On the controllability of micropolar fluids

Sergio Guerrero Rodríguez

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Plan of the talk

- Main results
- References
- Ideas of the proof
- Open problems

Main results (1/2)

$\Omega \subset \mathbb{R}^d$ smooth bounded domain ($d = 2, 3$), $T > 0$
$Q := (0, T) \times \Omega$, $\Sigma := (0, T) \times \partial\Omega$, $\mathcal{O} \subset\subset \Omega$ open set

Fluid linear velocity : y, pressure : p, angular velocity ω :

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = \nabla \times \omega + \mathbf{1}_{\mathcal{O}} u, \ \ \nabla \cdot y = 0 & \text{in } Q, \\ \omega_t - \Delta \omega - (d-2) \nabla(\nabla \cdot \omega) + (y \cdot \nabla)\omega = \nabla \times y + \mathbf{1}_{\mathcal{O}} v & \text{in } Q, \\ y = 0, \quad \omega = 0 & \text{on } \Sigma, \\ y|_{t=0} = y_0, \quad \omega|_{t=0} = \omega_0 & \text{in } \Omega. \end{cases}$$

Trajectory $(\overline{y}, \overline{p}, \overline{\omega})$

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + (\overline{y} \cdot \nabla) \overline{y} + \nabla \overline{p} = \nabla \times \overline{\omega}, \ \ \nabla \cdot \overline{y} = 0 & \text{in } Q, \\ \overline{\omega}_t - \Delta \overline{\omega} - (d-2) \nabla(\nabla \cdot \overline{\omega}) + (\overline{y} \cdot \nabla)\overline{\omega} = \nabla \times \overline{y} & \text{in } Q, \\ \overline{y} = 0, \quad \overline{\omega} = 0 & \text{on } \Sigma, \\ \overline{y}|_{t=0} = \overline{y}_0, \quad \overline{\omega}|_{t=0} = \overline{\omega}_0 & \text{in } \Omega. \end{cases}$$

Main results (2/2)

Local exact controllability to trajectories :

$$\exists \delta > 0 \text{ such that if } \|y_0 - \overline{y}_0\| + \|\omega_0 - \overline{\omega}_0\| < \delta$$

then

$$\exists (u, v) \text{ and } (y, p, \omega) \text{ such that } y|_{t=T} \equiv \overline{y}|_{t=T},\ \omega|_{t=T} \equiv \overline{\omega}|_{t=T}.$$

Assumptions :

$$y_0 \in V,\ \omega_0 \in H_0^1(\Omega),\ \overline{\omega} \in L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H_0^1(\Omega)).$$

MAIN RESULT 1 ([with Cornilleau 2015]) : $d = 2, 3$, local exact controllability to $(0, \overline{p}, \overline{\omega})$ with $u \equiv 0$.
We control $3d - 3$ variables with $2d - 3$ scalar controls.

MAIN RESULT 2 ([with Cornilleau 2015]) : $d = 2$, local exact controllability to 0 with $v \equiv 0$.
We control 3 variables with 2 scalar controls.

Remark

Difficulty when $v \equiv 0$ and $d = 3$

The unique continuation property for the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = \nabla \times \psi, \ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi - \nabla(\nabla \cdot \psi) = \nabla \times \varphi & \text{in } Q, \\ \varphi = \psi = 0 & \text{on } \Sigma, \\ \varphi|_{t=T} = \varphi_T, \quad \psi|_{t=T} = \psi_T & \text{in } \Omega, \end{cases}$$

is not satisfied. That is to say, there exists (φ, π, ψ) with

$$\varphi \equiv 0 \text{ in } (0,T) \times \mathcal{O} \ \text{ and } \ (\varphi|_{t=0}, \psi|_{t=0}) \not\equiv (0,0). \tag{1}$$

For instance, Ω the unit ball and z solution of

$$\begin{cases} -2\Delta z = \mu z & \text{in } \Omega, \\ \nabla z = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, $(\varphi, \pi, \psi) = (0, 0, e^{-\mu t}\nabla z)$ satisfies (1).

References

Works on the subject

• Micropolar fluids

- [Lukaszewicz 99]

- [G., Fernández-Cara 07]

• Controllability of heat-like systems

- [Ammar-Khodja, Benabdallah, Cristofol, De Teresa , Dupaix, Gaitan, González-Burgos 03-14]

- [Coron, G., Rosier 10]

• Controllability of Navier-Stokes-like systems

- [Carreño-Godoy, G., Gueye 12-14]

• Controllability of Navier-Stokes-like problems with a reduced number of controls

- [Carreño-Godoy, Coron, Fernández-Cara, G., Gueye, Imanuvilov, Lissy, Puel 06-14]

Idea of the proofs (1/1)

- Null controllability of

$$\begin{cases} y_t - \Delta y + \nabla p = \nabla \times \omega + \mathbf{1}_{\mathcal{O}} v + f_1, \ \nabla \cdot y = 0 & \text{in } Q, \\ \omega_t - \Delta \omega - (d-2)\nabla(\nabla \cdot \omega) = \nabla \times y + \mathbf{1}_{\mathcal{O}} v + f_2 & \text{in } Q, \\ y = 0, \quad \omega = 0 & \text{on } \Sigma, \\ y|_{t=0} = y_0, \quad \omega|_{t=0} = \omega_0 & \text{in } \Omega, \end{cases}$$

in appropriate weighted-spaces and for adequate f_1, f_2.

$\rightsquigarrow$ Carleman inequalities for the adjoint system.

- Inverse mapping theorem :

$$f_1 := -(y \cdot \nabla) y, \quad f_2 := -(y \cdot \nabla)\omega.$$

Proof $d = 3$, $u \equiv 0$ (1/2)

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = \nabla\times\psi + g_0,\ \ \nabla\cdot\varphi = 0 & \text{in } Q, \\ -\psi_t - \Delta\psi - \nabla(\nabla\cdot\psi) = \nabla\times\varphi + g_1 & \text{in } Q, \\ \varphi = \psi = 0 & \text{on } \Sigma, \\ \varphi|_{t=T} = \varphi_T,\ \psi|_{t=T} = \psi_T & \text{in } \Omega. \end{cases}$$

We prove

$$\int_Q \theta_1(|\varphi|^2 + |\psi|^2) \leq C\left(\int_{(0,T)\times\mathcal{O}} \theta_2|\psi|^2 + \int_Q \theta_3|g_0|^2 + \int_Q \theta_4|g_1|^2\right).$$

Proof $d=3$, $u\equiv 0$ (1/2)

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = \nabla\times\psi + g_0,\ \nabla\cdot\varphi = 0 & \text{in } Q, \\ -\psi_t - \Delta\psi - \nabla(\nabla\cdot\psi) = \nabla\times\varphi + g_1 & \text{in } Q, \\ \varphi = \psi = 0 & \text{on } \Sigma, \\ \varphi|_{t=T} = \varphi_T,\ \psi|_{t=T} = \psi_T & \text{in } \Omega. \end{cases}$$

We prove

$$\int_Q \theta_1(|\varphi|^2 + |\psi|^2) \le C\left(\int_{(0,T)\times\mathcal{O}} \theta_2|\psi|^2 + \int_Q \theta_3|g_0|^2 + \int_Q \theta_4|g_1|^2\right).$$

- Carleman inequalities for ψ and $\nabla\cdot\psi$:

$$\int_Q \rho_1|\psi|^2 \le C\left(\int_{(0,T)\times\mathcal{O}} \rho_1|\psi|^2 + \int_Q \rho_2(|\nabla\times\varphi|^2 + |g_1|^2) + \|\rho_3\nabla\cdot\psi\|^2_{H^{1/4,1/2}(\Sigma)}\right).$$

Regularity estimates for ψ :

$$\int_Q \rho_1|\psi|^2 \le C\left(\int_{(0,T)\times\mathcal{O}} \rho_4|\psi|^2 + \int_Q \rho_5(|g_1|^2 + |\nabla\times\varphi|^2)\right).$$

Proof $d = 3$, $u \equiv 0$ **(2/2)**

- Equation satisfied by $\nabla \times \varphi$:

$$-(\nabla \times \varphi)_t - \Delta(\nabla \times \varphi) = \nabla \times (\nabla \times \psi + g_0) \text{ in } Q.$$

Carleman inequality for $\nabla \times \varphi$ ($\widetilde{\mathcal{O}} \subset\subset \mathcal{O}$):

$$\int_Q \rho_5 |\nabla \times \varphi|^2 \leq C \left(\int_{(0,T)\times\widetilde{\mathcal{O}}} \rho_5 |\nabla \times \varphi|^2 + \|\rho_6 \nabla \times \varphi\|^2_{H^{1/4,1/2}(\Sigma)} + \int_Q \rho_7 (|\nabla \times \psi|^2 + |g_0|^2) \right).$$

Estimate of the local term $\rightsquigarrow$ equation of ψ
Estimate of the boundary term $\rightsquigarrow$ regularity estimates

$$\int_Q \rho_5 |\nabla \times \varphi|^2 \leq \varepsilon \int_Q \rho_1 |\psi|^2 + C \left(\int_{(0,T)\times\mathcal{O}} \rho_8 |\psi|^2 + \int_Q \rho_9 (|g_0|^2 + |g_1|^2) \right).$$

Proof $d = 2$, $v \equiv 0$ (1/1)

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = \nabla\times\psi + g_0, \ \nabla\cdot\varphi = 0 & \text{in } Q, \\ -\psi_t - \Delta\psi = \nabla\times\varphi + g_1 & \text{in } Q, \\ \varphi = \psi = 0 & \text{on } \Sigma, \end{cases}$$

We prove

$$\int_Q \sigma_1(|\varphi|^2 + |\psi|^2) \leq C\left(\int_{(0,T)\times\mathcal{O}} \sigma_2|\varphi|^2 + \|\sigma_3 g_0\|^2_{L^2(H^2)} + \|\sigma_4 g_1\|^2_{L^2(H^2)}\right).$$

Proof $d = 2$, $v \equiv 0$ (1/1)

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = \nabla\times\psi + g_0, \ \nabla\cdot\varphi = 0 & \text{in } Q, \\ -\psi_t - \Delta\psi = \nabla\times\varphi + g_1 & \text{in } Q, \\ \varphi = \psi = 0 & \text{on } \Sigma, \end{cases}$$

We prove

$$\int_Q \sigma_1(|\varphi|^2 + |\psi|^2) \le C\left(\int_{(0,T)\times\mathcal{O}} \sigma_2|\varphi|^2 + \|\sigma_3 g_0\|^2_{L^2(H^2)} + \|\sigma_4 g_1\|^2_{L^2(H^2)}\right).$$

Carleman inequalities for $\Delta\psi$ and $\Delta\varphi$ ($\widetilde{\mathcal{O}} \subset\subset \mathcal{O}$):

$$\int_Q \alpha_1(|\Delta\psi|^2 + |\Delta\varphi|^2) \le C\left(\int_{(0,T)\times\widetilde{\mathcal{O}}} \alpha_1(|\Delta\psi|^2 + |\Delta\varphi|^2) + \|\alpha_2 g_0\|^2_{L^2(H^1)}\right.$$
$$\left. + \|\alpha_3 g_1\|^2_{L^2(H^1)} + \|\alpha_4\Delta\psi\|^2_{H^{1/4,1/2}(\Sigma)} + \|\alpha_5\Delta\varphi\|^2_{H^{1/4,1/2}(\Sigma)}\right).$$

$\rightsquigarrow$ regularity estimates for the whole system

$\rightsquigarrow$ estimates of local terms

Open problems

- ($d = 2, 3$, $u \equiv 0$) Local controllability to trajectories with $\overline{y} \neq 0$?

- ($d = 3$, $v \equiv 0$) Local null controllability ?