equilibrium state result for small initial data. This controllability property is based on a series of results for the linearized system.

Definition of solutions

Definition

Given $T > 0$, $v_0 \in L^2[-1, 1]$, $h_0 \in (-1, 1)$, $g_0 \in \mathbb{R}$ and $u \in L^2[0, T]$, we say that

$$\begin{bmatrix} v \\ g \\ h \end{bmatrix} \in \left\{ \mathcal{C}([0, T]; L^2[-1, 1]) \cap L^2([0, T]; \mathcal{H}_0^1(-1, 1)) \right\} \times L^2[0, T] \times \mathcal{H}^1(0, T),$$

is a finite energy solution of (1) on $[0, T]$ if $h(0) = h_0$, $\dot{h}(t) = g(t) = v(t, h(t))$ and $h(t) \in (-1, 1)$, for almost every $t \in [0, T]$ and

$$\begin{aligned} \int_{-1}^{1} v(t, y)\psi(t, y)\,\mathrm{d}y - \int_{-1}^{1} v_0(y)\psi(0, y)\,\mathrm{d}y + g(t)l(t) - g_0 l(0) - \int_0^t g(\sigma)\dot{l}(\sigma)\,\mathrm{d}\sigma \\ - \int_0^t \int_{-1}^{1} v(\sigma, y)\dot{\psi}(\sigma, y)\,\mathrm{d}y\,\mathrm{d}\sigma + \int_0^t \int_{-1}^{1} v_y(\sigma, y)\psi_y(\sigma, y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ - \frac{1}{2}\int_0^t \int_{-1}^{1} v^2(\sigma, y)\psi_y(\sigma, y)\,\mathrm{d}y\,\mathrm{d}\sigma = \int_0^t u(\sigma)l(\sigma)\,\mathrm{d}\sigma, \end{aligned} \tag{3}$$

for every $t \in [0, T]$ and for every

$$\begin{bmatrix} \psi \\ l \end{bmatrix} \in \left\{ \mathcal{H}^1((0, T); L^2[-1, 1]) \cap L^2([0, T]; \mathcal{H}_0^1(-1, 1)) \right\} \times \mathcal{H}^1(0, T), \tag{4}$$

$$l(t) = \psi(t, h(t)) \qquad (t \in [0, T]). \tag{5}$$

The stabilizability result

Choose in (1) the feedback u given by

$$u(t) = -k_v \dot{h}(t) + k_p(h_1 - h(t)) \qquad (t \in [0, T]), \qquad (6)$$

where $k_v \geqslant 0$ and $k_p > 0$ are fixed constants and $h_1 \in (-1, 1)$.

- The total energy of the system is non increasing

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_{-1}^{1} v^2 \,\mathrm{d}y + \frac{1}{2}(\dot{h}(t))^2 + \frac{k_p}{2}|h(t) - h_1|^2\right]$$

$$= -\int_{-1}^{1} v_y^2 \,\mathrm{d}y - k_v \dot{h}^2(t) \leq 0$$

- k_v may be taken zero but k_p should be large enough to keep the point mass far from the boundary. The feedback u acts like a spring and a damper between $h(t)$ and h_1.

Global existence of solutions

Theorem

Let $v_0 \in L^2[-1,1]$, $g_0 \in \mathbb{R}$ *and* $h_0 \in (-1,1)$. *Moreover, assume that the constants* h_1 *and* k_p *in* (6) *verify*

$$0 < |h_1 - h_0| < \frac{1}{2\sqrt{2}} \min(1 - h_1, 1 + h_1), \tag{7}$$

$$k_p |h_0 - h_1|^2 \geqslant \|v_0\|^2_{L^2[-1,1]} + |g_0|^2. \tag{8}$$

Then equations (1) *with* u *given by* (6) *admit, for every* $T > 0$, *a unique finite energy solution* $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ *on* $[0, T]$, *such that*

$$\min(1 \pm h(t)) \geqslant \frac{1}{2} \min(1 - h_1, 1 + h_1) \quad (t \in [0, T]). \tag{9}$$

We have the following energy estimate

$$\frac{1}{2}\int_{-1}^{1} v^2(t,y)\mathrm{d}y + \frac{1}{2}g^2(t) + \frac{k_p}{2}(h(t)-h_1)^2$$
$$= \frac{1}{2}\int_{-1}^{1} v_0^2(y)\mathrm{d}y + \frac{1}{2}g_0^2 + \frac{k_p}{2}(h_0-h_1)^2 - \int_0^t\int_{-1}^{1} v_y^2(\sigma,y)\mathrm{d}y - k_v\int_0^t g^2(\sigma)\,\mathrm{d}\sigma. \tag{10}$$

From (10) and (8) it follows that, for any $t \in [0, T_{max})$,

$$\int_{-1}^{1} v^2(t,y)\mathrm{d}y + g^2(t) + k_p|h(t)-h_1|^2 \leqslant 2k_p|h_0-h_1|^2.$$

Hence, from the last inequality and (7), we deduce that

$$|h(t)-h_1| \leqslant \sqrt{2}|h_0-h_1| \leqslant \frac{1}{2}\min(1-h_1, 1+h_1) \quad (t \in [0, T_{max}))$$

which clearly yields that

$$\min(1-h(t), 1+h(t)) \geqslant \frac{1}{2}\min(1-h_1, 1+h_1) \quad (t \in [0, T_{max})).$$

The stability result

Theorem

Under the assumptions of Theorem 3, the finite energy solution of (1) satisfies

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{L^2[-1,1]} = 0, \quad \lim_{t\to\infty} g(t) = 0, \quad \lim_{t\to\infty} h(t) = h_1.$$

Let $W_1, \ W_2, \ W_3 : [0,\infty) \to [0,\infty)$ be defined by

$$W_1(t) = \frac{1}{2}\left(\int_{-1}^{1} \varphi^2 \,\mathrm{d}y + |g|^2\right), \qquad W_2(t) = \frac{k_p}{2}|h - h_1|^2,$$

$$W_3(t) = \int_{-1}^{1} \varphi_y^2(y)\,\mathrm{d}y + k_v g^2.$$

From the energy estimate we have that

$$\dot{W}_1 + \dot{W}_2 = -W_3.$$

Since $W_1 << W_3$, we deduce that W_1 lies in $L^1(0,\infty)$.

Let us assume, by contradiction, that W_1 does not converge to zero for $t \to \infty$: there exists $\varepsilon > 0$ and a sequence $(t_n)_{n\geq 0}$ of positive numbers such that $t_n \to \infty$ and

$$W_1(t_n) \geqslant \varepsilon \qquad (n \in \mathbb{N}).$$

$$\delta_n = \max\left\{\delta > 0 \ \mid \ W_1(t_n - \delta) \geqslant \frac{\varepsilon}{2}\right\} \qquad (n \in \mathbb{N}).$$

$$W_1 \in L^1[0,\infty) \ \Rightarrow \ \sum_{n\in\mathbb{N}} \delta_n < \infty \ \Rightarrow \ \lim_{n\to\infty} \delta_n = 0.$$

On the other hand, since $W_1 + W_2$ is non increasing

$$\frac{\varepsilon}{2} + \frac{1}{2}|h(t_n - \delta_n) - h_1|^2 = W_1(t_n - \delta_n) + W_2(t_n - \delta_n)$$

$$\geqslant W_1(t_n) + W_2(t_n) \geqslant \varepsilon + \frac{1}{2}|h(t_n) - h_1|^2 \qquad (n \in \mathbb{N}),$$

so that

$$|h(t_n - \delta_n) - h_1|^2 - |h(t_n) - h_1|^2 \geqslant \varepsilon \qquad (n \in \mathbb{N}).$$

From the mean value theorem and $|h(t) - h_1| \leqslant 2$ it follows that for every $n \in \mathbb{N}$ there exist $\alpha_n \in (0,1)$ such that

$$|g(t_n - \alpha_n\delta_n)| = |\dot{h}(t_n - \alpha_n\delta_n)| \geqslant \frac{\varepsilon}{4\delta_n} \to \infty.$$

The above estimate contradicts the fact that $W_1 \in L^1[0,\infty)$.

$(h(t))_{t\geq 0} \subset [-1+\varepsilon, 1-\varepsilon]$ is relatively compact in $\mathbb{R}$. Let $(t_n)_{n\geqslant 0}$ be a sequence of positive numbers such that

$$t_n \to \infty, \qquad \lim_{n\to\infty} h(t_n) = h^* \in [-1+\varepsilon, 1-\varepsilon]. \qquad (11)$$

It follows that, given $T > 0$, we have

$$\lim_{n\to\infty} \int_{t_n}^{T+t_n} W_3\left(\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix}, t\right) \mathrm{d}t = 0$$

$$\Rightarrow\ W_3\left(\begin{bmatrix} 0 \\ 0 \\ h^* \end{bmatrix}, t\right) \equiv 0 \Rightarrow h^* = h_1.$$

The controllability result

Theorem

Let $\tau > 0$ *and let* $h_1 \in (-1, 1)$ *be an irrational algebraic number. Then there exists* $\delta > 0$ *such that for every* $v_0 \in L^2(-1, 1)$, $h_0 \in (0, 1)$, $g_0 \in \mathbb{R}$ *satisfying*

$$\|v_0\|_{L^2(-1,1)} + |g_0| + |h_1 - h_0| < \delta,$$

there exists a control $u \in L^\infty(0, T)$ *such that the solution of the nonlinear system (1) verifies*

$$v(\tau) = 0, \quad \dot{h}(\tau) = 0 \text{ and } h(\tau) = h_1.$$

- We have an exact controllability result.
- The result holds for special targets h_1 (irrational algebraic numbers). The set of these numbers is dense in $\mathbb{R}$.
- The controllability result is local and depends on h_1.

The controllability result: Change of variable

Given $h_1 \in (-1,1)$, by using an appropriate change of variable we rewrite (1) in a fixed spatial domain

$$\begin{cases} \dot{z} - z_{xx} + zz_x = N_1(z, z_x, h, g) & x \in (-1,1) \setminus \{h_1\} \\ z(t,-1) = z(t,1) = 0 & t \in (0,T) \\ z(t,h_1) = g(t) & t \in (0,T) \\ \dot{g}(t) = [z_x](t,h_1) + u(t) + N_2(z,h,g) & t \in (0,T) \\ \dot{h}(t) = g(t) & t \in (0,T) \\ z(0,x) = z_0(x) & x \in (-1,1) \\ h(0) = h_0, \quad g(0) = g_0. \end{cases} \tag{12}$$

- In (12) the “discontinuity” point h_1 does not depend of t (fixed spatial domain).
- The change of variable is a $\mathcal{C}^\infty$ map $y \mapsto x = \Psi(y, h(t))$ with $\Psi(y,p) = y - p + h_1$ in a neighborhood of p and $\Psi(y,p) = y$ close to the boundaries $y = \pm 1$.

The controllability result: linearized system

The linearized version of (12) reads as follows

$$\begin{cases} \dot{z}(t,x) - z_{xx}(t,x) = 0 & x \in (-1,1) \setminus \{h_1\} \\ z(t,-1) = z(t,1) = 0 & t \in (0,T), \\ z(t,h_1) = g(t) & t \in (0,T), \\ \dot{g}(t) = [z_x](t,h_1) + w(t) & t \in (0,T), \\ \dot{h}(t) = g(t) & t \in (0,T), \\ z(0,x) = z_0(x) & x \in (-1,1) \\ h(0) = h_0, \ g(0) = g_0. \end{cases} \tag{13}$$

Given an initial data (z_0, h_0, g_0) and a final position h_1 we look for a control $w \in L^\infty(0,T)$ such that the solution of (13) verifies

$$z(T,\cdot) = 0, \quad g(T) = 0, \quad h(T) = h_1. \tag{14}$$

Controllability of the linearized system

Theorem

Let $T > 0$ and $h_1 \in (-1, 1)$ be an irrational algebraic number. Then for each $(z_0, g_0, h_0) \in L^2(-1, 1) \times \mathbb{R}^2$ there exists a control $w \in \mathcal{C}[0, T]$ such that the solution (z, g, h) of (13) verifies (14) and

$$\|w\|_{\mathcal{C}[0,T]} \leq \kappa_0 e^{\frac{\kappa_1}{T}} \left(\|z_0\|_{L^2(-1,1)} + |g_0| + |h_1 - h_0| \right), \qquad (15)$$

where κ_0 and κ_1 are two positive constants independent of T and of the data (z_0, g_0, h_0).

- The controllability result holds for a class of special targets h_1 (irrational algebraic numbers).
- The control cost degenerate exponentially as T goes to zero. The constants κ_0 and κ_1 may depend of h_1.

Abstract framework

We consider the Hilbert space $H = L^2(-1,1) \times \mathbb{R}$ and we define the unbounded operator $A_0 : \mathcal{D}(A_0) \to H$

$$\mathcal{D}(A_0) = \left\{ \begin{bmatrix} \varphi \\ p \end{bmatrix} \in H \;\middle|\; \varphi|_{(-1,h_1)} \in H^2(-1,h_1), \varphi|_{(h_1,1)} \in H^2(h_1,1), \right. \tag{16}$$

$$\left. \varphi \in H_0^1(-1,1), \varphi(h_1) = p \right\}$$

$$A_0 \begin{bmatrix} \varphi \\ p \end{bmatrix} = \begin{bmatrix} -\varphi_{xx} \\ -[\varphi_x](h_1) \end{bmatrix} \qquad \left(\begin{bmatrix} \varphi \\ p \end{bmatrix} \in \mathcal{D}(A_0) \right). \tag{17}$$

Abstract framework

Let $B_0 \in \mathcal{L}(\mathbb{C}, H)$ and $C \in \mathcal{L}(H, \mathbb{C})$ be defined by

$$B_0 w = \begin{bmatrix} 0 \\ w \end{bmatrix} \qquad (w \in \mathbb{C}) \qquad C \begin{bmatrix} \varphi \\ p \end{bmatrix} = p \qquad \left(\begin{bmatrix} \varphi \\ p \end{bmatrix} \in H \right).$$

For $t \geqslant 0$ we denote $Y(t) = \begin{bmatrix} z(t) \\ g(t) \end{bmatrix}$. The controlled system (13) can be written equivalently as

$$\begin{cases} \dot{Y}(t) + A_0 Y(t) = B_0 w(t) \\ \dot{h}(t) = CY(t) \\ Y(0) = Y_0 \\ h(0) = h_0. \end{cases} \tag{18}$$

We study the linear controllability problem: given $T > 0$, $h_1 \in (-1, 1)$ and $(Y_0, h_0) \in H \times \mathbb{R}$, find a control $w \in \mathcal{C}[0, T]$ such that

$$Y(T) = 0, \quad h(T) = h_1. \tag{19}$$

Spectral analysis of A_0

Theorem

Let $h_1 \in (-1, 1) \setminus \mathbb{Q}$. Then the eigenvalues of the operator A_0 are simple and they can be arranged as an increasing sequence $(\lambda_n)_{n \geqslant 1}$ such that each eigenvalue of A_0 is the square of a positive root of the equation

$$\frac{1}{\tan((1-h_1)x)} + \frac{1}{\tan((1+h_1)x)} = x. \tag{20}$$

Moreover, there exists a sequence of eigenvectors of A_0, $(\varphi_n)_{n \geq 1}$ which forms an orthonormal basis of H.

Spectral analysis of A_0

The corresponding eigenvectors $(\Phi_n)_{n\geq 1}$ are given by

$$\Phi_n = \frac{1}{\sqrt{D(\lambda_n)}} \begin{bmatrix} \varphi_n \\ 1 \end{bmatrix} \quad (n \geq 1), \tag{21}$$

where

$$\varphi_n(x) = \begin{cases} \frac{\sin(\sqrt{\lambda_n}(1+x))}{\sin\left(\sqrt{\lambda_n}(1+h_1)\right)} & x \in (-1, h_1), \\ \frac{\sin(\sqrt{\lambda_n}(1-x))}{\sin\left(\sqrt{\lambda_n}(1-h_1)\right)} & x \in (h_1, 1) \end{cases}$$

and

$$D(\lambda_n) = \frac{1+h_1}{2\sin^2\left(\sqrt{\lambda_n}(1+h_1)\right)} + \frac{1-h_1}{2\sin^2\left(\sqrt{\lambda_n}(1-h_1)\right)} + \frac{1}{2}. \tag{22}$$

Spectral analysis of A_0

The roots of (20) can be divided in two families which in an increasing order verify the following inequalities

$$y_1 < y_2 < ... < y_{n_1-1} < x_1 < y_{n_1} < ... < y_{n_k-1} < x_k < y_{n_k} < ...$$

Lemma

There exists $C > 0$ such that the following properties hold

$$y_{n+1} - y_n > C\beta, \;\; n \geq 1, \tag{23}$$

$$x_{k+1} - x_k > C\alpha, \;\; k \geq 1, \tag{24}$$

$$\max\left\{x_k - y_{n_k-1}, \, y_{n_k} - x_k\right\} > \frac{C}{k\pi}, \;\; k \geq 1. \tag{25}$$

The eigenvalues of our problem are

$$\{\lambda_m\}_{m\geq 1} = \{(x_k)^2\}_{k\geq 1} \cup \{(y_n)^2\}_{n\geq 1}.$$

The moment problem

Now, we have all the ingredients needed to translate the controllability problem (18)-(19) into a moments problem.

Theorem

Let $T > 0$, $h_0 \in \mathbb{R}$, $h_1 \in (-1,1) \setminus \mathbb{Q}$ and $Y_0 \in H$. A function $w \in L^2(0,T)$ is a control which leads the solution (Y,h) of (18) to verify (19) if and only if

$$\begin{cases} B_0^*\Phi_n \displaystyle\int_0^T e^{s\lambda_n} w(s)ds = -\langle Y_0, \Phi_n\rangle \qquad (n \geqslant 1) \\ \displaystyle\sum_{n\in\mathbb{N}^*} \frac{C\Phi_n B_0^*\Phi_n}{\lambda_n} \int_0^T w\, ds = h_1 - h_0 - \sum_{n\in\mathbb{N}^*} \frac{C\Phi_n}{\lambda_n}\langle Y_0, \Phi_n\rangle \end{cases} \tag{26}$$

Solution to the moment problem

We recall that $(F_m)_{m\geq 1} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is a biorthogonal sequence to the family of exponential functions $\left(e^{\lambda_n t}\right)_{n\geq 1}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ iff

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F_m(t)e^{\overline{\lambda}_n t}dt = \delta_{nm}. \tag{27}$$

A solution $w \in \mathcal{C}[0, T]$ to the moment problem (26) is given by

$$w(t) = a_0 F_0\left(t - \frac{T}{2}\right) - \sum_{n\geq 1} \frac{\langle Y_0, \Phi_n\rangle}{B_0^* \Phi_n} e^{-\lambda_n T/2} F_n\left(t - \frac{T}{2}\right) \tag{28}$$

and

$$a_0 = \frac{h_1 - h_0 - \sum_{n\geq 1} \frac{C\Phi_n}{\lambda_n}\langle Y_0, \Phi_n\rangle}{\sum_{n\geq 1} \frac{C\Phi_n B_0^* \Phi_n}{\lambda_n}}.$$

A biorthogonal sequence

Given $\alpha > \beta > 0$, consider two families of positive real numbers, $(\lambda_n^1)_{n\geq 1}$ and $(\lambda_n^2)_{n\geq 1}$, which verify

(Λ_1) $|\lambda_n^1 - \beta^2 n^2| \leq c_1 n, \; |\lambda_n^2 - \alpha^2 n^2| \leq c_1 n \qquad (n \geq 1);$

(Λ_2) $\sqrt{\lambda_{n+1}^1} - \sqrt{\lambda_n^1} \geq r, \; \sqrt{\lambda_{n+1}^2} - \sqrt{\lambda_n^2} \geq r \qquad (n \geq 1);$

(Λ_3) $\frac{c_2}{k} \leq \inf_{n\geq 1} \left| \sqrt{\lambda_k^2} - \sqrt{\lambda_n^1} \right| \qquad (k \geq 1).$

Under (Λ_1)-(Λ_2), there exists biorthogonal sequences to each of the families $(e^{\lambda_n^1 t})_{n\geq 1}$ and $(e^{\lambda_n^2 t})_{n\geq 1}$, separately (see, for instance, G. Tenenbaum and M. Tucsnak (JDE, 2007)).

We show that, for any $T > 0$, there exists a biorthogonal sequence to the family $(e^{\lambda_n^1 t})_{n\geq 1} \cup (e^{\lambda_n^2 t})_{n\geq 1} \cup \{e^{\lambda_0 t}\}$ in $L^2\left[-\frac{T}{2}, \frac{T}{2}\right]$. In order to do that, the separability condition (Λ_3) plays a fundamental role.

Existence of a biorthogonal sequence

Theorem

Let $T > 0$, $(\lambda_n^1)_{n\geq 1}$ *and* $(\lambda_n^2)_{n\geq 1}$ *be two sequences of positive numbers which verify properties* $(\Lambda_1) - (\Lambda_3)$ *and* $\lambda_0 = 0$*. Then there exist* $(F_n^1)_{n\geq 1} \cup (F_n^2)_{n\geq 1} \cup \{F_0^1\} \subset \mathcal{C}^\infty\left[-\frac{T}{2}, \frac{T}{2}\right]$ *which form a biorthogonal sequence to the family* $(e^{\lambda_n^1 t})_{n\geq 1} \cup (e^{\lambda_n^2 t})_{n\geq 1} \cup \{e^{\lambda_0 t}\}$ *in* $L^2\left[-\frac{T}{2}, \frac{T}{2}\right]$ *such that*

$$\|F_n^j\|_{\mathcal{C}[-\frac{T}{2},\frac{T}{2}]} \leq \frac{c}{\lambda_n^j} e^{w\sqrt{\lambda_n^j}+\frac{\kappa}{T}} \qquad (n \geq 1, \quad j \in \{1,2\}),$$

$$\|F_0^1\|_{\mathcal{C}[-\frac{T}{2},\frac{T}{2}]} \leq c e^{\frac{\kappa}{T}},$$

where the constants c*,* ω *and* κ *are independent of* n *and* T*.*

End of the proof of the controllability: linear case

$$w(t) = a_0 F_0\left(t - \frac{T}{2}\right) - \sum_{n\geq 1} \frac{\langle Y_0, \Phi_n\rangle}{B_0^*\Phi_n} e^{-\lambda_n T/2} F_n\left(t - \frac{T}{2}\right)$$

$$\|w\|_{\mathcal{C}[0,T]} \quad \leq \quad Ce^{\frac{\kappa}{T}}\left[|a_0| + \sum_{n\geq 1}\left|\frac{\langle Y_0, \Phi_n\rangle}{\lambda_n B_0^*\Phi_n}\right| e^{-\lambda_n T/2 + \omega\sqrt{\lambda_n}}\right].$$

To evaluate the last expression we take into account that

$$B_0^*\Phi_n = \frac{1}{m\sqrt{D(\lambda_n)}}, \quad |a_0|^2 \leq C\left(|h_1 - h_0|^2 + \sum_{n\geq 1} |\langle Y_0, \Phi_n\rangle|^2\right).$$

End of the proof of the controllability: linear case

Now, we have to evaluate the quantities

$$D(\lambda_n) = \frac{1+h_1}{2}\frac{1}{\sin^2\left(\sqrt{\lambda_n}(1+h_1)\right)} + \frac{1-h_1}{2}\frac{1}{\sin^2\left(\sqrt{\lambda_n}(1-h_1)\right)} + \frac{1}{2},$$

which depend of

$$l_k = \min\left\{k\alpha - (n_k - 1)\beta, n_k\beta - k\alpha\right\}$$

$$= k\beta \min\left\{\frac{\alpha}{\beta} - \frac{n_k - 1}{k}, \frac{n_k}{k} - \frac{\alpha}{\beta}\right\},$$

$$\frac{\alpha}{\beta} = \frac{1-h_1}{1+h_1}.$$

End of the proof of the controllability: linear case

Roth's Theorem: If θ is an irrational algebraic number and $\delta > 0$ arbitrarily small there are only a finite number of pairs of integers $q \in \mathbb{N}$, $p \in \mathbb{Z}$ such that

$$\left|\theta - \frac{p}{q}\right| < q^{-2-\delta}.$$

By choosing h_1 an irrational algebraic number and $\delta > 0$ there exists a positive constant C (depending on h_1) such that

$$l_k \geq Ck^{-1-\delta} \; (k \geq 1) \quad \Rightarrow \left|D(\lambda_k^i)\right| \leq Ck^{2+2\delta} \quad (k \geq 1, \; i \in \{1,2\}),$$

which implies the desired bound for the control

$$\|w\|^2_{\mathcal{C}[0,T]} \leq \frac{Ce^{\frac{2\kappa+12\omega^2}{T}}}{T^{2+2\delta}} \left(|h_1 - h_0|^2 + \sum_{n\geq 1} |\langle Y_0, \Phi_n\rangle|^2\right). \qquad (29)$$

Controllability with a source term

In this section we consider a control system derived from (18) by adding appropriate source terms. More precisely, for $f : [0, \infty) \to H$ we consider the system:

$$\begin{cases} \dot{Y}(t) + A_0 Y(t) = B_0 w(t) + f(t) \\ \dot{h}(t) = CY(t) \\ Y(0) = Y_0 \\ h(0) = h_0. \end{cases} \tag{30}$$

We introduce the weights

$$\rho_{\mathcal{F}}(t) = e^{-\frac{\alpha}{(\tau-t)^2}}, \quad \rho_0(t) = M_0 e^{\frac{M_1}{(q-1)(\tau-t)} - \frac{\alpha}{q^4(\tau-t)^2}} \tag{31}$$

where $q > 1$ and $\alpha > \frac{M_1 q^4 \tau}{2(q-1)}$.
Note that the constant α is chosen such that $\rho_0(\tau) = 0$.

Controllability with a source term

To these functions we associate the following Hilbert spaces

$$\mathcal{F} = \left\{ f \in L^2([0,\tau],H) \;\middle|\; \frac{f}{\rho_{\mathcal{F}}} \in L^2([0,\tau],H) \right\}, \tag{32}$$

$$\mathcal{W} = \left\{ w \in L^2(0,\tau) \;\middle|\; \frac{w}{\rho_0} \in L^2(0,\tau) \right\}, \tag{33}$$

$$\mathcal{Z} = \left\{ z \in L^2([0,\tau],H) \;\middle|\; \frac{z}{\rho_0} \in L^2([0,\tau],H) \right\}. \tag{34}$$

The inner product in $\mathcal{F}$ is defined by

$$\langle f_1, f_2 \rangle_{\mathcal{F}} = \int_0^\tau \rho_{\mathcal{F}}^{-2}(t) \langle f_1(t), f_2(t) \rangle dt$$

and similar definitions are considered in $\mathcal{W}$ and $\mathcal{Z}$. The induced norms are denoted by $\|\cdot\|_{\mathcal{F}}$, $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_{\mathcal{Z}}$, respectively.

Controllability with a source term

Theorem

In the above hypothesis let $\tau > 0$ *and* $f \in \mathcal{F}$ *be a source term in non homogeneous system* (30)*. Then, for any* $(Y_0, h_0) \in H \times \mathbb{R}$ *and* $h_1 \in \mathbb{R}$ *an irrational algebraic number there exists* $w \in \mathcal{W}$ *such that the solution* (Y, h) *of* (30) *satisfies* $Y \in \mathcal{Z}$ *and* $h(\tau) = h_1$*. Furthermore, there exists a positive constant* C*, not depending on* f*,* Y_0 *and* h_0*, such that*

$$\left\| \frac{Y}{\rho_0} \right\|_{\mathcal{C}([0,\tau],H)} + \left\| \frac{h - h_1}{\rho_0} \right\|_{\mathcal{C}[0,\tau]} + \left\| \frac{w}{\rho_0} \right\|_{\mathcal{C}[0,\tau]} \qquad (35)$$

$$\leq C \left(\|f\|_{\mathcal{F}} + \|Y_0\| + |h_0 - h_1| \right).$$

Controllability of the nonlinear system

Nonlinear system (12) can be written as

$$\begin{cases} \dot{Y}(t) + A_0 Y(t) = N \begin{bmatrix} Y(t) \\ h(t) \end{bmatrix} + B_0 w(t), \quad t \in (0, T) \\ \dot{h}(t) = CY(t), \quad t \in (0, T) \\ Y(0) = Y_0 \\ h(0) = h_0. \end{cases} \tag{36}$$

In order to control the nonlinear system (36) we consider its linearization (30), where $f : [0, \tau] \to H$. The source term control theorem gives the existence of a control $w = E_\tau(Y_0, h_0, h_1, f)$ such that the trajectory $\begin{bmatrix} Y \\ h \end{bmatrix}$ of (30) satisfies $Y(\tau) = 0$ and $h(\tau) = h_1$.

For every $\delta > 0$ small enough, we denote

$$\mathcal{B}_\delta = \{f \in \mathcal{F} \mid \|f\|_{\mathcal{F}} \leq \delta\}$$

and we define the operator $\mathcal{N}$ acting on $\mathcal{B}_\delta$ by

$$\mathcal{N}(f)(t) = N \begin{bmatrix} Y(t) \\ h(t) \end{bmatrix},$$

where N is the non linear operator from (36) and $\begin{bmatrix} Y(t) \\ h(t) \end{bmatrix}$ is the trajectory of (30) with the control $w = E_\tau(Y_0, h_0, h_1, f)$.

- $\mathcal{N}$ is a contraction on $\mathcal{B}_\delta$, where δ and $|h_0 - h_1|$ are small enough;
- $\mathcal{N}$ has a fixed point;
- controllability of the nonlinear system is a consequence of the controllability of it's linearization.

Thank you for your attention!