Control and Observation for Stochastic Partial Differential Equations

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- Do the controllability problems for stochastic ordinary differential equations(SDEs for short) and stochastic partial differential equations deserve to be studied?
- Yes. I can find many evidences to support my conclusion.
- I only point out that, in the control theory of deterministic ODEs and PDEs, both the optimal control problems and the controllability problems are studied extensively. However, for the stochastic counterpart, the situation is quite different.

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- Many excellent mathematicians have left their footprints in the stochastic optimal control theory.
- A very brief list includes J.-L. Lions, A. Bensoussan, W. H. Fleming, J.-M. Bismut, P.-L. Lions, etc.
- The stochastic controllability problems attract very few mathematicians. It seems that this area is almost an uncultivated land.

Outline

1. A toy model

2. Exact controllability of Stochastic transport equations

3. Some recent controllability results for SPDEs and open problems

1. A toy model

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- Consider the following controlled system:

$$\begin{cases} \dfrac{dy}{dt} = Ay + Bu, & t \in (0, T), \\ y(0) = y_0. \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $T > 0$. System (1) is said to be exactly controllable on $(0, T)$ if for any $y_0, y_1 \in \mathbb{R}^n$, there exists a $u \in L^2(0, T; \mathbb{R}^m)$ such that $y(T) = y_1$.

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- Theorem 1: System (1) is exactly controllable on $(0,T)\Leftrightarrow$

$$rank(B, AB, A^2B, \cdots, A^{n-1}B) = n.$$

- Consider a linear stochastic differential equation:

$$\begin{cases} dy = \big(Ay + Bu\big)dt + CydW(t), \quad t \geq 0, \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $A, C \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, $\{W(t)\}_{t \geq 0}$ is a standard one dimensional Brownian motion.

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where $A, C \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, $\{W(t)\}_{t\geq 0}$ is a standard one dimensional Brownian motion.

- System (2) is said to be exactly controllable if for any $y_0 \in \mathbb{R}^n$ and $y_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, there exists a control $u(\cdot) \in L^2_{\mathcal{F}}(\Omega; L^2(0, T; \mathbb{R}^m))$ such that the corresponding solution $y(\cdot)$ of (2) satisfies $y(T) = y_T$.

- $u(\cdot) \in L^2_{\mathcal{F}}(\Omega; L^2(0, T; \mathbb{R}^m))$ means that one can only utilize the information of the noise $\{W(s)\}_{0\leq s\leq t}$ which can be observed at time t. This is a key point in the study of stochastic control problems.

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- In virtue of a result by S. Peng (1994), system (2) is NOT exactly controllable for any B and any m!

- Thus, we consider the following alternative system:

$$
\begin{cases} dy = (Ay + Bu)dt + (Cy + Dv)dW(t), \quad t \geq 0, \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases} \tag{3}
$$

where $A, C, D \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $u \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ and $v \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$.

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- Theorem 2: System (4) is exactly controllable on $(0, T)\Leftrightarrow$

$$rank(B, AB, A^2B, \cdots, A^{n-1}B) = n \text{ and } rank(D) = n.$$

- Is Theorem 2 a trivial corollary of Theorem 1?

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- Taking $v(t) = D^{-1}Cy(t)$, we get

$$\begin{cases} dy = \big(Ay + Bu\big)dt, \quad t \geq 0, \\ y(0) = y_0 \in \mathbb{R}^n. \end{cases} \tag{4}$$

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Is this system exactly controllable when (A, B) satisfies the Kalman rank condition? NO.

- The reason is that we expect the final state could be any element in $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$.

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- The adjoint system of (2) reads:

$$\begin{cases} dz = -(A^\top z + C^\top Z)dt + ZdW, \quad t \in [0, T], \\ z(T) = z_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n). \end{cases} \tag{5}$$

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- Equation (5) is a backward stochastic ODE. Such kind of equations were first introduced by J. Bismut in 1970s and studied extensively in the last twenty years by E. Pardoux, S. Peng, N. El Karoui, R. Buckdahn, X. Zhou, J. Yong, etc.

- We only need to prove the following inequality:

$$|z_T|^2_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)} \leq C\mathbb{E}\int_0^T (|B^\top z(t)|^2 + Z^2)dt. \qquad (6)$$

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- Please note that we now deal with infinite dimensional space. One cannot simply mimic the proof for observability estimate for ODE to obtain (6). For example, in this case, unique continuation does not imply the observability estimate.

2. Exact controllability of Stochastic transport equations

- Let $T > 0$ and $G \subset \mathbb{R}^d (d \in \mathbb{N})$ a strictly convex bounded domain with the C^1 boundary Γ. Let $U \in \mathbb{R}^d$ with $|U|_{\mathbb{R}^d} = 1$. Denote by

$$\Gamma_{-S} = \{x \in \Gamma : \ U \cdot \nu(x) \leq 0\}, \quad \Gamma_{+S} = \Gamma \setminus \Gamma_{-S}.$$

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$$\Gamma_{-S} = \{x \in \Gamma : \ U \cdot \nu(x) \le 0\}, \quad \Gamma_{+S} = \Gamma \setminus \Gamma_{-S}.$$

- Consider a linear stochastic differential equation:

$$\begin{cases} dy + U \cdot \nabla y dt = a_1 y dt + \big[a_2 y + v \big] dW(t) & \text{in } (0,T) \times G, \\ y = u & \text{on } (0,T) \times \Gamma_{-S}, \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{7}$$

Here $y_0 \in L^2(G)$, $a_1, a_2 \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G))$.

- A solution to the system (7) is a stochastic process $y \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G)))$ such that for every $\tau \in [0,T]$ and every $\phi \in C^1(\overline{G})$, $\phi = 0$ on Γ_{+S}, it holds that

$$\begin{aligned}
&\int_G y(\tau,x)\phi(x)dx - \int_G y_0(x)\phi(x)dx \\
&-\int_0^\tau\int_G y(s,x)U\cdot\nabla\phi(x,U)dxds + \int_0^\tau\int_{\Gamma_{-S}} u(s,x)\phi(x)U\cdot\nu d\Gamma_{-S}ds \\
&= \int_0^\tau \int_G a_1(s,x)y(s,x)\phi(x)dxds \\
&\quad + \int_0^\tau\int_G \big[a_2(s,x)y(s,x)+v(s,x)\big]\phi(x)dxdW(s), \ P\text{-a.s.}
\end{aligned} \tag{8}$$

- For each $y_0 \in L^2(G)$, the system (7) admits a unique solution y. Further, there is a constant $C > 0$ such that for every y_0, it holds that

$$
\begin{aligned}
&|y|_{L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G)))} \\
&\leq e^{Cr_1}\Big(\mathbb{E}|y_0|_{L^2(G)} + |u|_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))} + |v|_{L^2_{\mathcal{F}}(0,T;L^2(G))}\Big).
\end{aligned} \tag{9}
$$

Here $r_1 = |a_1|^2_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G))} + |a_2|^2_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G))} + 1.$

- System (7) is said to be exactly controllable at time T if for every initial state $y_0 \in L^2(G)$ and every $y_1 \in L^2_{\mathcal{F}_T}(\Omega; L^2(G))$, one can find a pair of controls

$$(u, v) \in L^2_{\mathcal{F}}(0, T; L^2_w(\Gamma_{-S})) \times L^2_{\mathcal{F}}(0, T; L^2(G))$$

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- Put $R = \max_{x_1, x_2 \in \overline{G}} |x_1 - x_2|_{\mathbb{R}^d}$.

- **Theorem 3**: System (7) is exactly controllable at time T, provided that $T > R$.

- We put two controls on the system. Moreover, the control v acts on the whole domain. Compared with the deterministic transport solution, it seems that our choice of controls is too restrictive. One may consider the following four weaker cases for designing the control.

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- 1. Only one control is acted on the system, that is, $u = 0$ or $v = 0$ in (7).

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- 1. Only one control is acted on the system, that is, $u = 0$ or $v = 0$ in (7).
- 2. Neither u nor v is zero. But $v = 0$ in $(0, T) \times G_0$, where G_0 is a nonempty open subset of G.

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- 2. Neither u nor v is zero. But $v = 0$ in $(0, T) \times G_0$, where G_0 is a nonempty open subset of G.
- 3. Two controls are acted on the system. But both of them are in the drift term.

- **Theorem 4**: If $u \equiv 0$ or $v \equiv 0$ in the system (7), then the system (7) is not exactly controllable at any time T.

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- **Theorem 5**: Let G_0 be a nonempty open subset of G. If $v \equiv 0$ $(0, T) \times G_0$, then the system (7) is not exactly controllable at any time T.

- For the third case, we consider the following controlled equation:

$$\begin{cases} dy + U\cdot\nabla y dt = [a_1 y + \ell] dt + a_2 y dW(t) & \text{in } (0,T)\times G,\\ y = u & \text{on } (0,T)\times \Gamma_{-S},\\ y(0) = y_0 & \text{in } G. \end{cases} \tag{10}$$

Here $\ell \in L^2_{\mathcal{F}}(0,T;L^2(G))$ is a control.

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\tag{10}
$$

Here $\ell \in L^2_{\mathcal{F}}(0,T;L^2(G))$ is a control.

- **Theorem 6** The system (10) is not exactly controllable for any $T>0$.

- Let us introduce the adjoint system:

$$\begin{cases} dz+U\cdot\nabla z dt=\big(b_1 z+b_2 Z\big)dt+(b_3 z+Z)dW(t) & \text{in } (0,T)\times G,\\ z=0 & \text{on } (0,T)\times\Gamma_{+S},\\ z(T)=z_T & \text{in } G. \end{cases} \tag{11}$$

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- Here $z_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G))$, $b_1, b_2, b_3 \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G))$.

- A solution to (11) is a pair of stochastic processes

$$(z,Z)\in C_{\mathcal{F}}([0,T];L^2(\Omega;L^2(G)))\times L^2_{\mathcal{F}}(0,T;L^2(G))$$

such that for every $\psi\in C^1(\overline{G})$ with $\psi=0$ on Γ_{-S} and arbitrary $\tau\in[0,T]$, it holds that

$$\begin{aligned}
&\int_G z_T(x)\psi(x)dx-\int_G z(\tau,x)\psi(x)dx\\
&-\int_\tau^T\int_G z(s,x)U\cdot\nabla\psi(x)dxds\\
&=\int_\tau^T\int_G\big[b_1(s,x)z(s,x)+b_2(s,x)Z(s,x)\big]\psi(x)dxds\\
&+\int_\tau^T\int_G\big[b_3(s,x)z(s,x)+Z(s,x)\big]\psi(x)dxdW(s),\ \ P\text{-a.e.}
\end{aligned}$$

- For any $z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G))$, the equation (11) admits a unique solution (z, Z) satisfiing that

$$|z|_{C_{\mathcal{F}}([0,T];L^2(\Omega;L^2(G)))} + |Z|_{L^2_{\mathcal{F}}(0,T;L^2(G))} \leq e^{Cr_2} |z_T|_{L^2_{\mathcal{F}_T}(\Omega;L^2(G))}, \tag{12}$$

where

$$r_2 \triangleq \sum_{i=1}^{3} |b_i|^4_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G))} + 1.$$

- The observability estimate for (11) reads

$$
\begin{aligned}
&|z_T|_{L^2_{\mathcal{F}_T}(\Omega;L^2(G))}\\
&\leq \mathcal{C}(b_1,b_2,b_3)\big(|z|_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))}+|Z|_{L^2_{\mathcal{F}}(0,T;L^2(G))}\big).
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\end{aligned}
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- **Proposition 1**: Let $(z, Z) \in C_{\mathcal{F}}([0, T]; L^2(\Omega; L^2(G))) \times L^2_{\mathcal{F}}(0, T; L^2(G))$ solve the equation (11) with the terminal state z_T. Then

$$
|z|^2_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))} \leq e^{Cr_2}\mathbb{E}|z_T|^2_{L^2(G)}.
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$$|z|^2_{L^2_{\mathcal{F}}(0,T;L^2_w(\Gamma_{-S}))}\leq e^{Cr_2}\mathbb{E}|z_T|^2_{L^2(G)}.$$

- **Theorem 7**: If $T>R$, then the inequality (13) holds.

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- Let $0 < c < 1$ such that $cT > R$. Put

$$l = \lambda\Big[|x|^2 - c\Big(t - \frac{T}{2}\Big)^2\Big] \quad \text{and} \quad \theta = e^{l}. \tag{14}$$

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- **Proposition 2**: Assume that v is an $H^1(\mathbb{R}^n)$-valued continuous semi-martingale. Put $p = \theta v$. We have the following equality

$$\begin{aligned} &-\theta(l_t + U\cdot\nabla l)p\big[dv + U\cdot\nabla v dt\big] \\ =& -\frac{1}{2}d\big[(l_t + U\cdot\nabla l)p^2\big] - \frac{1}{2}U\cdot\nabla\big[(l_t + U\cdot\nabla l)p^2\big] \\ &+\frac{1}{2}\big[l_{tt} + U\cdot\nabla(U\cdot\nabla l) + 2U\cdot\nabla l_t\big]p^2 \\ &+\frac{1}{2}(l_t + U\cdot\nabla l)(dp)^2 + (l_t + U\cdot\nabla l)^2 p^2. \end{aligned} \tag{15}$$

Proof of the observability estimate: We apply Proposition 2 to the equation (11) with $v = z$, integrating (15) on $(0, T) \times G$, and taking mathematical expectation, then we get that

$$\begin{aligned}
&-\mathbb{E}\int_0^T\int_G \theta^2(l_t + U\cdot\nabla l)z(dz + U\cdot\nabla z dt)dxdt \\
&\geq \lambda\mathbb{E}\int_G (cT - 2U\cdot x)\theta^2(T)z^2(T)dx \\
&+2c\lambda\mathbb{E}\int_0^T\int_{\Gamma_{-S}} U\cdot\nu\big[c(T-2t) - 2U\cdot x\big]\theta^2 z^2 d\Gamma_{-S}dt \qquad (16)\\
&+2(1-c)\lambda\mathbb{E}\int_0^T\int_G \theta^2 z^2 dxdt \\
&+\mathbb{E}\int_0^T\int_G \theta^2(l_t + U\cdot\nabla l)\big[(b_3 z + Z)^2 + 2z^2\big]dxdt.
\end{aligned}$$

By virtue of that z solves the equation (11), we see

$$
\begin{aligned}
&2\lambda\mathbb{E}\int_G\int_{S^{d-1}}(cT-U\cdot x)\theta^2(T)z^2(T)dx\\
&+2(1-c)\lambda\mathbb{E}\int_0^T\int_G\theta^2z^2dxdt\\
&+\mathbb{E}\int_0^T\int_G\theta^2(l_t+U\cdot\nabla l)(b_3z+Z)^2dxdt\\
&+\mathbb{E}\int_0^T\int_G\theta^2(l_t+U\cdot\nabla l)^2z^2dxdt\\
&\le 3\mathbb{E}\int_0^T\int_G\theta^2(b_1^2z^2+b_2^2Z^2)dxdt\\
&-2c\lambda\mathbb{E}\int_0^T\int_{\Gamma_{-S}}U\cdot\nu\big[c(T-2t)-2U\cdot x\big]\theta^2z^2d\Gamma_{-S}dt.
\end{aligned}
\tag{17}
$$

Taking

$$\lambda_1 = \frac{3}{2(1-c)}\Big(\sum_{i=1}^{2} |b_i|^2_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G))} + |b_3|^4_{L^\infty_{\mathcal{F}}(0,T;L^\infty(G))} + 1\Big),$$

for any $\lambda \geq \lambda_1$, it holds that

$$\begin{aligned} &3\mathbb{E}\int_0^T\int_G \theta^2(b_1^2 + b_3^4 + b_3^2)z^2 dxdt \\ &\leq 2(1-c)\lambda\mathbb{E}\int_0^T\int_G \theta^2 z^2 dxdt. \end{aligned} \tag{18}$$

- Since $cT > R$, we find that

$$\mathbb{E}\int_G \theta^2(T,x)z^2(T,x)dx \leq \mathbb{E}\int_0^T\int_G \theta^2\big(b_2^2+2\lambda x-2c\lambda t\big)Z^2dxdt$$
$$-2c\lambda\mathbb{E}\int_0^T\int_{\Gamma_{-S}} U\cdot\nu\big[c(T-2t)-2U\cdot x\big]\theta^2 z^2 d\Gamma_{-S}dt. \qquad (19)$$

- Since $cT > R$, we find that

$$\mathbb{E}\int_G \theta^2(T,x)z^2(T,x)dx \leq \mathbb{E}\int_0^T\int_G \theta^2\big(b_2^2+2\lambda x-2c\lambda t\big)Z^2dxdt$$
$$-2c\lambda\mathbb{E}\int_0^T\int_{\Gamma_{-S}} U\cdot\nu\big[c(T-2t)-2U\cdot x\big]\theta^2z^2d\Gamma_{-S}dt. \tag{19}$$

- By the definition of θ, we have $e^{-\frac{1}{2}c\lambda T^2} \leq \theta \leq e^{\frac{1}{2}\lambda R^2}$. Thus,

$$e^{-\frac{1}{2}c\lambda T^2}\mathbb{E}\int_G z_T^2dx$$
$$\leq Ce^{\frac{1}{2}\lambda R^2}\left(\mathbb{E}\int_0^T\int_G Z^2dxdt + \mathbb{E}\int_0^T\int_{\Gamma_{-S}} U\cdot\nu z^2d\Gamma_{-S}dt\right),$$

Lack of exact controllability

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- Set

$$\eta(t) = \begin{cases} 1, \text{ if } t \in \big[(1 - 2^{-2i})T, (1 - 2^{-2i-1})T\big), \ i = 0, 1, \cdots, \\ -1, \text{ otherwise} \end{cases}$$

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- Lemma 1: Let $\xi=\int_0^T\eta(t)dW(t)$. It is impossible to find $(\varrho_1,\varrho_2)\in L^2_{\mathcal{F}}(0,T;\mathbb{R})\times L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and $x\in\mathbb{R}$ with

$$\lim_{t\to T}\mathbb{E}|\varrho_2(t)-\varrho(T)|^2=0,$$

such that

$$\xi=x+\int_0^T\varrho_1(t)dt+\int_0^T\varrho_2(t)dW(t). \tag{20}$$

- Put

$$\mathcal{V} \stackrel{\triangle}{=} \{v : v \in L^2_{\mathcal{F}}(0, T; L^2(0,1)),\ v = 0 \text{ in } (0, T) \times G_0\}.$$

Let ξ be given by Lemma 1. Choose a $\psi \in C_0^\infty(G_0)$ such that $|\psi|_{L^2(G)} = 1$ and set $y_T = \xi\psi$. We will show that y_T cannot be attained for any $y_0 \in \mathbb{R}$, $u \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$ and $v \in \mathcal{V}$.

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- If there exist a $u \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and a $v \in \mathcal{V}$ such that $y(T) = y_T$, then by Itô's formula, we obtain

$$\begin{aligned}\xi &= \int_G y_T\psi dx\\ &= \int_G y_0\psi dx - \int_0^T\int_G \psi y_x dxdt + \int_0^T\int_G \psi(y+v)dxdW(t)\\ &= \int_G y_0\psi dx + \int_0^T\Big(\int_G \psi_x y dx\Big)dt + \int_0^T\Big(\int_G \psi y dx\Big)dW(t).\end{aligned} \tag{21}$$

- It is clear that both $\int_G \psi_x y dx$ and $\int_G \psi y dx$ belong to $L^2_{\mathcal{F}}(0, T; \mathbb{R})$. Further,

$$\begin{aligned}&\lim_{t\to T}\mathbb{E}\Big|\int_G \psi y(t)dx - \int_G \psi y(T)dx\Big|^2\\&=\lim_{t\to T}\mathbb{E}\Big|\int_G \psi\big[y(t)-y(T)\big]dx\Big|^2=0.\end{aligned}$$

These, together with (21), contradict Lemma 1.

3. Some recent controllability results for SPDEs and open problems

Let us consider the following forward stochastic parabolic equation

$$
\begin{cases}
dy - \displaystyle\sum_{i,j}(a^{ij}y_i)_j dt = [\langle \alpha, \nabla y\rangle + \beta y + \chi_{G_0} f]dt \\
\qquad\qquad\qquad\qquad + (qy+g)\, dW(t) & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0 & \text{in } G,
\end{cases} \qquad (22)
$$

with suitable coefficients α, β, p and q. In (22), the initial state $y_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G))$. The control variable consists of the pair

$$
(f, g) \in L^2_{\mathcal{F}}(0, T; L^2(G_0)) \times L^2_{\mathcal{F}}(0, T; L^2(G)).
$$

- The null controllability of (22) is established by S. Tang and X. Zhang(2009, SICON). From which, they proved that system (22) is null controllable with two controls, i.e., one act on the drift term and the other act on the diffusion term. The control in the diffusion term have to be acted on the whole domain where the solution of system (22) defined!!!

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- Is it possible to use one control to drive the solution to zero? Positive answer is given by Lü(2011,JFA) for a special case of system (22).

Consider the following controlled stochastic heat equation with one control:

$$\begin{cases} dy - \sum_{i,j=1}^{n} (a^{ij} y_{x_i})_{x_j} dt = a(t) y dW + \chi_E \chi_{G_0} f dt & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \tag{23}$$

where $a(t) \in L^{\infty}_{\mathcal{F}}(0, T)$, E is a measurable subset in $(0, T)$ with a positive Lebesgue measure (i.e., $m(E) > 0$), χ_E is the characteristic function of E, $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, the control f belongs to $L^{\infty}_{\mathcal{F}}(0, T; L^2(\Omega; L^2(G)))$.

- **Theorem 8**(Qi Lü, 2011, JFA): System (1) is null controllable at time T, i.e., for each initial datum $y_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G))$, there is a control $f \in L^\infty_{\mathcal{F}}(0, T; L^2(\Omega; L^2(G)))$ such that the solution y of system (1) satisfies $y(T) = 0$ in G, P-a.s. Moreover, the control f satisfies the following estimate:

$$|f|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;L^2(G)))} \leq C\mathbb{E}|y_0|^2_{L^2(G)}. \tag{24}$$

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- For general stochastic heat equations, the answer is unknown.

- Consider the following stochastic Schrödinger equation:

$$\begin{cases} idy+\Delta ydt=(a_1\cdot\nabla y+a_2y)dt+(a_3y+g)dW(t) & \text{in } Q,\\ y=0 & \text{on } \Sigma\setminus\Sigma_0,\\ y=u & \text{on } \Sigma_0,\\ y(0)=y_0 & \text{in } G. \end{cases} \tag{25}$$

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- Here, the initial state $y_0\in H^{-1}(G)$, the control

$$u\in L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0)),\quad g\in L^2(0,T;H^{-1}(G)),$$

$a_k(k=1,2,3)$ are suitable coefficients.

- The solution to (25) is defined in the sense of transposition solution, which is not studied in the literature of stochastic PDEs.

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- **Theorem 9**(Qi Lü, 2013, JDE): System (1) is exactly controllable at any time $T > 0$.

The main difficulty of the study of the control and observation problems for SPDEs.

- 1. The lack of desired theory, i.e., the well-posedness results of the nonhomogeneous boundary value problems for SPDEs, the propagation of singularities results of the solution for SPDEs, etc.

The main difficulty of the study of the control and observation problems for SPDEs.

- 1. The lack of desired theory, i.e., the well-posedness results of the nonhomogeneous boundary value problems for SPDEs, the propagation of singularities results of the solution for SPDEs, etc.
- 2. The stochastic settings lead some useful methods invalid, for example, the lost of the compact embedding for the state spaces, i.e., although $L^2(\Omega; H_0^1(G)) \subset L^2(\Omega; L^2(G))$, the embedding is not compact, which violates the compactness -uniqueness argument. Another example is that the irregularity of the solution with respect to the time variable, for example, it is very hard to estimate the time derivative of the solutions, which is an obstacle to follow the method by Fernández Cara-Guerrero-Imanuvilov-Puel to study the null controllability of the stochastic Navier-Stokes equation.

The main difficulty of the study of the control and observation problems for SPDEs.

- 3. Generally speaking, the adjoint system is a backward SPDE, whose solution is constituted by two stochastic process. For example, let us recall the adjoint system of (4) is as follows:

$$\begin{cases} dz = -\big(A^\top z + C^\top Z\big)dt + ZdW, \quad t \in [0, T], \\ z(T) = z_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n). \end{cases} \tag{26}$$

When studying the observability problems, people now usually treat Z as a nonhomogeneous term. It is hard to use the fact that Z is a part of the solution.

This area is full of open problems. In my opinion, some of them are as follows.

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- The observability estimate for stochastic wave equations under the Geometric Control Conditions.
- The null controllability for stochastic heat equations with only one control.
- The null controllability of stochsatic wave equations, stochastic Schrödinger equations, etc. For this, one need to show that either one control is enough or, as the exact controllability problems, two controls are necessary.

Thank you!