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# Blowup Time Optimal Control for Some Evolution Differential Equations

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# 1 Case of ODE

## 1.1 Introduction

First, consider the following controlled system governed by the following ordinary differential equations:

$$\begin{cases} \dfrac{dy(t)}{dt} = f(y(t)) + B(t)u(t), \quad t > 0, \\ y(0) = y_0. \end{cases} \tag{1.1.1}$$

Here $y_0 \in \mathbb{R}^N$ and $B(\cdot) \in L^\infty([0, +\infty); \mathbb{R}^{N \times M})$. The function $f : \mathbb{R}^N \to \mathbb{R}^N$ is defined by $f(y) = \| y \|^{p-1} y$ with $p > 1$. The control function $u(\cdot)$ is taken from the set

$$\mathcal{U}_{ad} = \{u : [0, +\infty) \to \mathbb{R}^M | \ u \text{ is Lebesgue measurable and}$$
$$\| u(t) \| \leq \rho_0 \text{ a. e. } t \in [0, +\infty)\}.$$

Here $\rho_0$ is an arbitrary but fixed positive constant, and we use the notation $\| \cdot \|$ to denote the Euclidean norm.

The initial value problem $(1.1.1)$ has, corresponding to each initial data $y_0$ in $\mathbb{R}^N$ and each control $u(\cdot)$ in the set $\mathcal{U}_{ad}$, a unique solution, denoted by $y(\cdot\,; y_0, u)$, on its maximal interval of existence, denoted by $[0, T(y_0, u))$.

If $T(y_0, u) < +\infty$, then the solution of the system (1.1.1) blows up at $T(y_0, u)$, namely, $\| \; y(t; y_0, u) \; \| \to +\infty$ as $t \to T(y_0, u)$.

We shall consider the following time optimal control problem:

$$(P) \quad \min T(y_0, u).$$

The target set of the problem $(P)$ is $\{\infty\}$.

If $y_0 \neq 0$, then there exists a number $t^* \in (0, +\infty)$ such that

$$t^* = \inf_{u \in \mathcal{U}_{ad}} T(y_0, u).$$

We call the number $t^*$ the optimal time; the control $u^*$ in the set $\mathcal{U}_{ad}$ with $T(y_0, u^*) = t^*$ the optimal control; the function $y(\cdot\,; y_0, u^*)$, $t \in [0, T(y_0, u^*))$ the optimal state.

## 1.2 Extension

The results obtained in this study about the problem $(P)$ hold for the same time optimal control problems with more generic state systems in which the nonlinear term $f : \mathbb{R}^N \to \mathbb{R}^N$ is continuously differentiable and holds the following properties:
(a) There exist two positive constants $C_1$ and $p$ with $p > 1$ such that

$$\frac{< y, f(y) >}{\| y \|^{p+1}} \to C_1 \text{ as } \| y \| \to +\infty;$$

(b) There exists a positive constant $C_2$ such that

$$\| f(y) \| \leq C_2(1 + \| y \|^p), \quad \text{for each } y \in \mathbb{R}^n;$$

(c) There exists a positive number $C_3$ such that

$$|||f_y(y)||| \leq \frac{p < y, f(y) >}{\| y \|^2}, \quad \text{for all } \ y \in \mathbb{R}^n \text{ with } \ \| y \| \geq C_3.$$

Here $||| \cdot |||$ denotes the operator norm.

## 1.3 Motivation

**Point of view of mathematics.** The generic time optimal control problems governed by both ordinary differential equations and partial differential equations have their target sets in the state spaces (see [1], [2]).

[1] V. Barbu, Analysis and Control of Nonlinear Infinite Dimensional System, 1993.

[2] X. Li and J. Yong, Optimal Control Theory for Infinite Dimensional Systems, Birkhäuser, Boston, 1995;

While the time optimal control problem $(P)$ studied here possesses its target set $\{\infty\}$, which is outside of the state space $\mathbb{R}^N$.

**Point of view of applications.**

## 1.4 An mathematical example

Some controls will make the blowup time $T(y_0, u)$ ahead of $T(y_0, 0)$.

For example, consider a special case:

$$\begin{cases} \dfrac{dy(t)}{dt} = |y(t)|y(t) + u(t), \quad t > 0, \\ y(0) = 1. \end{cases}$$

It is obvious that

$$y(t; 1, 0) = 1/(1 - t),\ t \in [0, 1);$$

$$y(t; 1, 1) = \tan(t + \pi/4),\ t \in [0, \pi/4).$$

Hence, we have $T(1, 1) = \pi/4 < 1 = T(1, 0)$.

## 1.5 Main Results

P. Lin, G. Wang, Blowup time optimal control for ordinary differential equations, SIAM J. Control Optim., Vol. 49, No. 1, pp. 73 – 105.

The existence of the optimal control for the problem $(P)$.

**Theorem 1.5.1** Suppose that the initial data $y_0$ is a non-zero vector in $\mathbb{R}^N$. Then the problem $(P)$ has at least one optimal control.

## The Pontryagin maximum principle for the problem $(P)$.

**Theorem 1.5.2** Let $K_0 = (\text{esssup}_{s\in R^+}||B(s)||\rho_0)^{1/p}$. Suppose that the initial data $y_0$ has the following property: $||y_0|| > K_0$. Then Pontryagin's maximum principle holds for the problem $(P)$. Namely, if $t^*$ is the optimal time, $u^*$ is an optimal control and $y^*$ is the corresponding optimal state for the problem $(P)$, then there is a nontrivial function $\psi(\cdot)$ in the space $C([0,t^*];\mathbb{R}^N)$ holding the following properties:

$$\begin{cases} \psi(t) = \int_t^{t^*} f_y(y^*(\tau))\psi(\tau)d\tau, \quad \text{for all } \ t \in [0,t^*), \\ \psi(t^*) = 0 \end{cases}$$

and

$$\max_{\|u\|_M \leq \rho_0} < \psi(t), B(t)u > = < \psi(t), B(t)u^*(t) >, \quad \text{for a.e. } t \in [0,t^*].$$

**Corollary 1.5.3** Let $B(t) \equiv B$, for all $t \in R^+$, where $B$ is a matrix of $N \times M$. Suppose that the initial data $y_0$ has the following property: $0 < \| y_0 \| \le K_0$. Assume that $t^*$ is the optimal time, $u^*$ is an optimal control and $y^*$ is the corresponding optimal state for the problem $(P)$. Then there exist a number $s^* \in (0, t^*)$, a non-zero vector $\psi_0$ in $\mathbb{R}^N$ and two nontrivial functions $\psi_1(\cdot) \in C([0, s^*]; \mathbb{R}^N)$ and $\psi_2(\cdot) \in C([s^*, t^*]; \mathbb{R}^N)$ such that

$$\begin{cases} \psi_1(t) = \displaystyle\int_t^{s^*} f_y(y^*(\tau))\psi_1(\tau)d\tau, \quad \text{for all} \;\; t \in [0, s^*], \\ \psi_1(s^*) = \psi_0, \end{cases}$$

$$\begin{cases} \psi_2(t) = \displaystyle\int_t^{t^*} f_y(y^*(\tau))\psi_2(\tau)d\tau, \quad \text{for all} \;\; t \in [s^*, t^*), \\ \psi_2(t^*) = 0, \end{cases}$$

$$\max_{\|u\|_M \le \rho_0} < \psi(t), Bu > = < \psi(t), Bu^*(t) >, \quad \text{for a.e.} \;\; t \in [0, t^*],$$

where $\psi(t) = \begin{cases} \psi_1(t), \; t \in [0, s^*], \\ \psi_2(t), \; t \in (s^*, t^*]. \end{cases}$

## 1.6 Comparison with a reference

[1] E.N. Barron and W. Liu, Optimal control of the blowup time, SIAM J. Control Optim, (1996).

The differences between the current work and the work [1] are as follows:

(i) the work [1] is concerned with the optimal control to maximize the blowup time while our work deals with the optimal control to minimize the blowup time.

(ii) the state systems are different. Due to the involvement of the function $B(\cdot)$ in the state equation of the initial value problem (1.1.1), the state system for the problem $(P)$ can not be covered by the state systems in [1].

(iii) our work gives the existence of optimal controls while the work [1] did not discuss the existence of the optimal controls.

(iv) the ways to derive the Pontryagin maximum principles are totally different.

## 1.7 Main ideas of the proof

The main difficulty is caused by the fact that the target set in the current case is outside of the state space, and thus one can not directly use the methods in dealing with the general time optimal control problems, where the target sets are in the state space, to study this problem.

Idea of the proof of the existence of the time optimal control.

Step 1. Construct an approximate problem.

Let $R$ be a number such that $R > \| y_0 \|$. Write $B_R(0) = \{y \in \mathbb{R}^N | \ \| y \| \leq R\}$. We denote by $\partial B_R(0)$ the boundary of $B_R(0)$. Define $J_R$ be a map from $\mathcal{U}_{ad}$ to $\overline{R^+} \equiv [0, +\infty]$ by

$$J_R(u(\cdot)) = \inf\{t \geq 0 \ : \ y(t; y_0, u) \in \partial B_R(0)\}.$$

Consider the following approximate time optimal control problem:

$(P_R)$ Find $u_R \in \mathcal{U}_{ad}$ such that

$$J_R(u_R(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}_{ad}} J_R(u(\cdot)) =: t_R.$$

Step 2. We prove the existence of solutions to the problem $(P_R)$ for each $R$ with $R >\| y_0 \|$.

Step 3. Firstly, we prove that there exists two positive constants $t^*$, $T_0 > t^*$ and a control $u^* \in \mathcal{U}_{ad}$ such that

$$t_R \to t^*, \text{as } R \to +\infty$$

and there is a sequence of numbers $\{R_n\}$ with $R_n \to +\infty$ such that for $n \to \infty$,

$$u_{R_n} \rightharpoonup u^* \text{ weakly star in } \ L^\infty((0, T_0); R^M).$$

Secondly, we prove that $t^*$ is the optimal time and $u^*$ is the optimal control of the problem $(P)$.

The following lemmas play important roles in the proof of Theorem 1.1.

Lemma 1.7.1 There exists a positive constant $K_1$ such that for each number $K$ with $K > K_1$, we have that the solution for each initial data $y_0$ with $\| y_0 \| \geq K$,

$$\| \xi(t; t_0, y_0, u) \| \geq K,\ t \in [t_0, T(t_0; y_0, u)).$$

Here $\xi(\cdot; t_0, y_0, u)$ is the solution of the following system:

$$\begin{cases} \dfrac{d\xi(t)}{dt} = f(\xi(t)) + B(t)u(t), \quad t > t_0, \\ \xi(t_0) = y_0, \end{cases} \tag{1.7.1}$$

and $[t_0, T(t_0; y_0, u))$ is the maximal interval of existence for the solution $\xi(\cdot; t_0, y_0, u)$.

**Lemma 1.7.2** Let $y_0$ be a vector in $\mathbb{R}^N$. Then for each control $u(\cdot)$ in the set $\mathcal{U}_{ad}$ and for each time $t$ in the interval $[0, T(y_0, u))$,

$$\| y(t; y_0, u) \| \leq \max\{2^{\frac{1}{p-1}}(\frac{1}{p-1})^{\frac{1}{p-1}}(T(y_0, u) - t)^{-\frac{1}{p-1}},\ 2K_1\},$$

where $K_1$ is the constant given in Lemma 2.1.

Now, let $t_0 \geq 0$, $y_0 \in \mathbb{R}^N$ and $\varphi(\cdot) \in L^\infty((t_0, +\infty); \mathbb{R}^N)$. Consider the system:

$$\begin{cases} \dfrac{d\xi(t)}{dt} = f(\xi(t)) + \varphi(t), \quad t > t_0, \\ \xi(t_0) = y_0. \end{cases} \tag{1.7.2}$$

We have the following lemma:

**Lemma 1.7.3** Let $[t_0, t_1]$ be a closed interval with $0 \leq t_0 < t_1$. Let $\varphi(\cdot)$ and $\{\varphi_k(\cdot)\}_{k=1}^\infty$ be an element and a bounded sequence in $L^\infty((t_0, +\infty); \mathbb{R}^N)$, respectively. Suppose that

$$\lim_{k \to +\infty} \int_{t_0}^t \varphi_k(\tau) d\tau = \int_{t_0}^t \varphi(\tau) d\tau, \text{ for each } t \in [t_0, t_1].$$

Furthermore, assume that the solution $\xi(\cdot\,; t_0, y_0, \varphi)$ exists on the interval $[t_0, t_1]$. Then there is a natural number $k_0$ such that for each $k$ with $k \geq k_0$, the solution $\xi(\cdot\,; t_0, y_0, \varphi_k)$ to the system (1.7.2) exists on the interval $[t_0, t_1]$.

Idea of the proof of the maximum principle.
Step 1. To set up a penalty functional.
If $\lim_{t\to t^*} \| y(t;y_0,u) \| = +\infty$, then $\lim_{t\to t^*} \frac{1}{\|y(t;y_0,u)\|} = 0$.

Let $T$ be a fixed constant with $T > t^*$. Write $\mathcal{U}[0,T] = \{u|_{[0,T]} \mid u \in \mathcal{U}_{ad}\}$. We introduce the Ekeland distance $\overline{d}$ over the set $\mathcal{U}[0,T]$, which is given by

$$\overline{d} = \text{meas}\{t \in [0,T] \mid u(t) \neq v(t)\}.$$

For each number $\varepsilon$ with $0 < \varepsilon < t^*$, we define a penalty functional $J_\varepsilon : (\mathcal{U}[0,T], \overline{d}) \to \mathbb{R}^1$ by

$$J_\varepsilon(u(\cdot)) = \frac{1}{(p-1) \| y(t^* - \varepsilon; y_0, u) \|^{p-1}} \equiv \sigma(\varepsilon).$$

By Ekeland's variational principle, for each number $\varepsilon$ with $0 < \varepsilon < t^*$, there is a control $u^\varepsilon(\cdot)$ in the set $\mathcal{U}[0,T]$ such that the following hold:

$$\begin{cases} \overline{d}(\overline{u}(\cdot), u^\varepsilon(\cdot)) \leq \sqrt{\sigma(\varepsilon)}, \\ -\sqrt{\sigma(\varepsilon)}\, \overline{d}(\widetilde{u}(\cdot), u^\varepsilon(\cdot)) \leq J_\varepsilon(\widetilde{u}(\cdot)) - J_\varepsilon(u^\varepsilon(\cdot)), \ \forall\, \widetilde{u}(\cdot) \in \mathcal{U}[0,T]. \end{cases}$$

Step 2. To get certain approximate necessary conditions for each control $u_\varepsilon$.
Let the function $\psi^\varepsilon(\cdot)$ be the solution of the following system:

$$\begin{cases} \dfrac{d\psi^\varepsilon(t)}{dt} = -f_y(y(t;y_0,u^\varepsilon))\psi^\varepsilon(t), \quad t\in[0,t^*-\varepsilon], \\ \psi^\varepsilon(t^*-\varepsilon) = \dfrac{y(t^*-\varepsilon;y_0,u^\varepsilon)}{\| y(t^*-\varepsilon;y_0,u^\varepsilon) \|^{p+1}}. \end{cases}$$

Then, we have

$$\int_0^{t^*-\varepsilon} < \psi^\varepsilon(\tau), B(\tau)(u(\tau)-u^\varepsilon(\tau)) > d\tau \leq \sqrt{\sigma(\varepsilon)}T.$$

Step 3. To get certain estimate for the functions $\psi^\varepsilon(\cdot)$.
We obtain that

$$\| \psi^\varepsilon(t) \| \leq C, \quad \text{for each } t\in[0,t^*-\varepsilon],$$

where $C$ is a positive constant independent of $\varepsilon$.

Step 4. To get the convergence of a subsequence for the set of the functions $\{\psi^{\varepsilon}(\cdot)\}$ as $\varepsilon \to 0$.

We obtain that there exists a subsequence $\{\psi^{\varepsilon_n}(\cdot)\}$ of the set of the functions $\{\psi^{\varepsilon}(\cdot)\}$ and a function $\psi(\cdot)$ which is continuous on the interval $[0, t^*]$ and $\psi(t^*) = 0$, such that

$$\psi(t) = \lim_{n \to +\infty} \psi^{\varepsilon_n}(t), \quad \text{for each } \ t \in [0, t^*).$$

Step 5. To prove that

$$\| \, y(t^* - \varepsilon; y_0, u^{\varepsilon}) \, \| \to +\infty, \ \text{as } \varepsilon \to 0.$$

Step 6. To get the Pontryagin maximum principle of optimal controls for the problem $(P)$ by taking the limit in the approximate necessary conditions as $\varepsilon \to 0$.

Combining Step 2, Step 4 with Step 5, we obtain the maximum principle for the problem $(P)$ , namely, Theorem 1.2.

# 2 Case of PDE

## 2.1 Introduction and Motivation

Consider the following partial differential equation:

$$\begin{cases} y_t - \Delta y = |y|^{p-1}y + g, & \text{in } \Omega \times (0, \infty), \\ y = 0, & \text{on } \partial\Omega \times (0, \infty), \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{2.1.1}$$

Here $\Omega$ is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. $y_0 \in H_0^1(\Omega)$, $p > 1$, $g \in L^\infty(0, +\infty; L^\nu(\Omega))$ with

$$\nu > \max\{2, (2p+4)/(3p-3)\}.$$

We call $y(\cdot)$ to be a solution of equation (2.1) on $[0,T)$ if

$$(a)\ y \in C([0,T); H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)),\ \ y_t \in L^2(\Omega \times (0,T)),$$

and for each $t \in [0,T)$,

$$(b)\ y(t) = S(t)y_0 + \int_0^t S(t-s)|y(s)|^{p-1}y(s)ds + \int_0^t S(t-s)g(s)ds.$$

Here, $S(\cdot)$ denotes the semigroup generated by the operator $A = -\Delta$ with $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. It is proved that when $p$ satisfies certain conditions, $y_0$ is in $H_0^1(\Omega)$ and $g$ belongs to $L^\infty(0,+\infty; L^\nu(\Omega))$, equation (2.1.1) has a unique solution on the its maximal interval of existence. We denote this solution by $y(\cdot\ ; y_0, g)$ and write $[0, T_{max}(y_0, g))$ for its maximal interval of existence.

Let $\rho_0 > 0$. We define

$$G(\rho_0) = \{g \in L^\infty(0,+\infty; L^\nu(\Omega)) :\| g \|_{L^\infty(0,+\infty;L^\nu(\Omega))} \leq \rho_0\}.$$

For each $z_0 \in H_0^1(\Omega)$, set

$$E(z_0) = \int_\Omega [\frac{1}{2}|\nabla z_0|^2 - \frac{1}{p+1}|z_0|^{p+1}]dx, \quad z_0 \in H_0^1(\Omega),$$

$$C_*(z_0) = \frac{\rho_1^2(p+1)}{(p-1)^2}|\Omega|^{\frac{p-1}{2}} \| z_0 \|_2^{1-p} + \frac{1}{2}(\frac{p-1}{2(p+1)})^{-\frac{1}{p}}|\Omega|^{\frac{p-1}{2p}}\rho_1^{\frac{p+1}{p}},$$

and

$$C^*(z_0) = \frac{\rho_1^2(p+1)}{(p-1)^2}|\Omega|^{\frac{p-1}{2}}(\frac{\| z_0 \|_2}{2})^{1-p} + \frac{1}{2}(\frac{p-1}{2(p+1)})^{-\frac{1}{p}}|\Omega|^{\frac{p-1}{2p}}\rho_1^{\frac{p+1}{p}},.$$

Here $|\Omega|$ denotes the measure of $\Omega$ and

$$\rho_1 \equiv \rho_0|\Omega|^{(\nu-2)/2\nu}.$$

We define

$$Y_*(\rho_0) = \{z_0 \in H_0^1(\Omega) : E(z_0) \leq -C_*(z_0)\}$$

and

$$Y^*(\rho_0) = \{z_0 \in H_0^1(\Omega) : E(z_0) \leq -C^*(z_0)\}.$$

The main purpose of this paper is to study certain uniform properties for solutions $y(\cdot\ ; y_0, g)$ with respect to $g \in G(\rho_0)$ and $y_0 \in Y_*(\rho_0)$ (or $y_0 \in Y^*(\rho_0)$), and to provide some application of these properties.

## 2.2 Main Results

P. Lin, G. Wang, Some Properties for Blowup parabolic equations and their application. (preprint)

Theorem 2.2.1 Suppose that $1 < p < \min(3, 4/n + 1)$, $n \leq 4$; $1 < p \leq \frac{n}{n-2}$, $n > 4$. Let $\rho_0 > 0$. Let $y_0 \in Y_*(\rho_0)$ and $g \in G(\rho_0)$ be such that $y(\cdot\ ; y_0, g)$ exists on $[0, T_0]$. Assume that $\{y_{0k}\}_{k=1}^{\infty} \in Y_*(\rho_0)$ satisfies

$$y_{0k} \to y_0 \ \text{ strongly in } \ H_0^1(\Omega), \ \text{ as } \ k \to +\infty;$$

and that $\{g_k\}_{k=1}^{\infty} \subset G(\rho_0)$ satisfies

$$g_k \rightharpoonup g \ \text{ weakly star in } \ L^{\infty}((0, T_0); L^{\nu}(\Omega)) \ \text{ as } k \to +\infty.$$

Then there is a natural number $k_0$ such that for each $k$ with $k \geq k_0$, the solution $y(\cdot\ ; y_0, g_k)$ exists on the interval $[0, T_0]$.

**Theorem 2.2.2** Suppose that $\Omega$ is convex and that $1 < p < \min(3, 4/n+1),\ n \leq 4;\ 1 < p \leq \frac{n}{n-2},\ n > 4$. Let $\rho_0 > 0$ and $y_0 \in Y^*(\rho_0)$. Then there is a $C > 0$, (depending on $y_0$ and $\rho_0$), such that for each $t \in [0, T_{max}(y_0, g))$,

$$\| y(t; y_0, g) \|_{H_0^1(\Omega)} \leq C(T_{max}(y_0, g) - t)^{-(p+1)/[2(p-1)]},$$

for all $g \in G(\rho_0)$.

Let $\rho_0 > 0$. We set

$$\mathcal{U}_{ad} = \{u \in L^\infty(0, +\infty; L^\nu(\Omega)); \parallel u \parallel_{L^\infty(0,+\infty;\ L^\nu(\Omega))} \leq \rho_0\}.$$

Consider the following time optimal control problem:

$$(P) \quad t^* = \min_{u \in \mathcal{U}_{ad}} \{T_{max}(y_0, \chi_\omega u)\}.$$

$t^*$is called the optimal time; a control holding the property: $T_{max}(y_0, \chi_\omega u^*) = t^*$, is called an optimal control; and the solution $y(\cdot\,; y_0, \chi_\omega u^*)$ is called the optimal state.

Theorem 2.2.3 Suppose that $\Omega$ is convex and that $1 < p < \min(3, 4/n + 1)$, $n \leq 4$; $1 < p \leq \frac{n}{n-2}$, $n > 4$. Assume that $y_0 \in Y^*(\rho_0)$. Then the problem $(P)$ has at least one optimal control.

Let $R$ be a number such that $R > \| y_0 \|_2$. Write $B_R(0) = \{y \in L^2(\Omega);\ \| y \|_2 \leq R\}$. We denote by $\partial B_R(0)$ the boundary of $B_R(0)$. Define $J_R$ be a map from $\mathcal{U}_{ad}$ to $\overline{R^+} \equiv [0, +\infty]$ by

$$J_R(u) = \text{inf}\{t \geq 0 \ : \ y(t; y_0, u) \in \partial B_R(0)\}.$$

Consider the following approximate time optimal control problem:
$(P_R)$ Find $u_R \in \mathcal{U}_{ad}$ such that

$$J_R(u_R) = \inf_{u \in \mathcal{U}_{ad}} J_R(u) =: t_R.$$

We have the theorem on the approximate necessary conditions for the problem $(P)$ as follows.

**Theorem 2.2.4** Suppose that $\Omega$ is convex and that $1 < p < \min(3, 4/n+1),\ n \leq 4;\ 1 < p \leq \frac{n}{n-2},\ n > 4$. Assume that $y_0 \in Y^*(\rho_0)$. Then the Pontryagin's maximum principle holds for the problem $(P_R)$ with $R > \| y_0 \|_2$. Namely, if $t_R$ is the optimal time, $u_R$ is an optimal control and $y_R$ is the corresponding optimal state for the problem $(P_R)$ with $R > \| y_0 \|_2$, then there is a nontrivial function $\psi$ holding the following properties:

$$\begin{cases} \psi_t + \triangle\psi = -p|y_R|^{p-1}\psi, & x \in \Omega,\ t \in [0, t_R], \\ \psi(x,t) = 0, & x \in \partial\Omega, t \in [0, t_R], \\ \psi(x, t_R) = y_R(x, t_R)/\| y_R(x, t_R) \|_2, & x \in \Omega. \end{cases}$$

and

$$\max_{\|u\|_\nu \le \rho_0} < \psi(t), \chi_\omega u > = < \psi(t), \chi_\omega u_R(t) >, \quad \text{for a.e. } t \in [0, t_R].$$

Moreover, for a.e. $\in [0, t_R]$, we have

$$< -\frac{y_R(t)}{\| y_R(t) \|_2}, \Delta y_R(t) + |y_R(t)|^{p-1} y_R(t) + \chi_\omega u_R(t) > \ge 0.$$

Here $< \cdot , \cdot >$ denotes the inner product of the space $L^2(\Omega)$. Write

$$\widetilde{u}_R = \begin{cases} u_R, & t \in [0, t_R], \\ 0, & t \in [t_R, +\infty). \end{cases}$$

Then, it holds that

$$T(y_0, \widetilde{u}_R) \to t^*, \quad \text{as } R \to +\infty.$$

**Proposition 2.3.1** Suppose that $p>1$, $n=1,2$; $1<p\leq \frac{n}{n-2}$, $n\geq 3$. Let $y_0\in Y_*(y_0)$ and $g\in G(\rho_0)$. Then, it holds that

$$T_{max}(y_0,g)\leq \widetilde{T}_{max}<+\infty.$$

Here,

$$\widetilde{T}_{max}=\widetilde{T}_{max}(\Omega,p,y_0)=\frac{2(p+1)}{(p-1)^2}|\Omega|^{\frac{p-1}{2}}\parallel y_0\parallel_2^{1-p}.$$

It further stands that

$$\lim_{t\to T_{max}(y_0,u)}\parallel y(t;y_0,g)\parallel_{p+1}=+\infty.$$

**Proposition 2.3.2** Suppose that $1 < p < \min(3, 4/n + 1)$, $n \leq 4$; $1 < p \leq \frac{n}{n-2}$, $n > 4$. Let $y_0 \in C_*(y_0)$ and $g \in G(\rho_0)$. Then it holds that

$$\lim_{t \to T_{max}(y_0,g)} \| y(t; y_0, g) \|_2 = +\infty.$$

**Similarity variable:**

$$w_a(z, s) = (T - t)^{\beta} y(x, t),$$

where

$$x - a = (T - t)^{1/2} z, \; T - t = e^{-s}, \; \beta = 1/(p - 1)$$

$$(z, s) \in W_a = \{(z, s); \; s > s_0, \; e^{-s/2} z + a \in \Omega\},$$

and

$$s_0 = -\log T.$$

$w_a$ satisfies the equation:

$$\partial_s w_a - \Delta w_a + \frac{1}{2} z \cdot \nabla w_a + \beta w_a - |w_a|^{p-1} w_a = e^{-s\beta - s} \hat{h}(z, s), \quad (z, s) \in W_a.$$

## The existence of the optimal control for the problem $(P)$.

### Idea of the proof of Theorem 2.2.3

**Step 1.** Construct an approximate problem.

Let $R$ be a number such that $R > \| y_0 \|_{L^2(\Omega)}$. Write $B_R(0) = \{y \in L^2(\Omega) | \ \| y \|_{L^2(\Omega)} \leq R\}$. We denote by $\partial B_R(0)$ the boundary of $B_R(0)$. Define $J_R$ be a map from $\mathcal{U}_{ad}$ to $\overline{R^+} \equiv [0, +\infty]$ by

$$J_R(u(\cdot)) = \inf\{t \geq 0 \ : \ y(t; y_0, u) \in \partial B_R(0)\}.$$

Consider the following approximate time optimal control problem:

$(P_R)$ Find $u_R \in \mathcal{U}_{ad}$ such that

$$J_R(u_R(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{ad}} J_R(u(\cdot)) =: t_R.$$

Step 2. We prove the existence of solutions to the problem $(P_R)$ for each $R$ with $R > \| y_0 \|_{L^2(\Omega)}$.

Step 3. Firstly, we prove that there exists two positive constants $t^*$, $T$ and a control $u^* \in \mathcal{U}_{ad}$ such that

$$t_R \to t^*, \text{as } R \to +\infty$$

and there is a sequence of numbers $\{R_n\}$ with $R_n \to +\infty$ such that for $n \to \infty$,

$$u_{R_n} \rightharpoonup u^* \text{ weakly star in } L^\infty((0,T); L^\nu(\Omega)).$$

Secondly, we prove that $t^*$ is the optimal time and $u^*$ is the optimal control of the problem $(P)$.

The approximate maximum principle of the optimal control for the problem $(P)$.

Idea of the proof of Theorem 2.2.4

Step 1. To set up a penalty functional.

Let $T$ be a fixed constant with $T > t_R$. Write $\mathcal{U}[0,T] = \{u|_{[0,T]} \mid u \in \mathcal{U}_{ad}\}$. We introduce the Ekeland distance $\overline{d}$ over the set $\mathcal{U}[0,T]$, which is given by

$$\overline{d} = \text{meas}\{t \in [0,T] \mid u(t) \neq v(t)\}.$$

For each number $\varepsilon$ with $0 < \varepsilon < t_R$, we define a penalty functional $J_\varepsilon : (\mathcal{U}[0,T], \overline{d}) \to \mathbb{R}^1$ by

$$J_\varepsilon(u(\cdot)) = d_{\partial B_R(0)}(y(t_R - \varepsilon; y_0, u_\varepsilon)).$$

By Ekeland's variational principle, for each number $\varepsilon$ with $0 < \varepsilon < t_R$, there is a control $u^\varepsilon(\cdot)$ in the set $\mathcal{U}[0,T]$ such that the following hold:

$$\begin{cases} \overline{d}(\overline{u}(\cdot), u^\varepsilon(\cdot)) \leq \sqrt{\sigma(\varepsilon)}, \\ -\sqrt{\sigma(\varepsilon)}\, \overline{d}(\widetilde{u}(\cdot), u^\varepsilon(\cdot)) \leq J_\varepsilon(\widetilde{u}(\cdot)) - J_\varepsilon(u^\varepsilon(\cdot)),\ \forall\, \widetilde{u}(\cdot) \in \mathcal{U}[0,T]. \end{cases}$$

# Thank You