An introduction to quantum error correction

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Quantum information processing

Two perspectives

- **Physics**: Realization of devices, quantum logical gates, quantum memory, · · ·.
- **Computer science**: Algorithmes and quantum cryptography.

State of the art in physics

- Quantum communication over distances < 100km;
- Logical gates between a few qubits (less than 10);
- Still far from the requirements in universal computation: many thousands of qubits.

Main obstacle: decoherence!
State of the art in physics

1. Classical and quantum noise, decoherence

2. Basics of quantum error correction

3. Recent experimental development: major obstacles

4. Some new directions
Classical bit, classical noise

Classical bit: strongly dissipative bistable system

\[ m\ddot{x} + \eta \dot{x} + \frac{\partial}{\partial x} U(x) = 0 \]

\( U(x) \)

\( \eta: \text{friction coefficient} \)

Classical bit in the state 0 or 1:

1. Strong dissipation;
2. \( k_B T_{\text{noise}} \ll \Delta U \);
Classical noise, classical error correction

Classical noise: bit-flip errors

Classical error correction

- $0 \rightarrow 000$ and $1 \rightarrow 111$;
- Majority vote for correction;
- New error probability: $p \rightarrow 3p^2 - 2p^3$ (probability of two or more errors).
\[
\left(-\frac{\hbar^2}{2m}\nabla + U(x)\right)\psi_k(x) = E_k\psi_k(x),
\]

\[
|0\rangle = \psi_0(x), \quad |1\rangle = \psi_1(x).
\]

Quantum state: \[c_0|0\rangle + c_1|1\rangle.\]
Quantum noise: open quantum systems

1 Schrödinger: wave funct. \( |\psi\rangle \in \mathcal{H} \) or density op. \( \rho \sim |\psi\rangle \langle \psi| \)

\[
\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle, \quad \frac{d}{dt} \rho = -\frac{i}{\hbar} [H, \rho], \quad H = H_0 + uH_1
\]

2 Entanglement and tensor product for composite systems \((S, M)\):

- Hilbert space \( \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M \)
- Hamiltonian \( H = H_S \otimes I_M + H_{\text{int}} + I_S \otimes H_M \)

3 Randomness and irreversibility induced by the measurement of observable \( O \) with spectral decomp. \( \sum \mu \lambda_\mu P_\mu \):

- measurement outcome \( \mu \) with proba. \( P_\mu = \langle \psi | P_\mu | \psi \rangle = \text{Tr} (\rho P_\mu) \) depending on \( |\psi\rangle \), \( \rho \) just before the measurement
- measurement back-action if outcome \( \mu = y \):

\[
|\psi\rangle \mapsto |\psi\rangle_+ = \frac{P_y |\psi\rangle}{\sqrt{\langle \psi | P_y | \psi \rangle}}, \quad \rho \mapsto \rho_+ = \frac{P_y \rho P_y}{\text{Tr} (\rho P_y)}
\]

Quantum noise: interaction with environment

- **State space**: \( \{ \rho : \mathcal{H} \mapsto \mathcal{H} \mid \rho = \rho^\dagger, \rho \geq 0, \text{Tr} (\rho) = 1 \} \), where \( \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{\text{env}} \).

- **Dynamics**: \( U_\tau = \exp(-i \tau \mathbf{H}/\hbar), \mathbf{H} = \mathbf{H}_S \otimes \mathbf{l}_{\text{env}} + \mathbf{H}_{\text{int}} + \mathbf{l}_S \otimes \mathbf{H}_{\text{env}} \).

- **Error channel**: \( \mathcal{E}(\rho_S) = \text{tr}_{\text{env}} [U_{\tau}(\rho_S \otimes \rho_{\text{env}})U_{\tau}^\dagger] \).

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**Error channel: operator-sum representation**

Considering a basis \( \{ e_k \} \) of \( \mathcal{H}_{\text{env}} \):

\[
\mathcal{E}(\rho) = \sum_k \langle e_k | U_{\tau}(\rho \otimes |e_0\rangle \langle e_0|)U_{\tau}^\dagger | e_k \rangle
\]

\[
= \sum_k \mathbf{E}_k \rho \mathbf{E}_k^\dagger,
\]

where \( \mathbf{E}_k = (\mathbf{l}_S \otimes \langle e_k|)U_{\tau}(\mathbf{l}_S \otimes |e_0\rangle) \). As a consequence to the unitarity of \( U_{\tau} \), we have \( \sum_k \mathbf{E}_k^\dagger \mathbf{E}_k = \mathbf{l}_S \).
Quantum noise: interaction with environment

Discrete dynamics

\[ \mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger, \quad E_k = (I_S \otimes \langle e_k |) U_{\tau} (I_S \otimes | e_0 \rangle), \quad \sum_k E_k^\dagger E_k = I_S. \]

Continuous dynamics (\( \tau \to 0 \)): Lindblad equation

\[ \frac{d}{dt} \rho = \sum_k (L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k). \]

Correspondance with discrete case: \( E_0 = I - \frac{dt}{2} \sum_k L_k^\dagger L_k, \)
\[ E_k = \sqrt{dt} L_k, \quad k = 1, 2, \cdots. \]
Example: energy decay of a qubit

State space: \( \{ \rho \in \text{span}\{ |0 \rangle, |1 \rangle \} \mid \rho = \rho^\dagger, \rho \geq 0, \text{Tr}(\rho) = 1 \} \).

Energy decay channel

\[ \rho_{k+1} = \mathcal{E}(\rho_k) = E_0 \rho_k E_0^\dagger + E_1 \rho_k E_1^\dagger, \]

where

\[ E_0 = |0 \rangle \langle 0| + \sqrt{1 - p} |1 \rangle \langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - p} \end{pmatrix}, \]

\[ E_1 = \sqrt{p} |0 \rangle \langle 1| = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}. \]

Dissipation towards ground state:

\[ P_{1}^{k+1} = \langle 1| \rho_{k+1} |1 \rangle = (1 - p) \langle 1| \rho_k |1 \rangle = (1 - p)^{k+1} \langle 1| \rho_0 |1 \rangle \]

\[ P_{0}^{k+1} = \langle 0| \rho_{k+1} |0 \rangle = 1 - P_{1}^{k+1} = 1 - (1 - p)^{k+1} \langle 1| \rho_0 |1 \rangle. \]
Example: bit-flip and phase-flip of a qubit

**Bit-flip channel**

\[ \rho_{k+1} = \mathcal{E}(\rho_k) = E_0 \rho_k E_0^\dagger + E_1 \rho_k E_1^\dagger, \]

\[ E_0 = \sqrt{1-p} I = \sqrt{1-p}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \]

\[ E_1 = \sqrt{p} \sigma_x = \sqrt{p}(|0\rangle\langle 1| + |1\rangle\langle 0|) = \begin{pmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}. \]

**Phase decay channel**

\[ \rho_{k+1} = \tilde{\mathcal{E}}(\rho_k) = \tilde{E}_0 \rho_k \tilde{E}_0^\dagger + \tilde{E}_1 \rho_k \tilde{E}_1^\dagger, \]

\[ \tilde{E}_0 = \sqrt{1-p} I = \sqrt{1-p}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \]

\[ \tilde{E}_1 = \sqrt{p} \sigma_z = \sqrt{p}(|0\rangle\langle 0| - |1\rangle\langle 1|) = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix}. \]
1. Classical and quantum noise, decoherence
2. Basics of quantum error correction
3. Recent experimental development: major obstacles
4. Some new directions
Quantum error correction (QEC): bit-flip channel

Quantum error correction

Protect any superposition state $c_0 |0\rangle + c_1 |1\rangle$ without any knowledge of $c_0$ and $c_1$.

Idea

- Inspired by classical error correction, encode $c_0 |0\rangle + c_1 |1\rangle$ as
  
  $$c_0 |000\rangle + c_1 |111\rangle = c_0 |0\rangle \otimes |0\rangle \otimes |0\rangle + c_1 |1\rangle \otimes |1\rangle \otimes |1\rangle.$$

- Measuring a qubit: asking a qubit if it is in $|0\rangle$ or $|1\rangle$ erases the information by projecting the superposition on $|000\rangle$ or $|111\rangle$ (no majority vote).

- Parity measurements: asking instead if qubits 1 and 2 are in the same states (parity of qubits 1 and 2: $\sigma_z \otimes \sigma_z \otimes I$) and the same question for qubits 1 and 3 ($\sigma_z \otimes I \otimes \sigma_z$).

- Correction: if $m_0 = m_1 = 1$ then no correction; if $m_0 = -1, m_1 = -1$ then flip qubit 1; if $m_0 = -1, m_1 = 1$ then flip qubit 2; if $m_0 = 1, m_1 = -1$ then flip qubit 3.
QEC in practice

Quantum gates:

\[ X, Y, Z, H \text{ gates} \]

\[ U = \sigma_x, \sigma_y, \sigma_z, \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

C-NOT gate

\[ |c\rangle \quad |t\rangle \]

\[ U_{\text{C-NOT}} |c\rangle \otimes |t\rangle = |c\rangle \otimes (c \oplus t) \]

QEC steps:

\[ |\psi\rangle = c_0 |0\rangle + c_1 |1\rangle \]

Initialize

\[ |\psi\rangle \]
QEC in practice

Quantum gates:

\[ X, Y, Z, H \text{ gates} \]

\[ U = \sigma_x, \sigma_y, \sigma_z, \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) \]

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C-NOT gate

\[ |c\rangle \quad |t\rangle \]

\[ U_{\text{C-NOT}} |c\rangle \otimes |t\rangle = |c\rangle \otimes |c \oplus t\rangle \]

QEC steps:

\[ |\psi\rangle = c_0 |0\rangle + c_1 |1\rangle \]
QEC in practice

Quantum gates:

- $X, Y, Z, H$ gates
  
  $$U = \sigma_x, \sigma_y, \sigma_z, \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$$

- C-NOT gate
  
  $$U_{\text{C-NOT}} |c\rangle \otimes |t\rangle = |c\rangle \otimes |c \oplus t\rangle$$

QEC steps: $|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$
QEC in practice

Quantum gates:

\[ X, Y, Z, H \text{ gates} \]

\[ U = \sigma_x, \sigma_y, \sigma_z, \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

C-NOT gate

\[ U_{\text{C-NOT}} |c\rangle \otimes |t\rangle = |c\rangle \otimes |c \oplus t\rangle \]

QEC steps: \[ |\psi\rangle = c_0 |0\rangle + c_1 |1\rangle \]
QEC in practice

Failure modes?
Phase-flip vs bit-flip

- **Bit-flip error channel**: spanned by \{ \sqrt{1-p} \mathbb{I}, \sqrt{p} \sigma_x \}

  \[ E_{\text{bit-flip}}(\rho) = (1-p)\rho + p\sigma_x \rho \sigma_x. \]

- **Phase-flip error channel**: spanned by \{ \sqrt{1-p} \mathbb{I}, \sqrt{p} \sigma_z \}

  \[ E_{\text{phase-flip}}(\rho) = (1-p)\rho + p\sigma_z \rho \sigma_z. \]

Phase-flip QEC

- Similarly to bit-flip case, encode \( c_0 |0\rangle + c_1 |1\rangle \) as

  \[ c_0 |\mp\rangle + c_1 |\mp\rangle, \quad \text{where} \quad |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle). \]

- Parity measurements: We measure \( \sigma_x \otimes \sigma_x \otimes \mathbb{I} \) and \( \sigma_x \otimes \mathbb{I} \otimes \sigma_x \) instead of \( \sigma_z \otimes \sigma_z \otimes \mathbb{I} \) and \( \sigma_z \otimes \mathbb{I} \otimes \sigma_z \) for the bit-flip case.
Similarly to an error channel, the error correction (measurement and feedback) can be modeled by a quantum operation:

\[ \rho \mapsto \mathcal{R}(\rho) = \sum_k R_k \rho R_k^\dagger. \]

This corrects an error channel \( \rho \mapsto \mathcal{E}(\rho) \) if for any \( \rho \) in the code space

\[ \mathcal{R} \circ \mathcal{E}(\rho) = \rho. \]

**Theorem: discretization of error channels**

Assume that \( \mathcal{R} \) is the error-correction operation for the error channel \( \mathcal{E} \) modeled by operators \( \{E_k\} \). Suppose \( \mathcal{F} \) to be another error channel with operation elements \( F_j \) which are linear combinations (with complex coefficients) of operators \( E_k \). Then the error-correction operation \( \mathcal{R} \) also corrects the error channel \( \mathcal{F} \).

**Corollary: case of qubit**

It suffices to correct the operations \( \{I, \sigma_x, \sigma_z, \sigma_y = i\sigma_x\sigma_z\} \): any matrix \( \mathbf{E} \) on \( \mathbb{C}^2 \) is a linear combination of these operators.
QEC beyond bit-flip errors: a full QEC code

\[
|0_L\rangle = \frac{1}{\sqrt{8}} \left( |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \right) + |0011111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle
\]

\[
|1_L\rangle = \frac{1}{\sqrt{8}} \left[ |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |0011111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \right] + |1100000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle
\]

Single round of error correction
1. Classical and quantum noise, decoherence

2. Basics of quantum error correction

3. Recent experimental development: major obstacles

4. Some new directions
Google/UCSB: bit-flip error detection

- Only protects classical states $|000\rangle$ or $|111\rangle$ (no superpositions).
- Offline correction (no feedback).

IBM: bit-flip/phase-flip detection

- Only protects the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|--\rangle + |++\rangle)$.
- Offline correction (no feedback).

Data qubits: ${}^{13}\text{C}$ nuclear spins
Ancilla qubit: NV center electron spin

- Only protects phase-flip errors: in these qubits phase-flip errors are dominant.

Major obstacles to scaling

- Fault-tolerant QEC: up to a hundred physical qubits per logical qubit.
- Each qubit benefiting from state of art properties: lifetime, coupling strength, tunability.
- Avoid correlated errors: no undesired couplings.
- Short time-scales for real-time feedback: feedback delays.
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Directions followed by QUANTIC and collaborators

- **Hardware-efficient QEC**: encode, protect and manipulate quantum information on a quantum harmonic oscillator which benefits from an infinite dimensional Hilbert space instead of a multi-qubit register.

- **Controllability**: coupling the quantum harmonic oscillator to an ancilla qubit provides controllability over its infinite dimensional Hilbert space.

- **Autonomous correction**: replace real-time feedback by analog feedback circuits ensuring the stabilization of a desired manifold of quantum states by engineered dissipation.
Hardware-efficient QEC

- Encode information on a Schrödinger cat state of storage cavity\textsuperscript{1}.
- Measure continuously photon-number parity as error syndrom\textsuperscript{2}.
- Perform error correction with real-time feedback and using the ancilla qubit\textsuperscript{3}.

\textsuperscript{3} L. Sun et al., Nature 511, 444-448, 2014.
\textsuperscript{3} N. Ofek, A. Petrenko et al., In preparation.
Autonomous QEC

Engineer the dissipation of a quantum system by engineering its interaction with a reservoir to:

- Stabilize an entangled state\(^1\).
- Stabilize a manifold of quantum states\(^2\).
- Perform autonomous QEC\(^3\).

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