Moment method, cost of fast controls for parabolic equations and Gevrey functions with compact support

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Linear control system

Let $(H, \langle , \rangle_H)$ be some Hilbert space and $(A, \mathcal{D}(A))$ some closed unbounded operator with dense domain on H.
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Let $(U, \langle , \rangle_U)$ another Hilbert space . Let $B \in \mathcal{L}_c(U, \mathcal{D}(A^*)')$.
Let $T > 0$ some time, $y^0 \in H$ and $u \in L^2((0, T), U)$.

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Linear control system:

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We assume that there exists $C(t)$ such that for every $z \in D(A^*)$, (admissibility condition)

$$\int_0^t ||B^* e^{tA^*} z||_U^2 \leqslant C(t)||z||_H^2, \ \forall z \in \mathcal{D}(A^*), \forall t \in [0, T]. \qquad \text{(Adm)}$$

In this case, $B^* e^{tA^*}$ can be extended in $\mathcal{L}_C(H, U)$.

Admissibility and well-posedness

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A solution *of* (Syst-Cont) *with initial* $y^0 \in H$ *is a function* $y \in C^0([0,T],H)$ *verifying* $y(0) = y^0$ *and such that for every* τ *in* $[0,T]$ *and* z^τ *in* H*, we have*

$$\langle y(\tau), z^\tau \rangle_H - \langle y^0, e^{\tau A^*} z^\tau \rangle_H = \int_0^\tau \langle u(t), B^* e^{(\tau - t)A^*} z^\tau \rangle_U dt. \quad \text{(Def-Sol)}$$

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Théorème (Weiss'89)

We assume B admissible (i.e. (Adm) *holds). For every* $y^0 \in H$ *and* $u \in L^2((0,T),U)$*,* (Syst-Cont) *with initial condition* y^0 *admits a unique solution* $y \in C^0([0,T],H)$ *verifying*

$$||y||_{C^0([0,T],H)} \leqslant C(T)(||y^0||_H + ||u||_{L^2((0,T),U)}).$$

Null controllability

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Assume that (Syst-Cont) is controllable. In general, the control u driving y^0 to 0 at time T is not unique. Let us consider the set of controls

$$\mathcal{E}_{adm}(y^0) := \{\tilde{u} \text{ s.t. } (\tilde{y}, \tilde{u}) \text{ ver. (Syst-Cont)}, \tilde{y}(0) = y^0 \text{ and } \tilde{y}(T) = 0\}.$$

Let us consider the following optimization Problem:

$$\min_{\tilde{u} \in \mathcal{E}_{adm}(y^0)} J(\tilde{u}) := \min_{\tilde{u} \in \mathcal{E}_{adm}(y^0)} \frac{1}{2} \int_0^T ||\tilde{u}(t)||_U^2 dt. \qquad \text{(Cont-Opt)}$$

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Proposition

There exists a unique solution $u_{opt}(y^0)$ to the Problem (Cont-Opt).

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There exists a unique solution $u_{opt}(y^0)$ to the Problem (Cont-Opt).

Proof: $\mathcal{E}_{adm}(y^0)$ is an affine subspace of U and is also closed. u_{opt} is then the orthogonal projection of 0 $\mathcal{E}_{adm}(y^0)$.

Definition of the cost of the control

We consider $\Gamma : y^0 \mapsto u_{opt}(y^0)$. This application is linear continuous and we call C_T its operator norm, that we call the cost of the control at time T.

$$C_T := \sup_{||y^{\mathbf{0}}||=1} \min_{\tilde{u}\in\mathcal{E}_{adm}(y^{\mathbf{0}})} \int_0^T ||\tilde{u}(t)||_U^2 dt.$$

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C_T quantifies the minimal energy needed to bring any initial condition to 0. It is the smallest constant C such that for every $y^0 \in H$, there exists $(\tilde{y}, \tilde{u})$ verifying (Syst-Cont) such that $\tilde{y}(0) = y^0$ and $\tilde{y}(T) = 0$ with

$$||u||_U \leqslant C||y^0||_H.$$

Link with the inverse problems

If we call O_T the largest constant C such that

$$\int_0^T ||B^*\varphi(t)||^2 dt \geqslant C||\varphi^T||,$$

one can prove that we have the relation

$$O_T = \frac{1}{C_T}.$$

Hence, C_T also quantifies the accuracy of the reconstruction of the final data φ^T thanks to the observation $\int_O^T ||B^*\varphi(t)||^2 dt$.

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Goal

Estimate precisely the behavior of fast controls, i.e. the asymptotic of C_T as $T \to 0$.

Diagonal operators and scalar controls

From now on we assume A negative selfadjoint.
Normalized eigenvectors and eigenvalues: $(e_j, -\lambda_j)$, where $\lambda_j > 0$. From now on,

$$U = \mathbb{R}, \;\; Bu = bu,$$

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$$b_k := \langle b, e_k \rangle_{\mathcal{D}(A)',\mathcal{D}(A)}.$$

Going back to (Def-Sol), defining the solutions of (Syst-Cont) is equivalent to have for every k and every $t \in (0, T)$

$$\langle y(t,x), e_k \rangle_H = a_k e^{-\lambda_k t} + b_k \int_0^t e^{-\lambda_k (t-s)} u(s) ds. \qquad \text{(Def-Sol-Diag)}$$

A criterium to obtain admissibility

Définition

We say $(\lambda_n)_{n\in\mathbb{N}^*}$ *is* *regular* *if*

$$\gamma((\lambda_n)_{n\in\mathbb{N}^*}) := \inf_{m\neq n} |\lambda_m - \lambda_n| > 0. \qquad \text{(Regul)}$$

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Théorème (Ho-Russell'83)

Assume that $(\lambda_n)_{n\in\mathbb{N}^*}$ is regular. If $||(b_k)_{k\in\mathbb{N}^*}||_\infty < +\infty$, then b is admissible.

Ex: fractional heat equation with “boundary” control (1)

Consider Δ with domain $D(\Delta) := H_0^1(0, L)$ and state space $H := H^{-1}(0, L)$. $-\Delta : D(\Delta) \to H^{-1}(0, L)$ is a positive definite operator with compact resolvent, the k-th eigenvalue is $\frac{k^2\pi^2}{L^2}$, with eigenvector $e_k(x) := \frac{\sqrt{2}k\pi}{L^{3/2}} \sin\left(\frac{k\pi x}{L}\right)$. Thanks to the continuous functional calculus for positive self-adjoint operators, one can define any positive power of $-\Delta$. Let us consider here some $\alpha > 1$.

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We consider

$$\begin{cases} y_t = -(-\Delta)^{\alpha/2} y + bu & \text{in } (0,T)\times(0,L), \\ y(0,\cdot) = y^0 & \text{in } (0,L), \end{cases} \tag{Heat-Frac}$$

where b is a scalar control operator to be defined. We call

$$\lambda_k := \frac{k^\alpha \pi^\alpha}{L^\alpha}.$$

$\{\lambda_k\}_{k\in\mathbb{N}^*}$ is regular.

Ex: fractional heat equation with "boundary" control (2)

We set

$$b := (\varphi \in \mathcal{D}(A) \mapsto (\Delta^{-1}\varphi)_x(0)), \, i.e.$$

$$b := -(\partial_x \delta_0) \circ \Delta^{-1},$$

if $\alpha = 2n$ with $n \in \mathbb{N}$, b corresponds to impose $\Delta^n y_{|x=0} = u(t)$.

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$$\begin{aligned} b_k &= (\Delta^{-1} e_k)_x(0) \\ &= \frac{L^2}{k^2\pi^2} \frac{\sqrt{2}k\pi}{L^{3/2}} \sin\left(\frac{k\pi x}{L}\right)_x (0) \\ &= \frac{L^2}{k^2\pi^2} \frac{\sqrt{2}k\pi}{L^{3/2}} \frac{k\pi}{L} \\ &= \frac{\sqrt{2}}{\sqrt{L}} \end{aligned}$$

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In this case we have

$$||b_k||_\infty < \infty,$$

and the operator is admissible.

The moment method (Fattorini-Russell'71)

Taking into account (Def-Sol-Diag), imposing $y(T,x)=0$ is now equivalent to impose for every k that

$$a_k e^{-\lambda_k T} + b_k \int_0^T e^{-\lambda_k (T-t)} u(t) dt = 0.$$

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Consider a biorthogonal family to the family of exponentials $\{t \mapsto e^{\lambda_n t}\}$ in $L^2(0,T)$, i.e. a family $\{\psi_m\}_{m \in \mathbb{N}^*} \in L^2(0,T)$ such that

$$\langle e^{\lambda_k t}, \psi_l \rangle_{L^2(0,T)} = \delta_{kl}. \qquad \text{(Biorth)}$$

Creating bi-orthogonal functions

It is then enough to set

$$u(t) := -\sum_{k\in\mathbb{N}^*} (a_k/b_k)\psi_k(t),$$

as soon as this expression is meaningful (for example if $|b_k| \geqslant C > 0$).
(Biorth) is equivalent to prove

$$\mathcal{F}(\psi_j)(-i\lambda_k) = \delta_{jk},$$

(where $\mathcal{F}$ is the Fourier transform) as soon as the support of ψ_j is included in $(0, T)$ and ψ_j is in L^2.

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Simplest idea: create a function which has $-i\lambda_k$ as simple roots:

$$\Psi(z) := \prod_{k=1}^{\infty}(1 - iz/\lambda_k).$$

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We then set

$$J_k(z) := \frac{\Psi(z)}{\Psi'(-i\lambda_k)(z + i\lambda_k)}$$

and we look at $\mathcal{F}^{-1}(J_k)$.

The case of the fractional heat equation

We go back to (Heat-Frac) and to simplify we choose $L = \pi$, so that $\lambda_k = k^\alpha$.

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On the real axis we can prove

$$\sum_{k=1}^{\infty} \ln\left(1 + \frac{x^2}{k^{2\alpha}}\right) \leqslant |x|^{2/\alpha} \int_1^{\infty} \frac{dt}{t^{1/(2\alpha)(1+t)}} \leqslant |x|^{2/\alpha} \frac{\pi}{\sin\left(\frac{\pi}{2\alpha}\right)}.$$

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We deduce that

$$|J_k(x)| \simeq C(k) e^{\frac{\pi}{2\sin\left(\frac{\pi}{2\alpha}\right)} |x|^{1/\alpha}},$$

and this estimate is optimal at infinity.
$\Rightarrow$ cannot obtain a L^2 function by taking the inverse Fourier transform.

A remedy: the Paley-Wiener Theorem

Theorem (Paley-Wiener'33)

F is the Fourier transform of a L^2 function with compact support in $[-A, A]$ if and only if $F \in L^2(\mathbb{R})$ and F is of exponential type A, i.e. F is an entire function verifying

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The idea is then to multiply J_k by some entire function M_k (called multiplier) which ensures that $M_k J_k$ is still of exponential type, has the same roots as J_k and is sufficiently decreasing on the real axis, i.e. such that $M_k J_k \in L^2(\mathbb{R})$.

Link with Gevrey functions (1)

In the case of the fractional heat equation we know that

$$J_k \simeq \exp(K|x|^{1/\alpha})$$

for some $K > 0$. We want to obtain a function $M_k J_k$ which is in $L^2(\mathbb{R})$ $\Rightarrow$ We need $M_k \simeq \exp(-K|x|^{1/\alpha})$ at infinity. This exactly means that M_k has to be the Fourier transform of some function G_k which has to be Gevrey of order α, i.e. verifying

$$||G_k^{(j)}||_\infty \leqslant C_{Gev}(T,R)R^j j!^\alpha,$$

for some constant $R > 0$.

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for some constant $R > 0$.
Moreover, R is related to the exponential decreasing of M_k (i.e. the constant K). We also want M_k to be of exponential type $T/2$, hence applying the Paley-Wiener Theorem, it is equivalent to saying that G_k has to be of compact support $[-T/2, T/2]$.

Link with Gevrey functions (3)

The function ψ_k will also be bounded by $C_{Gev}(T, R)$ and then the cost of the control is also bounded by $C_{Gev}(T, R)$. Hence, if we have to make all our possible to obtain some constant $C_{Gev}(T, R)$ which is as small as possible. We replace $G_k(-i\lambda_k) = 1$ by the condition $\int |G_k| = 1$ (and some suitable inequality on $G_k(i\lambda_k)$).

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Main issue

Bound from above the cost of the control is closely related to the following problem: given some $T>0$, construct a Gevrey function G of order α, with support equal to $[-T/2, T/2]$, verifying $\int|G|=1$, with imposed coefficient R appearing in the growth of the derivative and minimizing the quantity $C_{Gev}(T,R)$.

The cost of fast controls for the fractional heat equation (1)

We denote by $C_H(T, L, \alpha)$ the cost of the control for (Heat-Frac).

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Some notations $f(x) \lesssim g(x)$ $(x \in \mathcal{S})$: there exists $C > 0$ independent of x (might depend on other parameters) such that for every $x \in \mathcal{S}$,

$$|f(x)| \leqslant C|g(x)|.$$

$f(x) \simeq g(x)$: $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$.

The cost of fast controls for the fractional heat equation (1)

We denote by $C_H(T, L, \alpha)$ the cost of the control for (Heat-Frac).
Some notations $f(x) \lesssim g(x)$ $(x \in \mathcal{S})$: there exists $C > 0$ independent of x (might depend on other parameters) such that for every $x \in \mathcal{S}$,

$$|f(x)| \leqslant C|g(x)|.$$

$f(x) \simeq g(x)$: $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$.
$f(x) \lesssim g(x, A^+)$: for every $K > A$, there exists a constant $C(K)$ such that

$$|f(x)| \leqslant C(K)g(x, K).$$

$(C(K)$ might explode when $K \to A^+)$

The cost of fast controls for the fractional heat equation (2)

We denote by

$$\beta_+(\alpha) := \limsup_{T\to 0} T^{\frac{1}{\alpha-1}} \frac{\ln(C_H(T,L,\alpha))}{L^{\frac{\alpha}{\alpha-1}}},$$

$$\beta_-(L,\alpha) := \liminf_{T\to 0} T^{\frac{1}{\alpha-1}} \frac{\ln(C_H(T,L,\alpha))}{L^{\frac{\alpha}{\alpha-1}}}.$$

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$\beta_-(L,2) > 0$ (Guïchal'85) and $\beta_+(L,2) < \infty$ (Seidman'84), i.e.
$C_H(T,L,2) \simeq e^{\frac{CL^2}{T}}$ for small T.
$\beta_-(\alpha) > 0$ and $\beta_+(\alpha) < \infty$ (Seidman-Avdonin-Ivanov'00), i.e.
$C_H(T,L,\alpha) \simeq e^{\frac{CL^{\alpha/(\alpha-1)}}{T^{1/(\alpha-1)}}}$ for small T.

The cost of fast controls for the fractional heat equation (2)

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$\beta_-(\alpha) > 0$ and $\beta_+(\alpha) < \infty$ (Seidman-Avdonin-Ivanov'00), i.e. $C_H(T,L,\alpha) \simeq e^{\frac{CL^{\alpha/(\alpha-1)}}{T^{1/(\alpha-1)}}}$ for small T. More precise estimations $\beta_-(2) \geqslant 1/4$ (Miller'04) and $\beta_+(2) \leqslant 3/4$ (Tenenbaum-Tucsnak'07). For $\alpha \geqslant 2$, precise estimates of $\beta_+(\alpha)$ in Lissy'14 (SICON).

Natural conjecture

$\alpha_+(2) = 1/4 = \alpha_-(2)$, i.e. for T small $C_H(T,L,2) \simeq e^{(L^2/4)^+/T}$.

A new lower bound (1)

This natural conjecture is false!

Theorem (Lissy'15, JDE)

$$\beta_-(\alpha) \gtrsim \frac{2^{\frac{1}{\alpha-1}}(\alpha-1)}{\left(\alpha \sin\left(\frac{\pi}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-1}}}.$$

For $\alpha = 2$, $\beta_-(\alpha) \geqslant 1/2$!

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Sketch of the proof

To simplify we restrict to the case $\alpha = 2$. We use a method introduced in Coron-Guerrero'05, based on the moment method and some tools from complex analysis. Also used in Lissy'15 (transport-diffusion), in Lissy-Gueye'16 (degenerate transport-diffusion) and in Lissy'16 (minimal time of control for parabolic systems).

We consider an optimal control u associated to the first eigenfunction $\sin(\pi x/L)$, which verifies

$$||u||_{L^2(0,L)} \leqslant C_H(T,L,2)||y^0||_{H^{-1}(0,L)} \leqslant C C_H(T,L,2)L^{3/2}.$$

A new lower bound (2)

By definition the control u has to solve to moment problem so we have

$$\frac{k\pi}{L}\int_0^T u(t)\exp\left(\frac{k^2\pi^2}{L^2}t\right)dt = -\int_0^L \sin\left(\frac{\pi}{L}\right)\sin\left(\frac{k\pi}{L}\right)dx.$$

Let us define the Fourier transform of u given by

$$v(z) := \int_{-T/2}^{T/2} u\left(t+\frac{T}{2}\right)\exp(-izt)dt.$$

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Let us define the Fourier transform of u given by

$$v(z) := \int_{-T/2}^{T/2} u\left(t+\frac{T}{2}\right)\exp(-izt)dt.$$

Let us consider some numerical parameter $\beta > 0$ to be chosen later. We introduce

$$f(z) := v\left(\frac{z-i\beta L^2}{T^2}\right).$$

It is easy to see that

$$|f(z)| \leqslant CC_H(T,L,2)\sqrt{T}\exp\left(\frac{|Im(z)-\beta L^2|}{2T}\right)L^{3/2}.$$

A new lower bound (3)

One has, for $k \in \mathbb{N}$ and $k > 1$,

$$f(b_k) = 0,$$

with

$$b_k := i\left(L^2\beta + \frac{T^2k^2\pi^2}{L^2}\right).$$

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We also have

$$f(b_1) = -\frac{L^2}{2\pi}\exp\left(-\frac{\pi^2 T}{2L^2}\right),$$

with

$$b_1 := i\left(L^2\beta + \frac{T^2 \pi^2}{L^2}\right).$$

A new lower bound (4)

We have the following equality, for every z such that $Im(z) > 0$:

$$\ln(|f(z)|) = \sum_{1}^{\infty} \ln\left(\frac{|z-a_l|}{|z-\overline{a_l}|}\right) + \sigma x_2 + \frac{x_2}{\pi}\int_{\mathbb{R}} \frac{\ln(|f(\tau)|)}{|\tau - z|^2} d\tau,$$

where the a_k are all the roots of f of positive imaginary part and σ is the type of f, which verifies

$$\sigma \leqslant \frac{1}{2T}.$$

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We apply this equality at point b_1, we obtain

$$\ln(|f(b_1)|) \leqslant \sum_1^{\infty} \ln\left(\frac{|b_1 - a_l|}{|b_1 - \overline{a_l}|}\right) + \frac{L^2\beta}{2T} + \frac{T\pi^2}{2L^2} + \frac{|b_1|}{\pi}\int_{\mathbb{R}} \frac{\ln(|f(\tau)|)}{\tau^2 + |b_1|^2} d\tau.$$

Let us study the right-hand side of this equality.

A new lower bound (4)

We have the following equality, for every z such that $Im(z) > 0$:

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Let us study the right-hand side of this equality.
First term of the right-hand side: We study

$$I := \sum_{l=1}^{\infty} \ln\left(\frac{|b_1 - a_l|}{|b_1 - \overline{a_l}|}\right).$$

A new lower bound (5)

It is easy to prove that

$$I \leqslant \sum_{2}^{\infty} \ln\left(\frac{|b_1 - b_l|}{|b_1 - \overline{b_l}|}\right) = \sum_{2}^{\infty} \ln\left(\frac{(k^2-1)T^2\pi^2/L^2}{2L^2\beta + (k^2+1)T^2\pi^2/L^2}\right)$$
$$\leqslant \int_2^{\infty} \ln\left(\frac{x^2T^2\pi^2/(2\beta L^4)}{1 + x^2T^2\pi^2/(2\beta L^4)}\right) dx.$$

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After some computations we obtain

$$\sum_{1}^{\infty} \ln \left(\frac{|b_1 - a_l|}{|b_1 - \overline{a_l}|} \right) \leqslant 2 \ln \left(1 + \frac{2\beta L^4}{(2\pi)^2 T^2} \right) - \frac{L^2 (2\beta)^{\frac{1}{2}}}{T} + 2.$$

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Concerning the third term of the right-hand side, an easy changing of variables gives

$$|b_1| \int_{\mathbb{R}} \frac{d\tau}{\tau^2 + |b_1|^2} = \pi.$$

A new lower bound (6)

Hence, we deduce that

$$\frac{b_1}{\pi}\int_{\mathbb{R}}\frac{\ln|f(\tau)|}{\tau^2+b_1^2}d\tau \leqslant \frac{\beta L^2}{2T}+\ln(CC_H(T,L)\sqrt{T}L^{3/2}).$$

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Hence, using all previous results we obtain an inequality on $C_H(T,L)$. Now, we optimize β by trying to maximize what is inside the exponential and we find the desired result.

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New conjecture

$\alpha_+(2)=1/2=\alpha_-(2)$, i.e. for T small enough $C_H(T,L,2)\simeq e^{(L^2/2)^+/T}$.

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Hence, we deduce that

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Hence, using all previous results we obtain an inequality on $C_H(T,L)$. Now, we optimize β by trying to maximize what is inside the exponential and we find the desired result.

New conjecture

$\alpha_+(2)=1/2=\alpha_-(2)$, i.e. for T small enough $C_H(T,L,2)\simeq e^{(L^2/2)^+/T}$.

In fact, this conjecture is natural if we look at the computations of Tenenbaum-Tucsnak'07.

Digression: the multiplier of Tenenbaum and Tucsnak (1)

Let $\nu > 0$ et $\beta > 0$ two parameters We call

$$\sigma_\nu(t) := e^{-\frac{\nu}{1-t^2}},$$

extended to 0 outside $(-1, 1)$. We set

$$H_\beta(z) := C_\nu \int_{-1}^{1} \sigma_\nu(t) e^{-i\beta tz} dt,$$

where

$$C_\nu := 1/||\sigma_\nu||_1.$$

Digression: the multiplier of Tenenbaum and Tucsnak (2)

This multiplier verifies

Lemme

$$
\begin{aligned}
H_\beta(0) &= 1,\\
H_\beta(ix) &\gtrsim \frac{e^{\beta|x|/(2\sqrt{\nu+1})}}{\sqrt{\nu+1}},\\
|H_\beta(z)| &\leqslant e^{\beta|Im(z)|}\\
H_\beta(x) &\lesssim \sqrt{\nu+1}e^{3\nu/4-(\pi+\delta/2)\sqrt{|x|}}.
\end{aligned}
$$

A remarquable point is that this is actually the multiplier that gives the best known result for estimation of the cost of the control for the heat equation!

Main estimates

Proposition

Let $(a_k)_{k \geqslant 0}$ be a decreasing sequence of positive numbers such that

$$a := \sum_k a_k < \infty.$$

Then, there exists a nonnegative function u with compact support included in $[-a, a]$ verifying

$$\int_{-a}^{a} u = 1, \;\; u(0) \leqslant \frac{1}{2a_0},$$

$$u \text{ is even}, \;\; u \text{ is increasing on } [-a, 0],$$

such that for every $j \in \mathbb{N}$, one has

$$||u^{(j)}||_\infty \leqslant \frac{1}{\prod_{i=0}^{j} a_i}.$$

Idea of the proof

For every $b > 0$ we call

$$H_b := \frac{1_{[-b,b]}}{2b}.$$

Let us remark that

$$\int_{\mathbb{R}} H_b = 1$$

H_b is even.

Idea of the proof

For every $b > 0$ we call

$$H_b := \frac{1_{[-b,b]}}{2b}.$$

Let us remark that

$$\int_{\mathbb{R}} H_b = 1$$

$$H_b \text{ is even.}$$

We then consider

$$u_n := H_{a_0} * H_{a_1} * \cdots * H_{a_n},$$

where $*$ represents the convolution product. u_n is of class C^{n-1} with support included in $[-\sum_1^n a_i, \sum_1^n a_i]$. Moreover we can prove

$$||u_n^{(j)}||_\infty \leqslant \frac{1}{\prod_{i=0}^{j} a_i}.$$

Idea of the proof (2)

Let us also remark that u_n is even and verifies

$$\int_{-a}^{a} u_n(x)dx = \int H_{a_1} \cdots \int H_{a_n} = 1,$$

moreover

$$u_n(0) \leqslant \frac{1}{2a_0}.$$

We can prove that u_n in increasing on $[-\sum_1^n a_i, 0]$.

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We can prove that u_n in increasing on $[-\sum_1^n a_i, 0]$.
We conclude as by letting $n \to \infty$, u being the limit (that exists) of the u_n.

An related Optimization Problem (1)

Minimizing $C_{Gev}(T,R)$ in our can be reformulated as follows. Let us set

$$\mathcal{A}_{adm}(a) := \{(a_k)_{k\in\mathbb{N}} | a_k > 0 \ \forall k, (a_k)_{k\in\mathbb{N}} \text{ is non-increasing}, \sum a_k = a\}.$$

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Then we are interested in

$$\inf_{(a_k)\in A_{adm}(a)} \sup_{j\in\mathbb{N}} \prod_{i=0}^{j} \left(\frac{\nu^{\alpha-1}}{(i+1)^{\alpha} a_i} \right),$$

for some large enough (compared to a) parameter $\nu > 0$ to be chosen later. This problem seems quite difficult.
As a toy model, we propose to investigate the behavior of

$$\prod_{i=0}^{\infty} \left(\frac{(i+1)^{\alpha} a_i}{\nu^{\alpha-1}} \right)$$

on the set $\mathcal{A}_{adm}$. Are there critical points?
Let

$$f((a_k)) := \sum_{k=0}^{\infty} \ln \left(\frac{(k+1)^{\alpha} a_k}{\nu^{\alpha-1}} \right).$$

An related Optimization Problem (2)

One has

$$\frac{\partial f}{\partial a_i}((a_k)) = \frac{1}{a_k}.$$

Let

$$g((a_k)) := \sum_{k=0}^{\infty} a_k.$$

One has

$$\frac{\partial g}{\partial a_i}((a_k)) = 1.$$

We want to study f under the constraint $g((a_k)) = a$ for every $j \in \mathbb{N}$ (the constraint $a_k > 0$ can be forgotten here). We also do not need here to use that we want $(a_k)_{k\in\mathbb{N}}$ to be non-increasing.
Let $\lambda \in \mathbb{R}$ and $(\mu_j)_{j\in\mathbb{N}}$, we consider the Lagrangian

$$\mathcal{L}((a_k)) := f((a_k)) - \lambda g((a_k)).$$

An related Optimization Problem (3)

We remark that for every $j \in \mathbb{N}$, we have $\frac{\partial \mathcal{L}}{\partial a_j} = 0$ if and only if $a_j = \lambda$.
In contradiction with constraint $g((a_k)) = a$

An related Optimization Problem (3)

We remark that for every $j \in \mathbb{N}$, we have $\frac{\partial \mathcal{L}}{\partial a_j} = 0$ if and only if $a_j = \lambda$. In contradiction with constraint $g((a_k)) = a$ Hence, there does not exist any critical point for the Lagrangian. One then might have the idea to "mimic" in some sense constants sequences for the initial optimization problem.Idea: take a sequence which is constant for small k and then decreases in some adequate way. We set $\nu > 0$ some parameter (that is destined to compensate the bad growth of the product J_k), and we consider the following sequence $(a_k)_{k\in\mathbb{N}}$ defined by

$$\begin{cases} a_k & = \frac{1}{\nu} \qquad \text{if } k \leqslant \lfloor \nu \rfloor - 1, \\ a_k & = \frac{\nu}{(1+k)^{\alpha}} \qquad \text{if } k \geqslant \lfloor \nu \rfloor \,, \end{cases}$$

An related Optimization Problem (3)

We remark that for every $j \in \mathbb{N}$, we have $\frac{\partial \mathcal{L}}{\partial a_j} = 0$ if and only if $a_j = \lambda$. In contradiction with constraint $g((a_k)) = a$ Hence, there does not exist any critical point for the Lagrangian. One then might have the idea to "mimic" in some sense constants sequences for the initial optimization problem.Idea: take a sequence which is constant for small k and then decreases in some adequate way. We set $\nu > 0$ some parameter (that is destined to compensate the bad growth of the product J_k), and we consider the following sequence $(a_k)_{k \in \mathbb{N}}$ defined by

$$\begin{cases} a_k = \frac{1}{\nu} & \text{if } k \leqslant \lfloor \nu \rfloor - 1, \\ a_k = \frac{\nu}{(1+k)^\alpha} & \text{if } k \geqslant \lfloor \nu \rfloor, \end{cases}$$

We call σ_ν the corresponding function constructed with the previous construction, its support is denoted a_ν and tends to $\alpha/(\alpha-1)$ as $\nu \to \infty$.

Proposition

$$||\sigma_\nu^{(j)}||_\infty \lesssim \frac{e^{\alpha\nu}(j!)^\alpha}{\nu^{(\alpha-1)j}}, \; j \in \mathbb{N}.$$

The multiplier

We have the following estimates:
We then set the multiplier

$$H_\beta(z) := \int_{-a_\nu}^{a_\nu} \sigma_\nu(t) e^{-i\beta tz} dt,$$

which will be our multiplier (up to some homothety). We have the following properties:

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which will be our multiplier (up to some homothety). We have the following properties:

Lemma

For every $\delta > 0$ small enough, one has

$$H_\beta(0) = 1, \quad (1)$$

$$|H_\beta(z)| \leqslant e^{a_\nu \beta |Im(z)|}, \quad (2)$$

$$|H_\beta(x)| \lesssim e^{\alpha\nu} e^{-\alpha(\beta\nu^{\alpha-\mathbf{1}}|x|)^{\mathbf{1}/\alpha} + \delta|x|^{\mathbf{1}/\alpha}}, \ x \in \mathbb{R}. \quad (3)$$

$$|H_\beta(ix)| \gtrsim e^{\frac{\beta x}{\mathbf{4}\nu}}, \ x > 0. \quad (4)$$

Main result

Adjusting the coefficients β and ν, (roughly, β as to be close to $T/4$ and ν should be chosen to absorb the bad growth of J_k), we obtain:

Theorem (Lissy'16, accepted in MCRF)

One has

$$\beta_+(\alpha) \leqslant \frac{1}{2(\alpha-1)^{\frac{1}{\alpha-1}} \sin(\frac{\pi}{2\alpha})^{\frac{\alpha}{\alpha-1}}}.$$

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One has

$$\beta_+(\alpha) \leqslant \frac{1}{2(\alpha-1)^{\frac{1}{\alpha-1}} \sin(\frac{\pi}{2\alpha})^{\frac{\alpha}{\alpha-1}}}.$$

Important point

- For $\alpha = 2$, $\beta_+(2) = 1$, worse that the result of Tenenbaum-Tucsnak'07 for the heat equation!
- But enables to treat much more general cases (notably all the possible cases of all controllable in arbitrary small time fractional heat or Schrödinger equations), under some modifications (eigenvalues in k^α for $\alpha > 1$). Hence, extends the results of Tenenbaum-Tucsnak '07 but also also the ones of Lissy'14 (SICON).

Perspectives

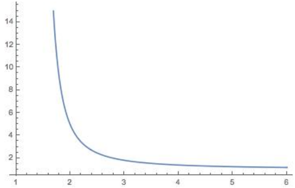

Figure: difference between the upper bound and the lower bound wrt α.

Perspectives

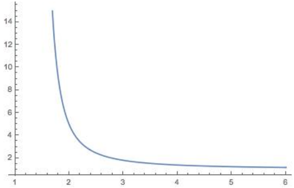

Figure: difference between the upper bound and the lower bound wrt α.

- Lower bound: study of first eigenfunction. Upper bound: linear combination of all modes. Does this explain that the lower bounds are far from the upper bounds?
- In the same spirit: is it possible to identify the “worst initial condition” for the cost of the control?
- Transmutation techniques as in Ervedoza-Zuazua‘11.
- Improve the Bray-Mandelbrojt construction presented here.
- Improve the multiplier of Tenenbaum-Tuscnak‘07.

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Thank you for your attention!