On an optimal control problem involving a free boundary

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A concrete problem

We consider the profit of an industry in the following context. The industry produces pollution which is discharged somewhere in the sea.

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Problem: maximize the income of the industry: best balance production rate/pollution expenses.

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To describe the stationary problem associated to the reaction and diffusion of this substance, set $0 < q < 1$

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- L is an elliptic linear operator of the form

$$Ly = \sum_{ij=1}^{N} a_{ij}(x) \frac{\partial^2 y}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial y}{\partial x_i} \tag{2}$$

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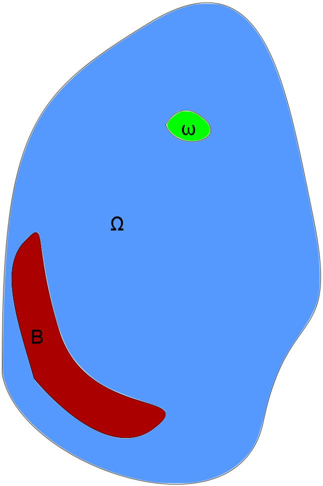
- $a_{ij}, b_i \in L^\infty(\Omega)$ such that there exist $\Lambda, \lambda \geq 0$, for which

$$\lambda|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \tag{3}$$

The data u

- u is the source term and represents the control variable: $u \geq 0 \Rightarrow y \geq 0$.



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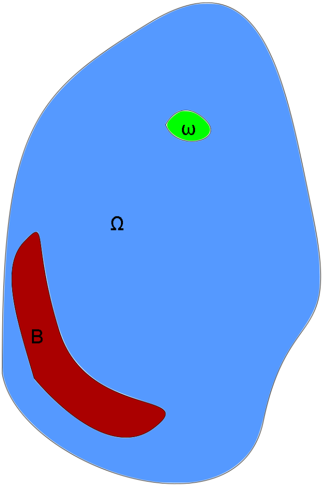

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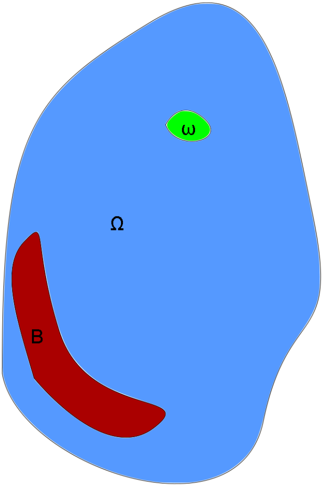
- The support of u is confined in $\omega \subset\subset \Omega$, the emission area.
- ω is very small compared to Ω and sufficiently far from the boundary :
  - one of the factors which cause a dead core.
  - allows us to assume that the pollution does not reach the boundary.



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We want to solve the problem $\min_{u \in U_{ad}} J(u)$ with

$$J(u) = \underbrace{\int_\Omega \chi_{S(y(x;u))\cap B}(x)\,dx}_{J_1(u)} + \underbrace{\int_\Omega \frac{1}{G(y(x;u))}\,dx}_{J_2(u)}. \tag{4}$$

Some remarks

This optimal control problem is non-standard:

- The cost functional does not depend only on some energy of $y \to$ very few example in the literature (J.F.Rodrigues et al.(1993)).

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- Possible existence of a dead core and the associated free boundary $\partial S(y) \cap \Omega \neq \emptyset$ (Díaz (1985), etc.).
- $S(y) \cap B \neq B$ for some y.
  - The first part of the functional is non trivial.

Set of admissible control

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The set of admissible controls

$$U_{\mathrm{ad}} = \{u \in H^1(\omega) \cap L^\infty(\omega) \,|\, 0 \leq u(x) \leq M,\ ||u||_{H^1(\omega)} \leq M^* \text{ and } u \text{ satisfies (5) for some } R > R_0 \text{ and } C > C_0\}$$

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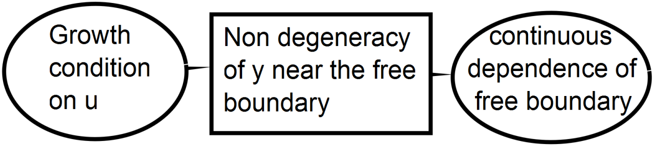
- The boundedness in H^1 is instead purely technical in order to be able to prove the existence of a minimum.
- The growth condition is the central element to obtain a continuous dependence of the free boundary from the control.



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Sufficient condition for continuity of the support

Let $\{y_n\}$ converging in $L^\infty(\Omega)$ to y. Suppose that the following non-degeneracy property holds uniformly for all $n \in \mathbb{N}$: there exist $\varepsilon_0 > 0$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $lim_{t\to 0} h(t) = 0$ and

$$|\{x \in \Omega \,:\, 0 < y_n(x) < \varepsilon\}| \leq h(\varepsilon) \quad \forall \varepsilon < \varepsilon_0. \tag{6}$$

Then $|N(y_n) \div N(y)| \to 0$

Diffusion of the support-Continuity of the free boundary

Condition (5) on u guarantees the support of y to be strictly bigger than the one of u: Diffusion of the support.

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On $S(u)$ a generalization of the results of Álvarez-Díaz (1987) for the Laplacian gives

$$|\{x \in S(u) : 0 < y < \varepsilon\}| \leq \varepsilon^{(1-q)/2} \tag{7}$$

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The new way to approach it utilizes the following tools:

- the well known Fleming-Rishel-Federer Co-area Formula,
- the Agmon-Douglis-Niremberg regularity result for the solution y,
- the Co-area regularity of y and the regularity of the its distribution function (Almgren-Lieb (1989)),
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to get the following absolutely continuous representation:

$$|\{x \in D : 0 < y < \varepsilon\}| = \underbrace{\int_0^{\varepsilon} \Big(\int_{\{y=s, \nabla y \neq 0\}} \frac{d\mathcal{L}^{N-1}}{|\nabla y|} \Big) ds}_{h_y(\varepsilon)}$$

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Theorem

Let $u_n \to u$ in $L^2(\omega)$ and weakly star in $L^\infty(\omega)$, $u_n \geq 0$, and let y_n and y the solutions of the associated problems (2). Then there exists a subsequence (still labeled as y_n) such that $y_n \to y$ in $W^{2,p}(\Omega)$ for any $p \in [1, +\infty)$, and there exist ε_0 and $\bar{h} : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, with $\bar{h}(0) = 0$ such that for all $\varepsilon < \varepsilon_0$ and for any n of this subsequence

$$|\{x \in N(u_n) : 0 < y_n(x) < \varepsilon\}| \leq \bar{h}(\varepsilon). \tag{8}$$

Counter example

The idea is to play with the following family of functions y_ε on $(-2, 2)$:

$$y_\varepsilon(r) = \begin{cases} 0 & r \in (1, 2) \\ e^{-\frac{1}{1-r}} & r \in (1-\varepsilon, 1) \\ C_2 - C_1 r^2 & r \in (0, 1-\varepsilon) \end{cases}$$

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- The controls associated to these solutions and the solutions themselves go to zero as $\varepsilon \to 0$,
- no uniform growth condition on u_ε,
- the support of y_ε is $[-1, 1]$ for all ε,
- the limit has empty support$\to$ no continuity of the free boundary.

Existence of a minimum

Minimizing sequence $u_n \subset U_{\mathrm{ad}}$: $J(u_n) \to \min$

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- u has the same bounds of u_n in $H^1(\omega) \cap L^\infty(\omega)$.
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- $\frac{1}{G(y)}$ is uniformly bounded$\Rightarrow$ $J_2(u_n)$ converges on a subsequence to $J_2(u)$.
- $J(u_{n_t}) = (J_1 + J_2)(u_{n_t})$ converges to $J(u) = $ min.

Numerics

Comparison of the values of the cost functional with respect to different indicative controls.

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Simplified problem in dimension 1

Setting of the numerical experiments:

- $\Omega = (0,4)$, $\omega = (1,2)$ and $B = [3,4]$,
- approximation of the reaction term y^q with a C^1 function $f_\delta(y) = \arctan(1000y)$
- $G(s) = s + 1$

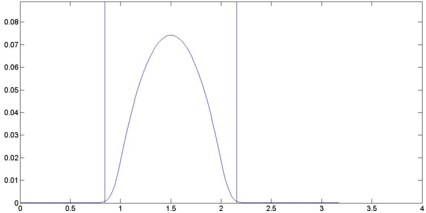
$$\begin{cases} -y(x)'' + f_\delta(y(x)) = u(x)\chi_\omega & \text{in } (0,4), \\ y(x) = 0 & x = 0, 4. \end{cases}$$

- $S(y(x;u)) \approx S_\delta(y(x;u)) = \{x \in (0,4) : \delta \leq y(x;u)\}$.

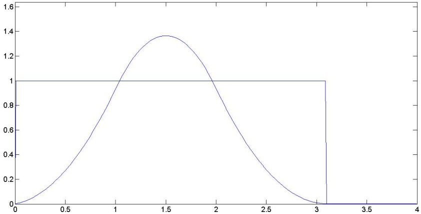
Experiment 1

Controls very concentrated near their supports.

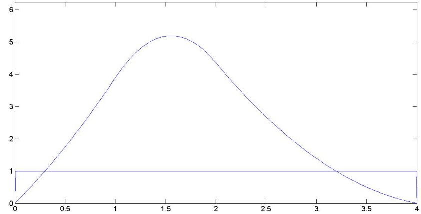
$$u(x)\chi_\omega(x) = \begin{cases} k & \text{if } x \in (1,2), \\ 0 & \text{if } x \notin (1,2). \end{cases}$$



$k = 2$
$J_1 = 0$
$J_2 = 1.978173132097462$
$J = 1.978173132097462$



$k = 5$
$J_1 = 0.099999999$
$J_2 = 1.507387461553110$
$J = 1.607387460553110$

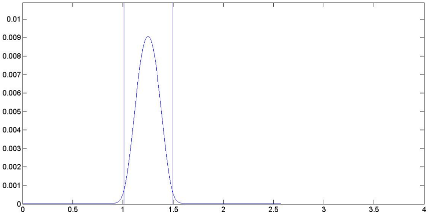


$k = 10$
$J_1 = 1$
$J_2 = 0.788557619378538$
$J = 1.788557619378538$

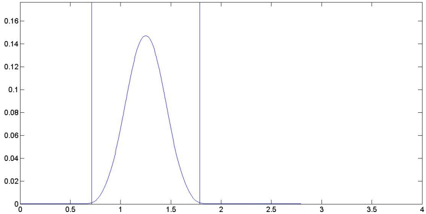
Experiment 2

Controls moderately concentrated near their supports

$$u(x)\chi_\omega(x) = \begin{cases} 16K(1.5-x)(x-1) & \text{if } x \in (1, 1.5), \\ 0 & \text{if } x \notin (1, 1.5). \end{cases}$$



$K = 2$
$J_1 = 0$
$J_2 = 2.004243820292839$
$J = 2.004243820292839$



$$K = 5$$
$$J_1 = 0$$
$$J_2 = 1.973120386756347$$
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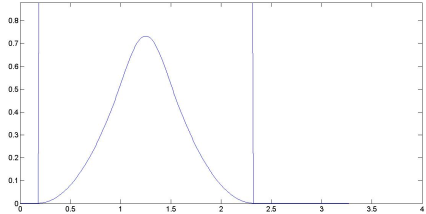

$$K = 10$$
$$J_1 = 0$$
$$J_2 = 1.796437715318067$$
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Thank you!