Contrôlabilité exacte globale simultanée d'un nombre arbitraire d'équations de Schrödinger bilinéaires

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Travail en collaboration avec Vahagn Nersesyan (UVSQ).

Model : 1D bilinear Schrödinger equation

$$\begin{cases} i\partial_t \psi = \left(-\partial_{xx}^2 + V(x)\right)\psi & x \in (0,1), \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,x) = \psi_0(x), \end{cases}$$

where

- $\psi : (0,1) \to \mathbb{C}$ wave function of the particle,
- $V : (0,1) \to \mathbb{R}$ potential,

Model : 1D bilinear Schrödinger equation

$$\begin{cases} i\partial_t \psi = \left(-\partial^2_{xx} + V(x)\right)\psi - u(t)\mu(x)\psi, & x \in (0,1), \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,x) = \psi_0(x), \end{cases}$$

where

- $\psi : (0,1) \to \mathbb{C}$ wave function of the particle,
- $V : (0,1) \to \mathbb{R}$ potential,
- $u : (0,T) \to \mathbb{R}$ amplitude of the external electric field,
- $\mu : (0,1) \to \mathbb{R}$ dipole moment.

Control problem : State $= \psi$, Control $= u$ i.e.

$T > 0$, ψ_0, ψ_f given. Is there u such that $\psi(T) = \psi_f$?

Simultaneous controllability setting

N identical independent particles

$$\begin{cases} i\partial_t \psi^j = \left(-\partial^2_{xx} + V(x)\right)\psi^j - u(t)\mu(x)\psi^j, & x \in (0,1), \\ \psi^j(t,0) = \psi^j(t,1) = 0, & j \in \{1,\dots,N\}, \\ \psi^j(0,x) = \psi^j_0(x). \end{cases} \qquad (\mathbf{S}_N)$$

State : $\left(\psi^1,\dots,\psi^N\right)$, **control :** $u : (0,T) \to \mathbb{R}$,

Goal : Simultaneous control of $(\psi^1,\dots,\psi^N)$ with a single control u.

Invariants :

$$\langle \psi^j(t), \psi^k(t) \rangle = \langle \psi^j_0, \psi^k_0 \rangle, \quad \forall t \in [0,T].$$

Outline

1. Introduction
   - Previous results
   - Main result
2. Approximate controllability towards finite sums of eigenvectors
3. Local exact controllability around finite sums of eigenvectors.
4. Global exact controllability

Infinite dimensional bilinear quantum system

- **Non exact controllability in H^2**
  - J.M. Ball, J.E. Marsden, M. Slemrod (1982) ; G. Turinici (2000)
- **Approximate controllability**
  - U. Boscain, M. Caponigro, T. Chambrion, P. Mason, M. Sigalotti (2009, 2012)
  - N. Boussaïd, M. Caponigro, T. Chambrion (2013)
- **Stabilization**
  - M. Mirrahimi (2009) ; K. Beauchard, M.Mirrahimi (2009)
  - V. Nersesyan (2009) ; K. Beauchard, V. Nersesyan (2010)
- **Local exact controllability**
  - K. Beauchard (2005) ; K. Beauchard, C. Laurent (2010), K. Beauchard, M. (2014)
  - K. Beauchard, J.-M. Coron (2006) ; K. Beauchard (2008)
- **Global exact controllability**
  - V. Nersesyan (2010)
  - V. Nersesyan, H. Nersisyan (2013)

Notations

- $\mathcal{S}$: $L^2((0,1),\mathbb{C})$ unit sphere.
- $\lambda_{k,V} \in \mathbb{R}$ and $\varphi_{k,V} \in \mathcal{S}$ eigenvalues and eigenvectors of

$$A_V\psi := \left(-\partial_{xx}^2 + V\right)\psi, \quad D(A_V) := H^2 \cap H_0^1((0,1),\mathbb{C})$$

- Eigenstates $\Phi_{k,V}(t,x) := e^{-i\lambda_{k,V}t}\varphi_{k,V}(x)$.
  $\left(\Phi_{1,V},\dots,\Phi_{N,V}\right)$ solution of $(\mathbf{S}_N)$ with $u \equiv 0$.
- Functional framework

$$H^s_{(V)} := D\left(A_V^{s/2}\right), \quad ||\cdot||^2_{H^s_{(V)}} := \sum_{k=1}^{\infty} |k^s\langle\cdot,\varphi_{k,V}\rangle|^2, \quad \forall s > 0.$$

$$H^3_{(V)} = \left\{\psi \in H^3 \cap H_0^1((0,1),\mathbb{C})\,;\, \psi''(0) = \psi''(1) = 0\right\}.$$

Well posedness

$$\begin{cases} i\partial_t \psi = \big(-\partial_{xx}^2 + V(x)\big)\psi - u(t)\mu(x)\psi, & (t,x) \in (0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,x) = \psi_0(x). \end{cases}$$

- K. Beauchard, C. Laurent (2010).
  $V, \mu \in H^3((0,1),\mathbb{R})$. $T > 0$. $u \in L^2((0,T),\mathbb{R})$, $\psi_0 \in H^3_{(V)}$:

  unique weak solution $\psi(\cdot,\psi_0,u) \in C^0([0,T], H^3_{(V)})$,

  and $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$.

Local exact controllability for $N = 1$

- K. Beauchard, C. Laurent (2010), local exact controllability

$V = 0$. $\mu \in H^3(0,1)$ satisfies $\exists c > 0$ such that

$$|\langle \mu\varphi_1, \varphi_k\rangle| \geq \frac{c}{k^3}, \quad \forall k \in \mathbb{N}^*.$$

$\forall T > 0$, $\exists \delta > 0$ such that

$$\forall \psi_f \in \mathcal{S} \cap H^3_{(0)} \quad \text{with} \quad ||\psi_f - \Phi_1(T)||_{H^3_{(0)}} < \delta,$$

there exists $u \in L^2((0,T), \mathbb{R})$ such that

$$\begin{cases} i\partial_t \psi = -\partial^2_{xx}\psi - u(t)\mu(x)\psi, \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,\cdot) = \varphi_1, \end{cases} \qquad \Longrightarrow \qquad \psi(T) = \psi_f.$$

- C^1 regularity of the map $\psi_f \mapsto u$.
- Optimal functional space.

Strategy of proof

- Local controllability : inverse mapping theorem to

$$u \in L^2((0,T),\mathbb{R}) \mapsto \psi(T) \in H^3_{(0)}$$

- Linear test

$$\begin{cases} i\partial_t \Psi = -\partial^2_{xx}\Psi - v(t)\mu(x)\Phi_1, & x \in (0,1), \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0,x) = 0. \end{cases}$$

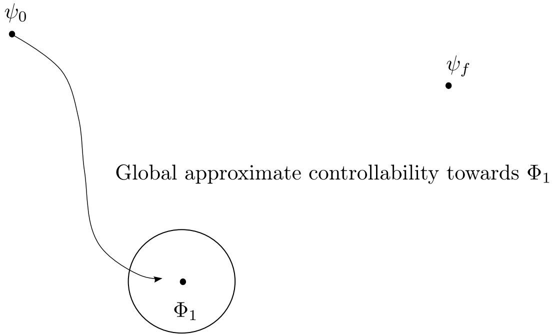
$$\Psi(T) = i \sum_{k=1}^{\infty} \left(\langle \mu\varphi_1, \varphi_k \rangle \int_0^T v(t) e^{i(\lambda_k - \lambda_1)t} \mathrm{d}t \right) \Phi_k(T).$$

  - Finding v such that $\Psi(T) = \Psi_f$ is equivalent to solve

$$\int_0^T v(t) e^{i(\lambda_k - \lambda_1)t} \mathrm{d}t = \frac{\langle \Psi_f, \varphi_k \rangle e^{i\lambda_k T}}{i\langle \mu\varphi_1, \varphi_k \rangle}, \ \forall k \in \mathbb{N}^*.$$

Global exact controllability for $N = 1$

- V. Nersesyan (2010), global exact controllability in $\mathcal{S} \cap H^{3+\epsilon}_{(0)}$ for generic μ.



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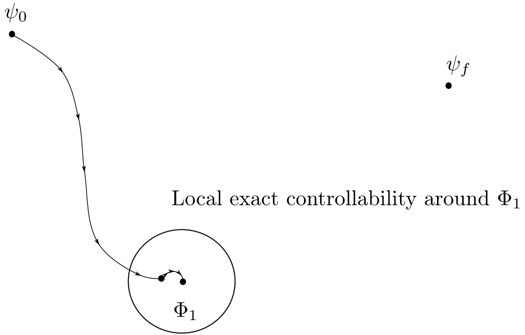

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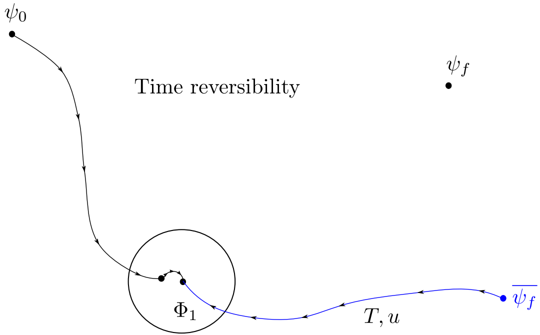
  Time reversibility :

$$\overline{\psi_f} \overset{u,T}{\rightsquigarrow} \overline{\psi_i} \Longrightarrow \psi_i \overset{u(T-\cdot),T}{\rightsquigarrow} \psi_f.$$



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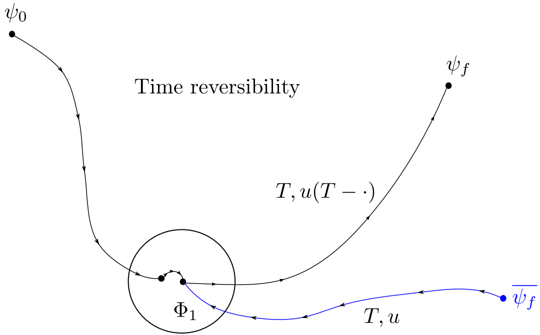


A first step towards simultaneous controllability

M. (2013) *Ann. Inst. H. Poincaré Anal. Non Linéaire*.

- $V = 0$. No local exact controllability in small time for $N \geq 2$.
- A first step towards simultaneous controllability : $N = 2, 3$.

$V = 0$. $\mu \in H^3(0,1)$ satisfies $\exists c > 0$ such that

$$|\langle \mu\varphi_j, \varphi_k \rangle| \geq \frac{c}{k^3}, \quad \forall j \in \{1, \dots, N\}, \forall k \in \mathbb{N}^*.$$

$$\left(\psi_0^1, \dots, \psi_0^N\right) = \left(\varphi_1, \dots, \varphi_N\right).$$

- $N = 2$: local controllability in arbitrary time up to a global phase i.e. around $e^{i\theta}(\Phi_1(T), \Phi_2(T))$
- $N = 2$: local exact controllability up to a global delay i.e. targets close to $(\Phi_1(T), \Phi_2(T))$ reached at time $T + T^*$
- $N = 3$: local controllability up to a global phase and a global delay.

Global exact simultaneous controllability

- Bold notations : $\boldsymbol{\psi} := (\psi^1, \dots, \psi^N)$, $\boldsymbol{H} := H^N$.
- Unitary equivalent vectors $\boldsymbol{\psi}_0$, $\boldsymbol{\psi}_f$: there exists $\mathcal{U} : L^2 \to L^2$ unitary map such that $\boldsymbol{\psi}_f = \mathcal{U}\boldsymbol{\psi}_0$ i.e.

$$\psi^j_f = \mathcal{U}\psi^j_0, \quad \forall j \in \{1, \dots, N\}.$$

Main Theorem

Let $N \in \mathbb{N}^*$. For every $V \in H^4((0,1),\mathbb{R})$, system $(\mathbf{S_N})$ is globally exactly controllable in $\boldsymbol{H}^4_{(V)}$, generically with respect to $\mu \in H^4((0,1),\mathbb{R})$. More precisely, there exists a set $\mathcal{Q}_V$ residual in $H^4((0,1),\mathbb{R})$ such that for every $\mu \in \mathcal{Q}_V$

$$\forall\, \boldsymbol{\psi}_0, \boldsymbol{\psi}_f \in \boldsymbol{H}^4_{(V)} \text{ unitarily equivalent, } \exists T > 0,\ \exists u \in L^2((0,T),\mathbb{R});$$
$$\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}_f.$$

- New result, even for $N = 1$.
- False if $\mu'(0) = \mu'(1) = 0$, J.M. Ball, J.E. Marsden and M. Slemrod (1982).

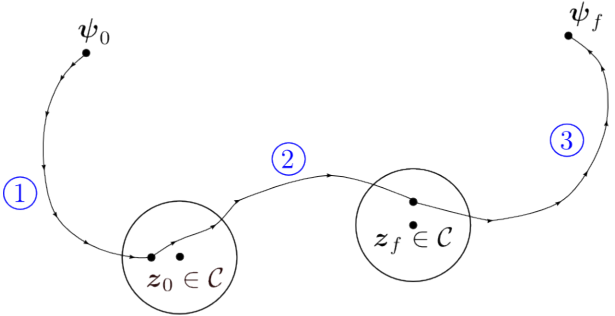
Strategy of proof

- Arbitrary potential V : consider the control $u(t) := \tilde{u}(t) - 1$.

$$
\begin{aligned}
i\partial_t \tilde{\psi}^j &= \left(-\partial^2_{xx} + V(x)\right)\tilde{\psi}^j - (\tilde{u}(t)-1)\mu(x)\tilde{\psi}^j, \\
&= \left(-\partial^2_{xx} + V(x) + \mu(x)\right)\tilde{\psi}^j - \tilde{u}(t)\mu(x)\tilde{\psi}^j, \quad x \in (0,1),
\end{aligned}
$$

'New potential' : $V + \mu$

- 'Pivot space' $\mathcal{C}$ instead of 'pivot vector'



Approximate controllability towards finite sums of eigenvectors

$N \in \mathbb{N}^*$. $V, \mu \in H^4((0,1),\mathbb{R})$ such that

(C₁) $\langle \mu \varphi_{j,V}, \varphi_{k,V} \rangle \neq 0$ for all $j \in \{1, \dots, N\}$, $k \in \mathbb{N}^*$.

(C₂) $\lambda_{j,V} - \lambda_{k,V} \neq \lambda_{p,V} - \lambda_{q,V}$ for all $j \in \{1, \dots, N\}$, $k, p, q \in \mathbb{N}^*$ such that $\{j,k\} \neq \{p,q\}$ and $k \neq j$.

Theorem

Let $\mathcal{C}_M := \text{Span}\{\varphi_{1,V}, \dots, \varphi_{M,V}\}$. Under Conditions **(C₁)** and **(C₂)**, for any $\boldsymbol{\psi}_0 \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ with $\langle \psi_0^j, \varphi_{j,V} \rangle \neq 0$, for all $j \in \{1, \dots, N\}$, there are $M \in \mathbb{N}^*$, $\boldsymbol{\psi}_f \in \boldsymbol{\mathcal{C}}_M$, sequences $T_n > 0$ and $u_n \in C_0^\infty((0,T_n),\mathbb{R})$ such that

$$\boldsymbol{\psi}(T_n, \boldsymbol{\psi}_0, u_n) \underset{n \to \infty}{\longrightarrow} \boldsymbol{\psi}_f \quad \text{in } \boldsymbol{H}^3.$$

$N = M = 1$: V. Nersesyan (2010).

Sketch of proof I

- Lyapunov type strategy.

$$\mathcal{L}(\boldsymbol{z}) := \alpha \sum_{j=1}^{N} \| \left(-\partial_{xx}^2 + V\right)^2 \mathcal{P}_N z^j \|_{L^2}^2 + 1 - \prod_{j=1}^{N} |\langle z^j, \varphi_{j,V} \rangle|^2,$$

with $\mathcal{P}_N$ orthogonal projection in L^2 onto $\overline{\mathrm{Span}\{\varphi_{k,V}\,;\, k \geq N+1\}}$.

- Decrease : $\boldsymbol{z} \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$ with $\langle z^j, \varphi_{j,V} \rangle \neq 0$, for all $j \in \{1, \dots, N\}$. Either

$$\boldsymbol{z} \in \bigcup_{M \in \mathbb{N}^*} \boldsymbol{\mathcal{C}}_M,$$

or $\exists T > 0$, $\exists u \in C_0^\infty((0,T), \mathbb{R})$ such that

$$\mathcal{L}\big(\psi(T, \boldsymbol{z}, u)\big) < \mathcal{L}(\boldsymbol{z}).$$

Sketch of proof II

idea : existence of T and $w \in C_0^\infty((0,T),\mathbb{R})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{L}\big(\psi(T,\boldsymbol{z},\sigma w)\big)\Big|_{\sigma=0} \neq 0.$$

- We define

$$\mathcal{K} := \left\{\boldsymbol{\psi} \in \boldsymbol{H}^4_{(V)}\,;\, \boldsymbol{\psi}(T_n,\boldsymbol{\psi}_0,u_n) \underset{n\to\infty}{\longrightarrow} \boldsymbol{\psi} \text{ in } \boldsymbol{H}^3,\ \text{for } T_n \geq 0,\ u_n \in C_0^\infty((0,T_n),\mathbb{R})\right\}.$$

- $\boldsymbol{e} \in \mathcal{K}$ such that $\mathcal{L}(\boldsymbol{e}) = \inf_{\boldsymbol{\psi}\in\mathcal{K}} \mathcal{L}(\boldsymbol{\psi})$. Then

$$\boldsymbol{e} \in \bigcup_{M\in\mathbb{N}^*} \boldsymbol{\mathcal{C}}_M.$$

Local exact controllability around finite sums of eigenvectors

$N \in \mathbb{N}^*$. $V, \mu \in H^3((0,1), \mathbb{R})$ such that

$(\mathrm{C_3})$ there exists $c > 0$ such that

$$|\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle| \geq \frac{c}{k^3}, \quad \forall j \in \{1, \dots, N\}, \forall k \in \mathbb{N}^*,$$

$(\mathrm{C_4})$ $\lambda_{k,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{n,V}$ for all $j, n \in \{1, \dots, N\}$, $k \geq j+1$, $p \geq n+1$ with $\{j,k\} \neq \{p,n\}$,

$(\mathrm{C_5})$ $1, \lambda_{1,V}, \dots, \lambda_{N,V}$ are rationally independent.

Theorem

Let $C_0, C_f \in U_N$ and $\boldsymbol{z}_0 := C_0\boldsymbol{\varphi}_V$, $\boldsymbol{z}_f := C_f\boldsymbol{\varphi}_V$. Under Conditions **$(\mathrm{C_3})$-$(\mathrm{C_5})$**, there exists $T > 0$, $\delta > 0$ such that, if

$$\mathcal{O}_{\delta,C} := \left\{\phi \in \boldsymbol{H}^3_{(V)}\,;\, \langle \phi^j, \phi^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^N \|\phi^j - (C\boldsymbol{\varphi}_V)^j\|_{H^3_{(V)}} < \delta\right\},$$

for every $\psi_0 \in \mathcal{O}_{\delta,C_0}$, $\psi_f \in \mathcal{O}_{\delta,C_f}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that the associated solution satisfies $\psi(T) = \psi_f$.

Local controllability around eigenvectors up to global phases

Proposition

$N \in \mathbb{N}^*$. $V, \mu \in H^3((0,1),\mathbb{R})$ such that Conditions $(\mathbf{C_3})$-$(\mathbf{C_4})$ are satisfied. Let $T > 0$, there are $\theta_1, \dots, \theta_N \in \mathbb{R}$, $\delta > 0$;

$$\forall \boldsymbol{\psi}_0 \in \boldsymbol{H}^3_{(V)};\ \langle \psi_0^j, \psi_0^k \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{N} \|\psi_0^j - \varphi_{j,V}\|_{H^3_{(\mathbf{V})}} < \delta,$$

$$\forall \boldsymbol{\psi}_f \in \boldsymbol{H}^3_{(V)};\ \langle \psi_f^j, \psi_f^k \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{N} \|\psi_f^j - e^{i\theta_j}\varphi_{j,V}\|_{H^3_{(\mathbf{V})}} < \delta,$$

there exists $u \in L^2((0,T),\mathbb{R})$ such that $\boldsymbol{\psi}(T, \boldsymbol{\psi}_0, u) = \boldsymbol{\psi}_f$.

Similar to M. (2013) for $N = 2, 3$. No condition on the phase terms θ_j.

- If $\theta_j = -\lambda_j T$, local exact controllability around

$$\Big(\Phi_{1,V}, \dots, \Phi_{N,V}, u \equiv 0\Big).$$

Natural strategy : linear test

- Linearized system around $\left(\Phi_{1,V}, \dots, \Phi_{N,V}, u \equiv 0\right)$

$$\begin{cases} i\partial_t \Psi^j = \left(-\partial_{xx}^2 + V(x)\right)\Psi^j - v(t)\mu(x)\Phi_{j,V}, \quad j \in \{1, \dots, N\} \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, \\ \Psi^j(0,x) = 0. \end{cases}$$

$$\Psi^j(T) = i\sum_{k=1}^{\infty}\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \int_0^T v(t)e^{i(\lambda_{k,V}-\lambda_{j,V})t}\mathrm{d}t\, \Phi_{k,V}(T).$$

Failure in the natural strategy

$$\Psi^j(T) = i \sum_{k=1}^{\infty} \langle \mu \varphi_{j,V}, \varphi_{k,V} \rangle \int_0^T v(t) e^{i(\lambda_{k,V} - \lambda_{j,V})t} \mathrm{d}t \, \Phi_{k,V}(T).$$

Controllability of $\langle \Psi^j(T), \Phi_{k,V}(T) \rangle$ for $j \in \{1, \dots, N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^j, \Phi_{k,V} \rangle$	1	2	$\dots$	N	$\dots$
Ψ^1					$\dots$
Ψ^2					$\dots$
$\vdots$	$\vdots$		$\ddots$		$\vdots$
Ψ^N					$\dots$

Failure in the natural strategy

$$\Psi^j(T) = i \sum_{k=1}^{\infty} \langle \mu \varphi_{j,V}, \varphi_{k,V} \rangle \int_0^T v(t) e^{i(\lambda_{k,V} - \lambda_{j,V})t} \mathrm{d}t \, \Phi_{k,V}(T).$$

Controllability of $\langle \Psi^j(T), \Phi_{k,V}(T) \rangle$ for $j \in \{1, \dots, N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^j, \Phi_{k,V} \rangle$	1	2	$\dots$	N	$\dots$
Ψ^1					
Ψ^2					
$\vdots$	$\vdots$		$\ddots$		$\ddots$
Ψ^N			$\dots$		

- Moment problem. Gap condition + null upper density (Conditions **(C$_3$)**-**(C$_4$)**) $\rightsquigarrow$ Solution of moment problem for non redundant frequencies

$$\left\{ \lambda_{k,V} - \lambda_{j,V} \,;\, j \in \{1, \dots, N\}, k \geq j+1 \right\}.$$

Failure in the natural strategy

$$\Psi^j(T) = i \sum_{k=1}^{\infty} \langle \mu \varphi_{j,V}, \varphi_{k,V} \rangle \int_0^T v(t) e^{i(\lambda_{k,V} - \lambda_{j,V})t} \mathrm{d}t \, \Phi_{k,V}(T).$$

Controllability of $\langle \Psi^j(T), \Phi_{k,V}(T) \rangle$ for $j \in \{1, \dots, N\}$ and $k \in \mathbb{N}^*$.

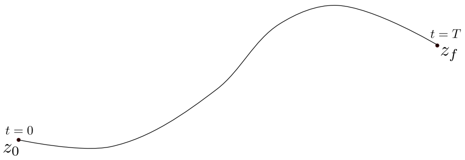
$\langle \Psi^j, \Phi_{k,V} \rangle$	1	2	$\dots$	N	$\dots$
Ψ^1					
Ψ^2					
$\vdots$	$\vdots$		$\ddots$		$\ddots$
Ψ^N			$\dots$		

- Moment problem.
- Lost directions.

$$\frac{\langle \Psi^j(T), \Phi_{j,V}(T) \rangle}{\langle \mu \varphi_{j,V}, \varphi_{j,V} \rangle} = \frac{\langle \Psi^k(T), \Phi_{k,V}(T) \rangle}{\langle \mu \varphi_{k,V}, \varphi_{k,V} \rangle} = i \int_0^T v(t) \mathrm{d}t, \quad \forall j, k \in \{1, \dots, N\}.$$

The return method

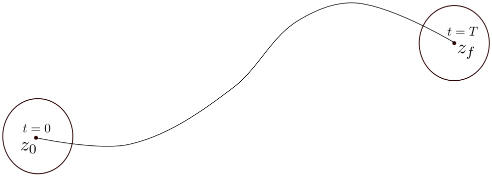
Introduced by J.-M. Coron (1992).



The return method

Introduced by J.-M. Coron (1992).

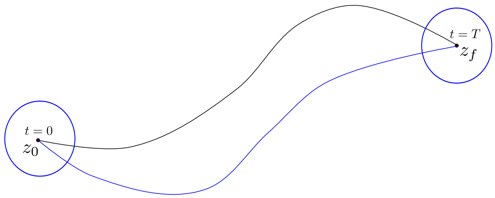
- Local exact controllability : non controllable linearized system.



The return method

Introduced by J.-M. Coron (1992).

- Local exact controllability : non controllable linearized system.
- Design of a 'reference' trajectory such the linearized system around this trajectory is controllable.



Difficulty : design of such a reference trajectory.

Very useful : J.-M. Coron, T. Horsin, O. Glass, K. Beauchard, S. Guerrero, M. Chapouly, L. Rosier, P. Lissy... for Euler, Navier-Stokes, Burgers, Schrödinger equations...

Reaching $\left(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V}\right)$

Design of the reference trajectory through partial controllability results. Local controllability of $\Big(\mathcal{P}_1(\psi^1(T)), \dots, \mathcal{P}_N(\psi^N(T))\Big)$

$$\mathcal{P}_j(\psi) = \sum_{k=j+1}^{+\infty} \langle \psi, \varphi_{k,V}\rangle \varphi_{k,V}.$$

- Linearized system :

$$\Psi^j(T) = \sum_{k=1}^{\infty} \left(\langle \Psi_0^j, \varphi_{k,V}\rangle + i\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \int_0^T v(t) e^{i(\lambda_{k,V}-\lambda_{j,V})t} \mathrm{d}t \right) \Phi_{k,V}(T).$$

$\langle \Psi^j, \Phi_{k,V}\rangle$	1	2	$\dots$	N	$\dots$
Ψ^1		controllable quantities	controllable quantities	controllable quantities	controllable quantities
Ψ^2			controllable quantities	controllable quantities	controllable quantities
$\vdots$	$\vdots$		$\ddots$	controllable quantities	$\ddots$
Ψ^N			$\dots$		controllable quantities

Reaching $\left(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V}\right)$

Design of the reference trajectory through partial controllability results. Local controllability of $\Big(\mathcal{P}_1\big(\psi^1(T)\big), \dots, \mathcal{P}_N\big(\psi^N(T)\big)\Big)$

$$\mathcal{P}_j(\psi) = \sum_{k=j+1}^{+\infty} \langle \psi, \varphi_{k,V}\rangle \varphi_{k,V}.$$

- Bilinear system : $\left(\psi_0^1, \dots, \psi_0^N\right) \approx \left(\varphi_{1,V}, \dots, \varphi_{N,V}\right)$ with $\langle \psi_0^j, \psi_0^k\rangle = \delta_{j=k}$,

$\langle \psi^j, \Phi_{k,V}\rangle$	1	2	$\dots$	N	$\dots$
ψ^1					
ψ^2			$\mathcal{P}_1\big(\psi^1(T)\big) = \dots = \mathcal{P}_N\big(\psi^N(T)\big) = 0$		
$\vdots$	$\vdots$		$\ddots$		$\ddots$
ψ^N			$\dots$		

i.e.

$$\left(\psi^1(T), \dots, \psi^N(T)\right) = \left(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V}\right).$$

The reference trajectory

Let $T > 0$ and $0 < \varepsilon_1 < \cdots < \varepsilon_{N-1} < T$.

Under Conditions **(C$_3$)** and **(C$_4$)**, there exist $\overline{\eta} > 0$, $C > 0$ such that $\forall \eta \in (0, \overline{\eta})$, $\exists \theta_1^\eta, \ldots, \theta_N^\eta \in \mathbb{R}$, $\exists u_{ref}^\eta \in L^2((0,T),\mathbb{R})$ with

$$\|u_{ref}^\eta\|_{L^2} \leq C\eta,$$

such that $\forall j \in \{1, \ldots, N\}$, $\forall k \in \{1, \ldots, N-1\}$,

$$\langle \mu \psi_{ref}^{j,\eta}(\varepsilon_k), \psi_{ref}^{j,\eta}(\varepsilon_k)\rangle = \langle \mu \varphi_{j,V}, \varphi_{j,V}\rangle + \eta \delta_{j=k},$$

and

$$\boldsymbol{\psi}_{ref}^\eta(T) = \left(e^{i\theta_1^\eta}\varphi_{1,V}, \ldots, e^{i\theta_N^\eta}\varphi_{N,V}\right).$$

Main ideas : Small perturbations + partial control result

skip proof

Proof of the construction of the reference trajectory

- $[0, \varepsilon_{N-1}]$: Small perturbation (partial control result) such that

$$\langle \mu \psi^{j,\eta}_{ref}(\varepsilon_k), \psi^{j,\eta}_{ref}(\varepsilon_k) \rangle = \langle \mu \varphi_{j,V}, \varphi_{j,V} \rangle + \eta \delta_{j=k}, \forall j \in \{1, \dots, N\}, \forall k \in \{1, \dots, N-1\}.$$

- $[\varepsilon_{N-1}, T]$: Reaching the target.

$$\boldsymbol{\psi}^{\eta}_{ref}(T) = \left(e^{i\theta^{\eta}_{\mathbf{1}}} \varphi_{1,V}, \dots, e^{i\theta^{\eta}_{\mathbf{N}}} \varphi_{N,V} \right) \iff \mathcal{P}_j\big(\psi^{j,\eta}_{ref}(T)\big) = 0, \quad \forall j \in \{1, \dots, N\},$$

where

$$\mathcal{P}_j(\psi) = \sum_{k \geq j+1} \langle \psi, \varphi_{k,V} \rangle \varphi_{k,V}.$$

Inverse mapping theorem at $\big(0, \Phi_{1,V}(\varepsilon_{N-1}), \dots, \Phi_{N,V}(\varepsilon)\big)$ to

$$\Theta\big(u, \boldsymbol{\psi}_0\big) := \Big(\boldsymbol{\psi}_0, \mathcal{P}_1\big(\psi^1(T)\big), \dots, \mathcal{P}_N\big(\psi^N(T)\big) \Big)$$

Continuous right inverse of $\mathrm{d}\Theta\big(0, \Phi_{1,V}(\varepsilon_{N-1}), \dots, \Phi_{N,V}(\varepsilon)\big)$: solve a trigonometric moment problem with frequencies

$$\{\lambda_{k,V} - \lambda_{j,V} \,;\, j \in \{1, \dots, N\}, k \geq j+1\}.$$

Controllability of the linearized system around the reference trajectory

$$\begin{cases} i\partial_t \Psi^{j,\eta} = \left(-\partial^2_{xx} + V(x)\right)\Psi^{j,\eta} - u^{\eta}_{ref}(t)\mu(x)\Psi^{j,\eta} - v(t)\mu(x)\psi^{j,\eta}_{ref}, \\ \Psi^{j,\eta}(t,0) = \Psi^{j,\eta}(t,1) = 0, \\ \Psi^{j,\eta}(0,x) = \Psi^{j,\eta}_0(x). \end{cases}$$

Linearization of the invariants :

$$\mathrm{Re}(\langle \Psi^{j,\eta}, \psi^{j,\eta}_{ref}(t)\rangle) = 0, \quad \forall\, 1 \leq j \leq N,$$
$$\langle \Psi^{j,\eta}, \psi^{k,\eta}_{ref}(t)\rangle + \overline{\langle \Psi^{k,\eta}, \psi^{j,\eta}_{ref}(t)\rangle} = 0, \quad \forall\, 1 \leq k < j \leq N.$$

Controllability : There exists $\hat{\eta} \in (0, \overline{\eta})$ such that for any $\eta \in (0, \hat{\eta})$, for any suitable $(\boldsymbol{\Psi}_0, \boldsymbol{\Psi}_f) \in \boldsymbol{H}^3_{(V)}$, there exists $v \in L^2((0,T),\mathbb{R})$ such that the solution initiated from $\boldsymbol{\Psi}_0$ satisfies

$$\boldsymbol{\Psi}^{\eta}(T) = \boldsymbol{\Psi}_f.$$

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T) \rangle$ for $j \in \{1, \dots, N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V} \rangle$	1	2	$\dots$	N	$\dots$
$\Psi^{1,\eta}$					$\dots$
$\Psi^{2,\eta}$					$\dots$
$\vdots$	$\vdots$		$\ddots$		$\vdots$
$\Psi^{N,\eta}$					$\dots$

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T)\rangle$ for $j \in \{1, \dots, N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V}\rangle$	1	2	$\dots$	N	$\dots$
$\Psi^{1,\eta}$					
$\Psi^{2,\eta}$					
$\vdots$	$\vdots$		$\ddots$		$\ddots$
$\Psi^{N,\eta}$			$\dots$		

■ Choice of η small enough + moment problem

- For $\eta = 0$: $j \in \{1, \dots, N\}$, $k \geq j+1$ and $k = j = N$

$$\langle \Psi^{j,0}(T), \Phi_{k,V}(T)\rangle = i\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \int_0^T v(t) e^{i(\lambda_{k,V} - \lambda_{j,V})t} \mathrm{d}t,$$

solve a trigonometric moment problem (Conditions $(\mathbf{C_3})$ and $(\mathbf{C_4})$).

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T)\rangle$ for $j \in \{1,\dots,N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V}\rangle$	1	2	$\dots$	N	$\dots$
$\Psi^{1,\eta}$		■	■	■	■
$\Psi^{2,\eta}$			■	■	■
$\vdots$	$\vdots$		$\ddots$	■	■ $\ddots$
$\Psi^{N,\eta}$			$\dots$	■	■

■ Choice of η small enough + moment problem

- For $\eta = 0$: $j \in \{1,\dots,N\}$, $k \geq j+1$ and $k = j = N$

$$\langle \Psi^{j,0}(T), \Phi_{k,V}(T)\rangle = i\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle \int_0^T v(t) e^{i(\lambda_{k,V}-\lambda_{j,V})t}\mathrm{d}t,$$

  solve a trigonometric moment problem (Conditions $(\mathbf{C_3})$ and $(\mathbf{C_4})$).
- Choice of η sufficiently small $\Longrightarrow$ controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T)\rangle$.

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,v}(T)\rangle$ for $j \in \{1,\dots,N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,v}\rangle$	1	2	...	N	...
$\Psi^{1,\eta}$					
$\Psi^{2,\eta}$					
$\vdots$	$\vdots$		$\ddots$		$\ddots$
$\Psi^{N,\eta}$			...		

- Choice of η small enough + moment problem
- Minimal family for diagonal directions.
  - For $\eta = 0$:

$$\langle \Psi^{j,0}(T), \Phi_{j,v}(T)\rangle \rightsquigarrow \langle \mu\varphi_{j,v}, \varphi_{j,v}\rangle \int_0^T v(t)\mathrm{d}t, \quad \forall j \in \{1,\dots,N\}.$$

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T)\rangle$ for $j \in \{1,\dots,N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V}\rangle$	1	2	$\dots$	N	$\dots$
$\Psi^{1,\eta}$					
$\Psi^{2,\eta}$					
$\vdots$	$\vdots$		$\ddots$		$\ddots$
$\Psi^{N,\eta}$			$\dots$		

- Choice of η small enough + moment problem
- Minimal family for diagonal directions.
  - For $\eta = 0$:
    $$\langle \Psi^{j,0}(T), \Phi_{j,V}(T)\rangle \rightsquigarrow \langle \mu\varphi_{j,V}, \varphi_{j,V}\rangle \int_0^T v(t)\mathrm{d}t, \quad \forall j \in \{1,\dots,N\}.$$
  - For $\eta > 0$:
    $$\langle \Psi^{j,\eta}(T), \Phi_{j,V}(T)\rangle \rightsquigarrow \int_0^T v(t)\langle \mu\psi_{ref}^{j,\eta}(t), \psi_{ref}^{j,\eta}(t)\rangle\mathrm{d}t, \quad \forall j \in \{1,\dots,N\}.$$

    Independence condition on $\langle \mu\psi_{ref}^{j,\eta}(t), \psi_{ref}^{j,\eta}(t)\rangle$ in the construction of ψ_{ref}^{η}.

Sketch of proof

Controllability of $\langle \Psi^{j,\eta}(T), \Phi_{k,V}(T) \rangle$ for $j \in \{1, \dots, N\}$ and $k \in \mathbb{N}^*$.

$\langle \Psi^{j,\eta}, \Phi_{k,V} \rangle$	1	2	$\dots$	N	$\dots$
$\Psi^{1,\eta}$					
$\Psi^{2,\eta}$					
$\vdots$		$\ddots$	$\ddots$		$\ddots$
$\Psi^{N,\eta}$					

- Choice of η small enough + moment problem
- Minimal family for diagonal directions.
- Invariants

$$\langle \Psi^{j,\eta}, \psi^{k,\eta}_{ref}(t) \rangle + \overline{\langle \Psi^{k,\eta}, \psi^{j,\eta}_{ref}(t) \rangle} = 0, \quad \forall\, 1 \leq k < j \leq N.$$

$N \in \mathbb{N}^*$. $V, \mu \in H^3((0,1),\mathbb{R})$ such that

(C$_3$) there exists $c > 0$ such that

$$|\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle| \geq \frac{c}{k^3}, \quad \forall j \in \{1,\dots,N\}, \forall k \in \mathbb{N}^*,$$

(C$_4$) $\lambda_{k,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{n,V}$ for all $j, n \in \{1,\dots,N\}$, $k \geq j+1$, $p \geq n+1$ with $\{j,k\} \neq \{p,n\}$,

(C$_5$) $1, \lambda_{1,V}, \dots, \lambda_{N,V}$ are rationally independent.

Theorem

Let $C_0, C_f \in U_N$ and $\boldsymbol{z}_0 := C_0\boldsymbol{\varphi}_V$, $\boldsymbol{z}_f := C_f\boldsymbol{\varphi}_V$. Under Conditions **(C$_3$)**-**(C$_5$)**, there exists $T > 0$, $\delta > 0$ such that, if

$$\mathcal{O}_{\delta,C} := \left\{ \phi \in \boldsymbol{H}^3_{(V)} \, ; \, \langle \phi^j, \phi^k \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^N \|\phi^j - (C\boldsymbol{\varphi}_V)^j\|_{H^3_{(\boldsymbol{V})}} < \delta \right\},$$

for every $\psi_0 \in \mathcal{O}_{\delta,C_0}$, $\psi_f \in \mathcal{O}_{\delta,C_f}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that the associated solution satisfies $\psi(T) = \psi_f$.

Proof : rotation and linearity

1. **Proof in the case** $C_0 = C_f = I_N$**.** $\psi_0, \psi_f \approx \varphi_V$.
   - Use of the proposition.

$$\psi_0 \approx \varphi_V \quad \overset{T^*,u}{\rightsquigarrow} \quad \left(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V}\right).$$

   - Rotation and rational independence of eigenvalues : Condition **(C$_5$)**.

$$\left(e^{i\theta_1}\varphi_{1,V}, \dots, e^{i\theta_N}\varphi_{N,V}\right) \overset{T_r, u=0}{\rightsquigarrow} \zeta := \left(e^{i(\theta_1 - \lambda_{1,V} T_r)}\varphi_{1,V}, \dots, e^{i(\theta_N - \lambda_{N,V} T_r)}\varphi_{N,V}\right)$$
$$\approx \left(e^{-i\theta_1}\varphi_{1,V}, \dots, e^{-i\theta_N}\varphi_{N,V}\right)$$

   - Use of the proposition.

$$\overline{\psi_f} \approx \varphi_V \quad \overset{T^*,v}{\rightsquigarrow} \quad \overline{\zeta} \quad \Longrightarrow \quad \zeta \quad \overset{T^*, v(T^*-\cdot)}{\rightsquigarrow} \quad \psi_f.$$

2. **Proof in the case** $C_0 = C_f = C \in U_N$**.** Let $z := C\varphi_V$. $\psi_0, \psi_f \approx z$.
   - Linearity with respect to the state

3. **Conclusion :** $C_0, C_f \in U_N$.
   - Connectedness in the set of unitary matrices and compactness.

3. Conclusion : $C_0, C_f \in U_N$.

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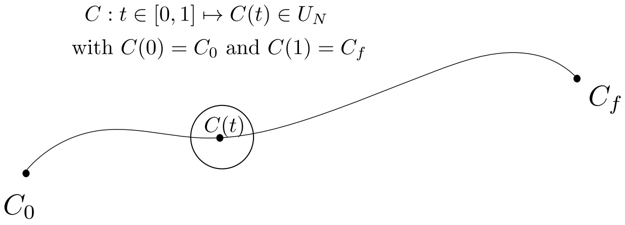

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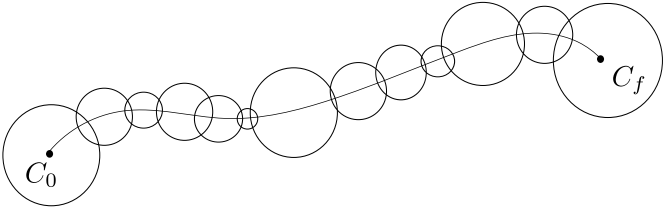

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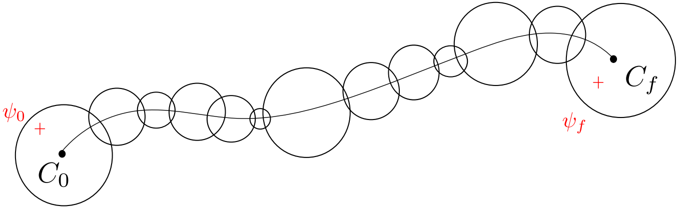

Proof : connectedness and compactness

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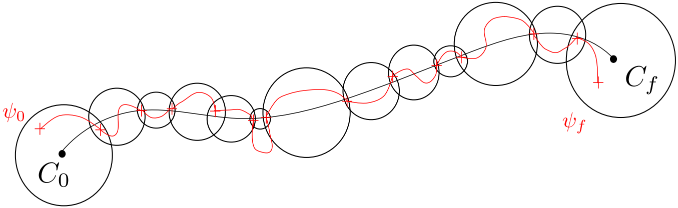


Global exact controllability under favourable hypotheses

$V, \mu \in H^4((0,1),\mathbb{R})$ such that

(C_6) for any $j \in \mathbb{N}^*$, $\exists c_j > 0$;

$$|\langle \mu\varphi_{j,V}, \varphi_{k,V}\rangle| \geq \frac{c_j}{k^3}, \quad \forall k \in \mathbb{N}^*,$$

(C_7) $\{1, (\lambda_{j,V})_{j\in\mathbb{N}^*}\}$ are rationally independent : $\forall M \in \mathbb{N}^*$, $\forall \boldsymbol{r} \in \mathbb{Q}^{M+1}\backslash\{\mathbf{0}\}$,

$$r_0 + \sum_{j=1}^{M} r_j \lambda_{j,V} \neq 0.$$

Conditions **(C_6)**-**(C_7)** $\Longrightarrow$ Conditions **(C_1)**-**(C_5)**, for any $N \in \mathbb{N}^*$.

Theorem

Let $N \in \mathbb{N}^*$. Under Conditions **(C_6)**-**(C_7)**, for any unitarily equivalent vectors $\psi_0, \psi_f \in \boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V)}$, there are $T > 0$, $u \in L^2((0,T),\mathbb{R})$ such that

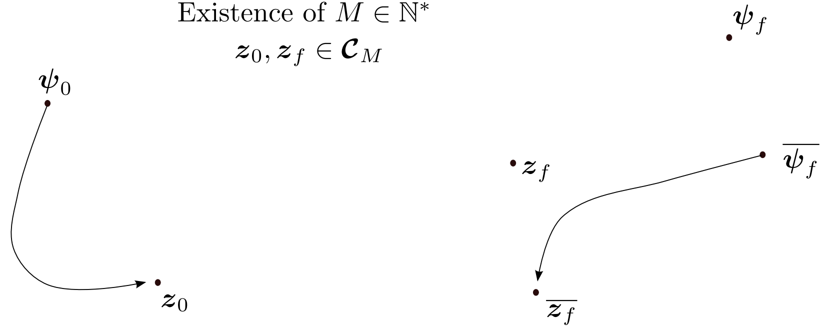
$$\psi(T, \psi_0, u) = \psi_f.$$

Sketch of proof

Global approximate controllability

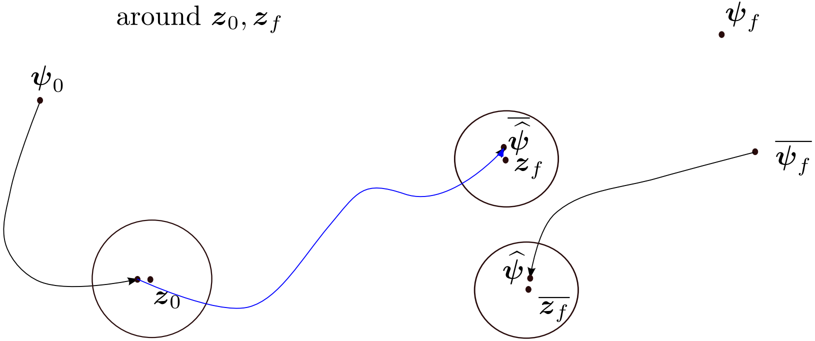
Existence of $M \in \mathbb{N}^*$

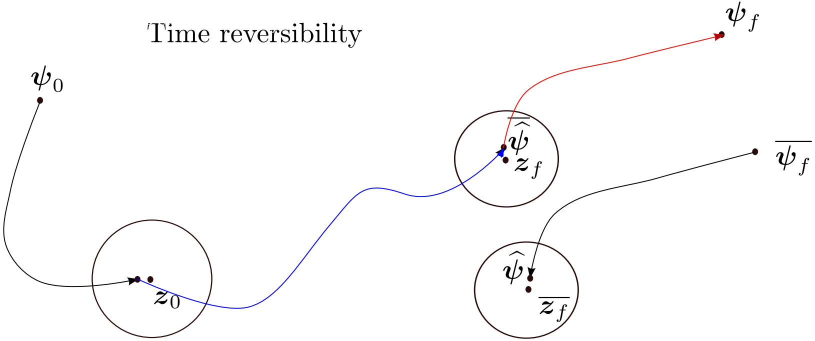
$\boldsymbol{z}_0, \boldsymbol{z}_f \in \boldsymbol{\mathcal{C}}_M$



Sketch of proof

Exact controllability of $(\mathbf{S}_{\boldsymbol{M}})$
around $\boldsymbol{z}_0, \boldsymbol{z}_f$




Time reversibility
ψ_0
z_0
$\widehat{\overline{\psi}}$
z_f
ψ_f
$\overline{\psi_f}$
$\widehat{\psi}$
$\overline{z_f}$

Dealing with an arbitrary potential

$V \in H^4((0,1),\mathbb{R})$ arbitrary

$$\begin{cases} i\partial_t \psi^j = -\left(\partial_{xx}^2 + V(x) + \mu(x)\right)\psi^j - u(t)\mu(x)\psi, & (t,x) \in (0,T)\times(0,1), \\ \psi^j(t,0) = \psi^j(t,1) = 0, & j \in \{1,\dots,N\}, \end{cases} \qquad (\tilde{\mathbf{S}}_N)$$

Link between propagators of $(\mathbf{S}_N)$ and $(\tilde{\mathbf{S}}_N)$:

$$\tilde{\psi}(T,\psi_0,u) = \psi(T,\psi_0,u-1).$$

- $\mathcal{Q}_V$: set of $\mu \in H^4((0,1),\mathbb{R})$ such that Conditions $(\mathbf{C_6})$ and $(\mathbf{C_7})$ are satisfied for V replaced by $V+\mu$ i.e.

$$\forall j \in \mathbb{N}^*, \exists c_j > 0; \ |\langle \mu\varphi_{j,V+\mu}, \varphi_{k,V+\mu}\rangle| \geq \frac{c_j}{k^3}, \quad \forall k \in \mathbb{N}^*,$$

$$\{1, (\lambda_{j,V+\mu})_{j\in\mathbb{N}^*}\} \text{ are rationally independent.}$$

- $\mu \in \mathcal{Q}_V$: global exact controllability of $(\tilde{\mathbf{S}}_N)$ in $\boldsymbol{\mathcal{S}} \cap \boldsymbol{H}^4_{(V+\mu)}$.
- $\mathcal{Q}_V$ is residual in $H^4((0,1),\mathbb{R})$.

Open problems and perspectives

Conclusion

- Global exact controllability
- Arbitrary number of equations
- No restriction on the potential

Open problems

- Large time : Lyapunov strategy, rotation (Kronecker diophantine approximation), compactness argument.
- Optimal spaces : $\boldsymbol{H}^4_{(V)}$, $\boldsymbol{H}^3_{(V)}$ (Lyapunov strategy in infinite dimension)

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- Optimal spaces : $\boldsymbol{H}^4_{(V)}$, $\boldsymbol{H}^3_{(V)}$ (Lyapunov strategy in infinite dimension)

Thank you for your attention.

Beauchard, K. and Laurent, C. (2010).
Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control.
J. Math. Pures Appl. (9), 94(5):520–554.

Morancey, M. (2013).
Simultaneous local exact controllability of 1D bilinear Schrödinger equations.
Ann. Inst. H. Poincaré Anal. Non Linéaire.
DOI : 10.1016/j.anihpc.2013.05.001.

Morancey, M. and Nersesyan, V. (2013).
Simultaneous global exact controllability of an arbitrary number of 1D bilinear Schrödinger equations.
preprint, arXiv:1306.5851.

Nersesyan, V. (2010).
Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications.
Ann. Inst. H. Poincaré Anal. Non Linéaire, 27(3):901–915.