Stable determination of coefficients in the dynamical Schrödinger equation in a magnetic field

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Outline

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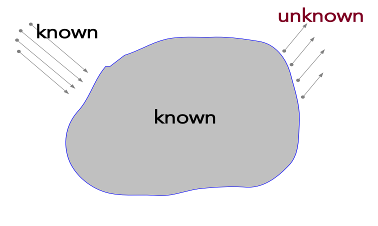
Direct problem VS Inverse problem

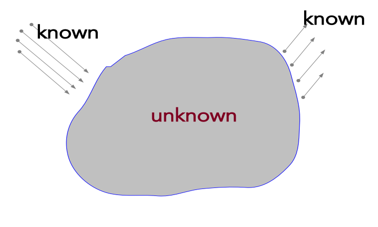
Direct problem

Given inputs and the property of the medium, find the output



Inverse problem

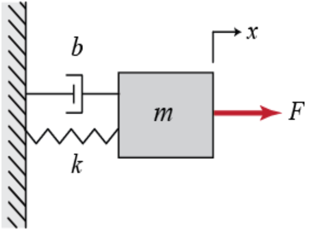
Given (a collection of) input(s) and output(s), find the property of the medium



Can one determine the internal properties of a medium by making measurments outside the medium?

Mass-spring-dashpot system

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx(t) = F$$
$$x(0) = x_0, \quad \frac{dx}{dt}(0) = v_0$$



x is the equilibrium displacement of mass m, k is the spring stifness.

Direct problem:

Given F, k, m and b find the position $x(t)$ for any time $t \geqslant 0$.

Inverse problem:

Given $x(t)$ for any $t \geqslant 0$, m and b find the coefficient k.

The dynamical Dirichlet-to-Neumann map(s)

Let $(\mathcal{M}, \mathrm{g})$ be a compact Riemannian manifold with boundary of dimension $n \geqslant 2$.

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We denote by Δ the Laplace-Beltrami.

$$\Delta = \frac{1}{\sqrt{|\mathrm{g}|}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{|\mathrm{g}|}\, \mathrm{g}^{jk} \frac{\partial}{\partial x_k} \right).$$

The dynamical Dirichlet-to-Neumann map(s)

Let $(\mathcal{M}, \mathrm{g})$ be a compact Riemannian manifold with boundary of dimension $n \geqslant 2$.
We denote by Δ_A the magnetic Laplace operator.

$$\Delta_A = \frac{1}{\sqrt{|\mathrm{g}|}} \sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_j} - i a_j \right) \sqrt{|\mathrm{g}|}\, \mathrm{g}^{jk} \left(\frac{\partial}{\partial x_k} - i a_k \right).$$

Here $A = a_j dx^j$ is a covector field (1-form) with real-valued coefficients $a_j \in \mathcal{C}^\infty(\mathcal{M})$ is the magnetic potential.

The dynamical Dirichlet-to-Neumann map(s)

Let $(\mathcal{M}, \mathrm{g})$ be a compact Riemannian manifold with boundary of dimension $n \geqslant 2$.
We consider the boundary value problem on $[0,T] \times \mathcal{M}$

$$(\mathrm{S}) \begin{cases} (i\partial_t + \Delta_A)u = 0, \\ u|_{t=0} = 0, \\ u|_{[0,T]\times\partial\mathcal{M}} = f \end{cases}$$

Let $A \in \mathcal{C}^\infty$ and $f \in \mathscr{H} = L^2(0,T;H^2(\partial\mathcal{M})) \cap H^1(0,T;L^2(\partial\mathcal{M}))$ such that $f(0,x) = 0$ for all $x \in \partial\mathcal{M}$. Then there exist a unique solution $u = u_{A,f}$ to (S) within the following class: $u \in \mathscr{C}([0,T];H^1(\mathcal{M})) \cap \mathscr{C}^1([0,T];H^{-1}(\mathcal{M}))$, and

$$\|\partial_\nu u\|_{L^2(\Sigma)} \leqslant C\|f\|_{\mathscr{H}(\Sigma)}, \quad \Sigma = (0,T) \times \partial\mathcal{M}$$

where ν is the exterior unit normal and u is a solution of (S).

The inverse problem

- Therefore we may define the Dirichlet-to-Neumann map

$$\Lambda_{g,A} : \mathscr{H}(\Sigma) \to L^2(\Sigma); \quad \Lambda_{g,A}(f) = (\partial_\nu + iA(\nu))u|_\Sigma.$$

The inverse problem is to determine the magnetic potential A from the DN map $\Lambda_{g,A}$.

- The first question is the identifiability of the magnetic potential, i.e. the injectivity of the map $A \mapsto \Lambda_{g,A}$.
- First of all, let us observe that there is an obstruction to uniqueness. The DN map is invariant under the gauge transformation of the magnetic potential. Namely, given $\varphi \in \mathcal{C}^1(\overline{\mathcal{M}})$ such that $\varphi|_{\mathcal{M}} = 0$ one has

$$e^{-i\varphi}\Delta_A e^{i\varphi} = \Delta_{A+d\varphi}, \quad e^{-i\varphi}\Lambda_A e^{i\varphi} = \Lambda_{A+d\varphi} = \Lambda_A, \quad d\varphi = \frac{\partial\varphi}{\partial x^j}dx^j.$$

Therefore, the magnetic potential A cannot be uniquely determined by the DN map Λ_A.

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Invariance of the DN-map

Let $\varphi \in \mathcal{C}^\infty(\mathcal{M};\mathbb{R})$, $\varphi_{|\partial\mathcal{M}} = 0$. Let $u = e^{i\varphi}v$, then we get that for any $f \in \mathcal{C}_0^1([0,T]\times\partial\mathcal{M})$

$$\begin{cases} (i\partial_t + \Delta_A)u = 0 \\ u|_{t=0} = 0 \\ u|_{[0,T]\times\partial\mathcal{M}} = f \end{cases} \quad \Leftrightarrow \quad \begin{cases} (i\partial_t + \Delta_{A+d\varphi})v = 0 \\ v|_{t=0} = 0 \\ v|_{[0,T]\times\partial\mathcal{M}} = f \end{cases},$$

$$\begin{aligned} \Lambda_A(f) = (\partial_\nu + iA(\nu))u_{|\Sigma} &= (\partial_\nu + iA(\nu))(e^{i\varphi}v)_{|\Sigma} \\ &= (\partial_\nu + i(A + d\varphi)(\nu))v_{|\Sigma} = \Lambda_{A+d\varphi}(f) \end{aligned}$$

Then

$$\Lambda_A = \Lambda_{A+d\varphi}$$

The inverse problem

From a geometric view point this can be seen as follows. Since $\mathcal{M}$ is a compact Riemannian manifold with boundary, then for every covector $A \in H^k(\mathcal{M})$, there exist uniquely determined $A^s \in H^k(\mathcal{M})$ and $\varphi \in H^{k+1}(\mathcal{M})$ such that:

$$A = A^s + d\varphi, \quad \delta A^s = 0, \quad \varphi|_{\mathcal{M}} = 0$$

We call the fields A^s and $d\varphi$ the solenoidal and potential parts of the covector A. The non-uniqueness manifested in the last slide says that the best we could hope to reconstruct from the DN map $\Lambda_{\mathrm{g},A}$ is the solenoidal part A^s of the covector A.

Stability estimates

Although the boundary control method (BCM) provides a form of uniqueness, it relies on a unique continuation property (UCP) (due to Tataru, Robbiano-Zuily...) and the Control theory and is not well suited to get explicit stability estimates.

We are interested in a method which will provide stability estimates of the form

$$\|A_1^s - A_2^s\|_{L^2(\mathcal{M})} \leqslant \Phi\big(\|\Lambda_{g,A_1} - \Lambda_{g,A_2}\|\big).$$

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Proof Strategies

How does one relate the information on the boundary of the manifold provided by the DN map, and what happens in the interior of the manifold? Suppose that we have two solutions u_1, u_2

$$\begin{cases} (i\partial_t + \Delta_{A_1})u_1 = 0 \\ u_1|_{t=0} = 0 \\ u_1|_{[0,T]\times\partial\mathcal{M}} = f_1 \end{cases}, \quad \begin{cases} (i\partial_t + \Delta_{A_2})u_2 = 0 \\ u_2|_{t=T} = 0 \\ u_2|_{[0,T]\times\partial\mathcal{M}} = f_2 \end{cases},$$

then for $A = A_1 - A_2$ and $V_A = i\delta(A) - |A_2|^2 + |A_1|^2$, we have

$$\int_0^T\int_{\partial\mathcal{M}} \left(\Lambda_{g,A_1} - \Lambda_{g,A_2}\right) f_1 \overline{f}_2 \, d\sigma^{n-1} \, dt = -2i \int_0^T\int_{\mathcal{M}} \langle A(x), du_2\rangle \, \overline{u}_1 \, dv^n \, dt - \int_0^T\int_{\mathcal{M}} V_A(x) u_2 \overline{u}_1 \, dv^n \, dt$$

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$$\Lambda_{g,A_1} - \Lambda_{g,A_2} = 0 \qquad \Longrightarrow \qquad A^s = 0\,?$$

Proof Strategies (Uniqueness)

Construct a familly of solutions u_1^λ, u_2^λ, $\lambda > 0$, of

$$\begin{cases} (i\partial_t + \Delta_{A_1})u_1^\lambda = 0 \\ u_1^\lambda|_{t=0} = 0 \end{cases}, \quad \begin{cases} (i\partial_t + \Delta_{A_2})u_2^\lambda = 0 \\ u_2^\lambda|_{t=T} = 0 \end{cases},$$

such that:

$$2i\int_0^T\int_{\mathcal{M}} \left\langle A(x), du_2^\lambda \right\rangle \overline{u}_1^\lambda \,\mathrm{dv}^n \,\mathrm{d}t + \int_0^T\int_{\mathcal{M}} V(x) u_2^\lambda \overline{u}_1^\lambda \,\mathrm{dv}^n \,\mathrm{d}t = 0, \;\; \forall \lambda > 0$$

$$\Downarrow$$

$$A^s(x) = 0, \;\; x \in \mathcal{M}.$$

Outline

Simple geometries

In the construction by geometrical optics, we will need to solve the eikonal equation

$$|\nabla\psi(x)|^2 = \mathrm{g}^{jk}\partial_j\psi\,\partial_k\psi = 1.$$

To have a well-defined solution, it is convenient to work in geometries where no caustics occur.

Definition

A manifold $(\mathcal{M}, \mathrm{g})$ is said to be simple if the boundary $\partial\mathcal{M}$ is strictly convex and if for all $x \in \mathcal{M}$, the exponential map $\exp_x : \exp_x^{-1}(\mathcal{M}) \mapsto \mathcal{M}$ is a diffeomorphism.

In particular, the cut locus is empty for all points of the manifold and there is a unique distance minimizing geodesics between two points.

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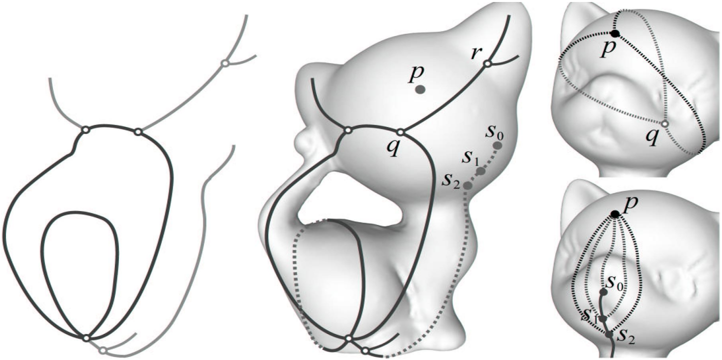
Cut Locus

- $\mathcal{M}$ is compact then $\mathcal{M}$ is geodesically complete: any two points admit a minimizing geodesic.
- The cut locus of a point p in a manifold is roughly the set $\mathrm{Cut}(p)$ of all other points for which there are multiple geodesics connecting them from p.



Smoothness of Distance Function

Let $z \in \mathcal{M}$ and consider the distance function

$$\psi : \mathcal{M} \to \mathbb{R}; \quad \psi(x) = d_{\mathrm{g}}(z, x).$$

- ψ is a continuous function. However, it is not hard to see that ψ is not smooth on $\mathcal{M}$. In fact, ψ is never smooth at z.
- Consider $(\mathbb{S}^2; \mathrm{g}_{\mathbb{S}^2})$. Let $\bar{z} = -z$ be the antipodal point of z. Then for x near $\bar{z}$, $d_{\mathrm{g}}(z, x) = \pi - d_{\mathrm{g}}(\bar{z}, x)$. It follows that d_{g} is also not smooth at $\bar{z}$.

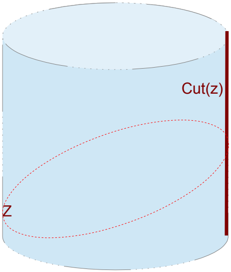
Theorem

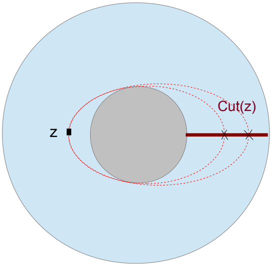
The function ψ is smooth on $\mathcal{M} \backslash Cut(z) \cup \{z\}$. Moreover, for each $x \in \mathcal{M} \backslash Cut(z) \cup \{z\}$, if we let $\gamma_{z,\theta}$ be the unique normal minimizing geodesic from z to x, then the gradient of $\nabla\psi(x)$ at x is

$$\nabla\psi(x) = \dot{\gamma}_{z,\theta}(r), \quad r = d_{\mathrm{g}}(z, x)$$

Smoothness of Distance Function

- Spheres (or Hemispheres) are not simple!!: The cut locus of every point on the sphere consists of exactly one point, namely the antipodal one.
- One can always increase a simple manifold with boundary into a bigger simple manifold $\mathcal{M}_1$.
- Any simple manifold is diffeomorphic to a ball in $\mathbb{R}^n$.
- Simple manifolds are non-trapping.





The eikonal equation: $|\nabla\psi|^2 = 1$

$$|\nabla\psi|^2 = g^{jk}\partial_j\psi\partial_k\psi = 1, \quad x \in \mathcal{M}.$$

In simple manifolds, it is easy to give explicit solutions of the eikonal equation

$$\psi(x) = d_g(x,z), \quad z \in \mathcal{M}_1 \backslash \mathcal{M}$$

where d_g is the geodesical distance ($\mathcal{M}_1$ is a simple extension of $\mathcal{M}$).
In fact, one can use geodesical polar coordinates

$$x = \exp_z(r\theta), \quad r = d_g(x,z) > 0, \quad \theta \in S_z\mathcal{M}.$$

In those coordinates, the metric reads

$$g = dr^2 + g_0(r,\theta)$$

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Geodesical X-ray transform

The resolution of the eikonal equation is not the only reason why we are working in simple geometries. We will need the injectivity of an X-ray transform which is only true in simple geometries. The set of incoming/outgoing vectors is defined as follows

$$\partial_{\pm}S\mathcal{M} = \left\{(x,\theta) \in S\mathcal{M},\, x \in \partial\mathcal{M} : \pm\langle \nu, \theta\rangle < 0\right\}.$$

The scattering relation:

$$\mathscr{S} : \partial_+ S\mathcal{M} \longrightarrow \partial_- S\mathcal{M}, \quad \mathscr{S}(x,\theta) = \Phi_{\tau_+}(x,\theta).$$

For $(x,\theta) \in \partial_+ S\mathcal{M}$, we denote by $\gamma_{x,\theta} : [0, \tau_+(x,\theta)] \to \mathcal{M}$ the maximal geodesic satisfying the initial conditions $\gamma_{x,\theta}(0) = x$ and $\dot\gamma_{x,\theta}(0) = \theta$. For each smooth covector $A \in \mathcal{C}^\infty(\mathcal{M}, T^*\mathcal{M})$, $A = a_j dx^j$ we introduce the smooth function $F_A \in \mathcal{C}^\infty(S\mathcal{M})$ given by

$$F_A(x,\theta) = \sum_{j=1}^{n} a_j(x)\theta^j = \left\langle A^\sharp(x), \theta\right\rangle, \quad (x,\theta) \in S\mathcal{M}.$$

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Geodesical X-ray transform

The ray transform of 1-forms on a simple Riemannian manifold $(\mathcal{M}, \mathrm{g})$ is the linear operator:

$$\mathcal{I} : \mathcal{C}^\infty(\mathcal{M}, T^*\mathcal{M}) \to \mathcal{C}^\infty(\partial_+ S\mathcal{M})$$

defined by

$$\mathcal{I}(A)(x,\theta) = \int_0^{\tau_+(x,\theta)} a_j(\gamma_{x,\theta}(t))\dot{\gamma}^j_{x,\theta}(t)dt = \int_0^{\tau_+(x,\theta)} F_A(\Phi_t(x,\theta))dt,$$

It is easy to see that $\mathcal{I}(d\varphi) = 0$ for any smooth function φ in $\mathcal{M}$ with $\varphi_{|\partial\mathcal{M}} = 0$. It is known that $\mathcal{I}$ is injective on the space of solenoidal 1-forms satisfying $\delta A = 0$ for simple metric g. In other words, $A \in H^1(\mathcal{M}, T^*\mathcal{M})$ and $\mathcal{I}(A) = 0$ implies $A^s = 0$, i.e., $A = d\varphi$ with some φ vanishing on $\partial\mathcal{M}$.

The adjoint of the geodesical X-ray transform

We will now, determine the adjoint $\mathcal{I}^*$ of $\mathcal{I}$. Let $L^2_\mu(\partial_+ S\mathcal{M})$ be the space of square integrable functions with respect to the measure $\mu(x,\theta)\,\mathrm{d}\sigma^{2n-2}$ with $\mu(x,\theta) = |\langle \theta, \nu(x)\rangle|$.
The ray transform $\mathcal{I}$ is a bounded operator from $L^2(\mathcal{M})$ into $L^2_\mu(\partial_+ S\mathcal{M})$.
We now recall the Santaló formula

$$\int_{S\mathcal{M}} F(x,\theta)\,\mathrm{dv}^{2n-1} = \int_{\partial_+ S\mathcal{M}} \left(\int_0^{\tau_+(x,\theta)} F\left(\Phi_t(x,\theta)\right) dt \right) \mu\,\mathrm{d}\sigma^{2n-2}$$

for any $F \in \mathcal{C}(S\mathcal{M})$.

The adjoint of the geodesical X-ray transform

For $A \in L^2(\mathcal{M}, T^*\mathcal{M})$ and $\Psi \in L^2_\mu(\partial_+ S\mathcal{M})$, we get

$$\begin{aligned}
(\mathcal{I}(A), \Psi)_\mu &= \int_{\partial_+ S\mathcal{M}} \mathcal{I}(A)(x,\theta)\overline{\Psi}(x,\theta)\mu(x,\theta)\,\mathrm{d}\sigma^{2n-2} \\
&= \int_{\partial_+ S\mathcal{M}} \left(\int_0^{\tau_+(x,\theta)} F_A\left(\Phi_t(x,\theta)\right) dt \right) \overline{\Psi}(x,\theta)\mu\,\mathrm{d}\sigma^{2n-2} \\
&= \int_{S\mathcal{M}} F_A(x,\theta)\overline{\check{\Psi}}(x,\theta)\,\mathrm{dv}^{2n-1}(x,\theta) = (A, \mathcal{I}^*(\Psi))
\end{aligned}$$

where the adjoint $\mathcal{I}^* : L^2_\mu(\partial_+ S\mathcal{M}) \to L^2(\mathcal{M}, T^*\mathcal{M})$ is given by

$$(\mathcal{I}^*\Psi(x))_j = \int_{S_x\mathcal{M}} \theta^j \check{\Psi}(x,\theta)\,\mathrm{d}\omega_x(\theta)$$

where $\check{\Psi}$ is the extension of the function Ψ from $\partial_+ S\mathcal{M}$ to $S\mathcal{M}$ constant on every orbit of the geodesic flow, i.e.

$$\check{\Psi}(x,\theta) = \Psi\big(\gamma_{x,\theta}(\tau_-(x,\theta)), \dot{\gamma}_{x,\theta}(\tau_-(x,\theta))\big) = \Psi(\Phi_{\tau_-(x,\theta)}(x,\theta)).$$

Injectivity of the X-ray transform

The geodesical ray transform is a bounded operator from $H^k(\mathcal{M}, T^*\mathcal{M})$ to $H^k(\partial_+ S\mathcal{M})$. We may extend $\mathcal{I}$ to $\partial_+ S\mathcal{M}_1$. The normal operator $N = \mathcal{I}^*\mathcal{I}$ satisfy

Theorem

If the manifold $(\mathcal{M}, \mathrm{g})$ is simple then the geodesical X-ray transform is s-injective. In fact, we have the estimate

$$C_1 \|A^s\|_{L^2(\mathcal{M})} \leqslant \|N(A)\|_{L^2(\mathcal{M}_1)} \leqslant C_2 \|A^s\|_{L^2(\mathcal{M})} \quad N = \mathcal{I}^*\mathcal{I},$$

for all $A \in L^2(\mathcal{M}, T^*\mathcal{M})$.

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for all $A \in L^2(\mathcal{M}, T^*\mathcal{M})$.

The normal operator

Now, we compute the composition $N = \mathcal{I}^*\mathcal{I}$. Let $A \in L^2(\mathcal{M}, T\mathcal{M})$, we have

$$(N(A))_j(x) = \mathcal{I}^*(\mathcal{I}(A)) = \int_{S_x\mathcal{M}} \theta^j \widetilde{\mathcal{I}(A)}(x,\theta)d\omega_x(\theta)$$

Since, for $F_A(x,\theta) = \langle A^\sharp, \theta\rangle$, we get

$$\begin{aligned}\widetilde{\mathcal{I}(A)}(x,\theta) &= \int_0^{\tau_+(\Phi_{\tau_-(x,\theta)}(x,\theta))} F_A(\Phi_t(\Phi_{\tau_-(x,\theta)}(x,\theta)))dt \\ &= \int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} F_A(\Phi_t(x,\theta))dt.\end{aligned}$$

Then

$$\begin{aligned}(N(A))_j(x) &= \int_{S_x\mathcal{M}} \theta^j \int_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} F_A(\Phi_t(x,\theta))dtd\omega_x(\theta) \\ &= 2\int_{S_x\mathcal{M}} \theta^j \int_0^{\tau_+(x,\theta)} F_A(\Phi_t(x,\theta))dtd\omega_x(\theta)\end{aligned}$$

The normal operator

We denote by $dv_x(\xi)$ the volume form on $T_x\mathcal{M}$ for a fixed $x \in \mathcal{M}$, we consider the following change integration varibales in $T_x\mathcal{M}$ as follows $\xi = t\theta$.

$$(N(A))_j(x) = 2\int_{T_x\mathcal{M}} \frac{v^j}{|v|^{n+1}} F_A(\exp_x v, J_{(x,\exp_x v)}(v))dv_x$$

Here $J_{(x,\exp_x v)} : T_x\mathcal{M} \to T_{\exp_x v}\mathcal{M}$ the parallel transport along the geodesic $t \to \exp_x tv$.
If $\mathcal{O}$ is an open set of the simple Riemannian manifold $(\mathcal{M}_1, \mathrm{g})$, the normal operator $N = \mathcal{I}^*\mathcal{I}$ is an elliptic pseudodifferential operator of order -1 on $\mathcal{O}$. Therefore for each $k \geqslant 0$ there exists a constant $C_k > 0$ such that for all $A \in H^k(\mathcal{M}, T^*\mathcal{M})$ compactly supported in $\mathcal{O}$

$$\|N(A)\|_{H^{k+1}(\mathcal{M}_1)} \leqslant C_k \|A^s\|_{H^k(\mathcal{O})}.$$

Outline

WKB expansion

We are looking for solutions of the wave equation of the form

$$u = e^{i\lambda(\psi-\lambda t)}\alpha(2\lambda t,x)\beta(2\lambda t,x) + \mathcal{O}(\lambda^{-1}).$$

We compute

$$(i\partial_t + \Delta_A)\big(e^{i\lambda(\psi-\lambda t)}\alpha(2\lambda t,x)\beta(2\lambda t,x)\big) = e^{i\lambda(\psi-\lambda t)}$$
$$\times\Big[\lambda^2\big(\underbrace{1-|\nabla\psi|^2}_{\text{eikonal}}\big)(\alpha\beta) + 2i\lambda\beta\big(\underbrace{\partial_t\alpha + \langle d\psi, d\alpha\rangle + \tfrac{\alpha}{2}\Delta\psi}_{\text{transport I}}\big)$$
$$+ 2i\lambda\alpha\big(\underbrace{\partial_t\beta + \langle d\psi, d\beta\rangle - i\,\langle A, d\psi\rangle\,\beta}_{\text{transport II}}\big) + \lambda^0\Delta_A(\alpha\beta)\Big](2\lambda t,x)$$

and this provides

- the eikonal equation: $|\nabla\psi| = 1,\ \psi(x) = d_g(z,x)\ ,\ z\in\mathcal{M}_1\backslash\mathcal{M}$
- the transport equation I: $\partial_t\alpha + \langle d\psi, d\alpha\rangle + \frac{\alpha}{2}\Delta\psi = 0$.
- the transport equation II: $\partial_t\beta + \langle d\psi, d\beta\rangle - i\,\langle A, d\psi\rangle\,\beta = 0$.

The transport equation

Transport equation I: $\partial_t \alpha + \langle d\psi, d\alpha \rangle + \frac{\alpha}{2}\Delta\psi = 0$

In polar coordinates

$$\langle d\psi, d\alpha \rangle = \partial_r \alpha,$$

$$\Delta\psi = \frac{1}{|\mathrm{g}|^{1/2}} \frac{\partial}{\partial r}\left(|\mathrm{g}|^{1/2} \frac{\partial \psi}{\partial r} \right) = \tfrac{1}{2}\partial_r(\log|\mathrm{g}|)$$

hence the transport equation reads

$$\partial_t \alpha + \partial_r \alpha + \tfrac{1}{4}\partial_r(\log|\mathrm{g}|)\,\alpha = 0$$

and can easily be solved

$$\alpha = \kappa(t-r)|\mathrm{g}|^{-1/4}\Psi(z,\theta).$$

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Transport equation II: $\partial_t \beta + \langle d\psi, d\beta \rangle - i \langle A, d\psi \rangle \beta = 0$

In polar coordinates

$$\langle d\psi, d\beta \rangle = \partial_r \beta,$$
$$\langle A, d\psi \rangle = F_A(\Phi_r(z,\theta)) := F_A(r,z,\theta)$$

hence the transport equation II reads

$$\partial_t \beta + \partial_r \beta - iF_A(r,z,\theta)\beta = 0$$

and can easily be solved

$$\beta = \exp\left(i \int_0^t F_A(r-s,z,\theta)ds \right).$$

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WKB expansion

It remains to construct the correction term, solution of the equation

$$\begin{cases} (i\partial_t + \Delta_A)v = -e^{i\lambda(\psi-\lambda t)}\Delta_A(\alpha\beta)(2\lambda t, x) \\ v|_{t=\tau} = 0 \\ v|_{[0,T]\times\partial\mathcal{M}} = 0 \end{cases}, \quad \tau = 0 \text{ or } T$$

for which we have (energy) estimates

$$\lambda\|v_\lambda(t,\cdot)\|_{L^2(\mathcal{M})} + \|\nabla v_\lambda(t,\cdot)\|_{L^2(\mathcal{M})} = \mathcal{O}(1).$$

Hence our solutions look like

$$u = e^{i\lambda(\psi-\lambda t)} \underbrace{\varphi(2\lambda t - r)|\mathrm{g}|^{-1/4}\Psi(z,\theta)}_{\alpha(2\lambda t,x)} \underbrace{\exp\left(i\int_0^{2\lambda t} F_A(r-s,z,\theta)ds\right)}_{\beta(2\lambda t,x)} + v_\lambda$$

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Outline

Integration by parts

The metric g is fixed, and we have two potentials A_1, A_2.

How does one relate the information on the boundary of the manifold provided by the DN map, and what happens in the interior of the manifold? Suppose that we have two solutions u_1, u_2

$$\begin{cases} (i\partial_t + \Delta_A)u_j = 0 \\ u_j|_{t=\tau_j} = 0 \\ u_j|_{[0,T]\times\partial\mathcal{M}} = f_j \end{cases}, \quad \tau_1 = 0,\ \tau_2 = T$$

then for $A = A_1 - A_2$ and $V_A = i\delta(A) - |A_2|^2 + |A_1|^2$, we have

$$\int_0^T\int_{\partial\mathcal{M}} \left(\Lambda_{g,A_1} - \Lambda_{g,A_2}\right) f_1 \overline{f}_2 \, d\sigma^{n-1}\, dt = -2i\int_0^T\int_{\mathcal{M}} \langle A(x), du_2\rangle\, \overline{u}_1\, dv^n\, dt - \int_0^T\int_{\mathcal{M}} V_A(x) u_2 \overline{u}_1\, dv^n\, dt$$

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Determining a magnetic potential

We use our geometrical optics solutions

$$u_1 = e^{i\lambda(\psi-\lambda t)}(\alpha_1\beta_1)(2\lambda t, x) + v_1(t,x,\lambda)$$
$$u_2 = e^{i\lambda(\psi-\lambda t)}(\alpha_2\beta_2)(2\lambda t, x) + v_2(t,x,\lambda)$$

with $\kappa \in \mathscr{C}_0^\infty(\mathbb{R})$ with support so small that

$$\kappa(t-r)|_{t\leqslant 0} = 0, \quad \kappa(t-r)|_{t\geqslant T} = 0$$

and obtain in the integration by parts formula

$$2\lambda \int_0^T\int_{\mathcal{M}} \langle A, d\psi\rangle \,(\alpha_2\overline{\alpha}_1)(2\lambda t,x)(\beta_2\overline{\beta}_1)(2\lambda t,x)\,\mathrm{dv}^n dt = \mathcal{O}(\lambda^{-1})$$

Determining a magnetic potential (conclusion)

Using polar coordinates, we get

$$\int_{S_z\mathcal{M}} \left[\exp\left(i\int_0^{\tau_+(z,\theta)} F_A(s,z,\theta)ds\right) - 1\right]\Psi(z,\theta)\mu(z,\theta)d\omega_z(\theta) = \mathcal{O}(\lambda^{-1})$$

Letting λ tend to ∞, A^s small and varying $\Psi(z,\theta)$ we get

$$\mathcal{I}(A)(z,\theta) = \int_0^{\tau_+(z,\theta)} F_A\big(r,z,\theta\big)\,\mathrm{d}r = 0$$

and we conclude with the s-injectivity of the geodesical X-ray transform:

$$A^s = 0 \quad \text{hence } A_1^s = A_2^s.$$

Main Theorems (stability from the DN map)

Let us now introduce the admissible set of magnetic potentials A. Let $M_0 > 0$ and $k \geqslant 1$ be given, set

$$\mathscr{A}(m_0, k) = \left\{ A \in \mathcal{C}^\infty(\mathcal{M}, T^*\mathcal{M}),\ \|A\|_{\mathcal{C}^k(\mathcal{M})} \leqslant m_0 \right\}.$$

Theorem

Let $(\mathcal{M}, \mathrm{g})$ be a simple compact Riemannian manifold with boundary of dimension $n \geqslant 2$ and let $T > 0$. There exist $k \geqslant 1$, $\varepsilon > 0$, $C > 0$ and $\sigma \in (0,1)$ such that for any $A_1, A_2 \in \mathscr{A}(m_0, k)$ with $\left\|A_1^s - A_2^s\right\|_{\mathcal{C}(\mathcal{M})} \leqslant \varepsilon$, the following estimate holds true

$$\|A_1^s - A_2^s\|_{L^2(\mathcal{M})} \leqslant C \left\|\Lambda_{\mathrm{g},A_1} - \Lambda_{\mathrm{g},A_2}\right\|^\sigma.$$

where C depends on $\mathcal{M}$, m_0, n, and ε.

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Electric potential

We consider the boundary value problem on $[0, T] \times \mathcal{M}$

$$(\mathrm{S}) \begin{cases} (i\partial_t + \Delta_A + q)u = 0, \\ u|_{t=0} = 0, \\ u|_{[0,T]\times\partial\mathcal{M}} = f \end{cases}$$

Theorem

Let $(\mathcal{M}, \mathrm{g})$ be a simple compact Riemannian manifold with boundary of dimension $n \geqslant 2$ and let $T > 0$. There exist $k \geqslant 1$, $\varepsilon > 0$, $C > 0$ and $\sigma \in (0, 1)$ such that for any $A_1, A_2 \in \mathscr{A}(m_0, k)$ with $\left\|A_1^s - A_2^s\right\|_{\mathcal{C}(\mathcal{M})} \leqslant \varepsilon$, the following estimate holds true

$$\|q_1 - q_2\|_{L^2(\mathcal{M})} + \|A_1^s - A_2^s\|_{L^2(\mathcal{M})} \leqslant C \left\|\Lambda_{\mathrm{g},A_1,q_1} - \Lambda_{\mathrm{g},A_2,q_2}\right\|^{\sigma}.$$

where C depends on $\mathcal{M}$, m_0, n, and ε.

Outline

1-dimensional Borg-Levinson Theorem

Let us consider the simplest case of the Dirichlet boundary value problem on $(0, 1)$

$$\begin{aligned} &-u'' + q(x)u = \lambda u, \quad 0 < x < 1, \\ &u(0) = u(1) = 0. \end{aligned}$$

If $q(x)$ is real-valued, this problem has a set of eigenvalues

$$\lambda_1(q) < \lambda_2(q) < \cdots < \lambda_n(q) < \cdots$$

The first question of the inverse eigenvalue problem is:
Question: If $\lambda_n(q_1) = \lambda_n(q_2)$ for all $n \geqslant 1$, does $q_1 = q_2$?
The answer is easily seen to be negative. You have only to take $q_2(x) = q_1(1-x) \neq q_1(x)$ as a counter example.

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A potential $q(x)$ is said to be symmetric if $q(x) = q(1-x)$.

Theorem (Borg-Levinson 1949)

Suppose q_1 and q_2 are symmetric and $\lambda_n(q_1) = \lambda_n(q_2)$ for all $n \geqslant 1$. Then $q_1 = q_2$.

If q is not symmetric, one needs some auxiliary condition to prove the uniqueness: Let us consider the case of the following BVP on $(0,1)$

$$-u'' + q(x)u = \lambda u, \quad 0 < x < 1,$$
$$u(0) = 0, \quad u'(0) = 1.$$

We put $\varphi_n(\cdot, q)$ the solution with $\lambda = \lambda_n$

Theorem

Suppose $\lambda_n(q_1) = \lambda_n(q_2)$ and $\varphi_n'(1, q_1) = \varphi_n'(1, q_2)$ for all $n \geqslant 1$. Then $q_1 = q_2$.

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n-dimensional Borg-Levinson theorem

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with smooth boundary Γ. For real $q \in L^\infty(\Omega)$ and $q \geqslant 0$, we denote by A_q the unbounded operator $A_q = -\Delta + q$, $\mathscr{D}(A_q) = H_0^1(\Omega) \cap H^2(\Omega)$.
The spectrum of A_q consists of a sequence of eigenvalues, counted according to their multiplicities:

$$0 \leqslant \lambda_1(q) \leqslant \lambda_2(q) \leqslant \ldots \leqslant \lambda_k(q) \leqslant \ldots$$

The corresponding eigenfunctions are denoted by $(\varphi_k(\cdot, q))$. Since $\varphi_k(\cdot, q)$ is the solution of the following BVP

$$\begin{cases} (-\Delta + q)\varphi = \lambda_{k,q}\varphi & \text{in} \quad \Omega \\ \varphi = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

n-dimensional Borg-Levinson theorem

Theorem (Nachman, Sylvester and Uhlmann; Isozaki)

Let $q_1, q_2 \in \mathscr{C}^\infty(\Omega;\mathbb{R})$, suppose that, for each k

$$\lambda_k(q_1) = \lambda_k(q_2) \quad \textit{and} \quad \frac{\partial\varphi_k}{\partial\nu}(x;q_1) = \frac{\partial\varphi_k}{\partial\nu}(x;q_2), \ \forall x \in \Gamma$$

then

$$q_1(x) = q_2(x) \quad \textit{for all} \quad x \in \Omega.$$

There is a one-to-one correspondence between the potential and the pair of all eigenvalues and the normal derivatives of eigenfunctions.

Gel'fand inverse problem

Taking $z \neq \lambda_j(q)$. Let v_z solves the boundary value problem

$$\begin{cases} (-\Delta + q)v = zv & \text{in} \quad \Omega \\ v = f, & \text{on} \quad \partial\Omega, \end{cases}$$

the (fixed-frequency) Dirichlet-to-Neumann map Π_z^q is defined by

$$\Pi_z^q(f) = \partial_\nu v_z$$

IP: Given Gel'fand spectral data $\{\Pi_z^q,\, z \in \varrho(-\Delta + q)\}$, determine q.

Equivalence of inverse problems

(a)- Given boundary spectral data $\mathrm{BSD} = \{\lambda_k(q), \partial_\nu \varphi_k\}$, determine q.

(b)- Given Gel'fand spectral data $\mathrm{GSD} = \left\{\Pi_z^q, z \in \mathbb{C}\setminus\{\lambda_k\}\right\}$, determine q.

(c)- Given the D-to-N map Λ_q, determine q.

Theorem (Katchalov, Kurylev, Lassas and Mandache)

Inverse problems (a)-(b)-(c) are equivalent, i.e. any of the data (a)-(b)-(c) determine all other data.

The BSD determines the derivative of the D-to-N map,

$$\frac{d}{dz}(\Pi_z^q(f)) = -\sum_{k=0}^{\infty} \frac{1}{(z-\lambda_k(q))^2} \langle f, \partial_\nu \varphi_k \rangle \, \partial_\nu \varphi_k$$

$$\Lambda_q(f) = \frac{1}{2\pi}\int_{i\mu+\mathbb{R}} e^{-its}\Pi_{-s}^q \mathscr{F}(f)(x,s)ds.$$

$$\left[\Pi_z^{q_1} = \Pi_z^{q_2}, \forall z\right] \Rightarrow \left[q_1 = q_2\right], \qquad \left[R^{q_1} = R^{q_2}\right] \Rightarrow \left[q_1 = q_2\right].$$

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Equivalence of inverse problems

(a)- Given boundary spectral data $\mathrm{BSD} = \{\lambda_k(q), \partial_\nu \varphi_k\}$, determine q.
(b)- Given Gel'fand spectral data $\mathrm{GSD} = \left\{\Pi_z^q, z \in \mathbb{C} \backslash \{\lambda_k\}\right\}$, determine q.
(c)- Given the D-to-N map Λ_q, determine q.

Theorem (Katchalov, Kurylev, Lassas and Mandache)

Inverse problems (a)-(b)-(c) are equivalent, i.e. any of the data (a)-(b)-(c) determine all other data.

The BSD determines the derivative of the D-to-N map,

$$\frac{d}{dz}(\Pi_z^q(f)) = -\sum_{k=0}^{\infty} \frac{1}{(z-\lambda_k(q))^2} \langle f, \partial_\nu \varphi_k \rangle \, \partial_\nu \varphi_k$$

$$\Lambda_q(f) = \frac{1}{2\pi} \int_{i\mu+\mathbb{R}} e^{-its} \Pi_{-s}^q \mathscr{F}(f)(x,s) ds.$$

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Main Theorems (stability of SIP from the DN map)

$$\mathcal{Q}(M_0) = \left\{ q \in H^\alpha(\mathcal{M}),\ \alpha > \frac{n}{2} + 1,\ \|q\|_{H^\alpha(\mathcal{M})} \leqslant M_0 \right\}.$$

Theorem

Let $(\mathcal{M}, \mathrm{g})$ be a simple Riemannian compact manifold with boundary of dimension $n \geqslant 2$. There exist $C > 0$ and $s \in (0,1)$ such that the following estimate holds

$$\|q_1 - q_2\|_{L^2(\mathcal{M})} \leqslant C\,\epsilon^s$$

for any non-negative $q_1, q_2 \in \mathcal{Q}(M_0)$ which are equal on the boundary $\partial\mathcal{M}$, where

$$\epsilon = |\lambda_{A,q_1} - \lambda_{A,q_2}|_{\ell^1_\omega(\mathbf{C})} + \|\partial_\nu \varphi_{A,q_1} - \partial_\nu \varphi_{A,q_2}\|_{\ell^1_\omega(H^{1/2}(\partial\mathcal{M}))}$$

is assumed to be small and $\partial_\nu \varphi_{A,q_j} = \left(\partial_\nu \varphi_{k,A,q_j}\right)_k$, $j = 1, 2$.

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Thank you