On the optimization of stabilizing time-varying feedback controls

Martin Gugat

1. Optimal Dirichlet Boundary Control

2. Optimal Neumann Boundary Control
   - Example: Solution of (EC)

3. Stabilization
   - Example: Stationary Feedback Law
   - Example: Time-varying Feedback Control

4. Optimized Feedback
   - Examples for optimized feedback
   - Examples for optimized feedback: Korteweg-de Vries Equation

5. Stabilization of semilinear systems: Telegraph equation

6. Stabilization of a quasilinear wave equation

7. Conclusion
On the optimization of stabilizing time-varying feedback controls

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Optimal Boundary Control of the Wave Equation
Optimal Dirichlet Boundary Control

\[ y(t, 1) = u(t) \]
The Problem of Optimal Exact Control: The 1d-case

Assume that $T = 2k$ for some natural number $k$. 
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We consider the wave equation on $[0, T] \times [0, 1]$. 
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- Initial state $y_0 \in L^2(0, 1)$. 
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- Assume that $T = 2k$ for some natural number $k$.
- We consider the wave equation on $[0, T] \times [0, 1]$.
- Initial state $y_0 \in L^2(0, 1)$.
- Initial velocity $y_1 \in H^{-1}(0, 1)$.

\[
\begin{align*}
\text{(EC)} & \quad \begin{cases}
\text{minimize } \|u\|_{L^2(0,T)}^2 \text{ subject to } \\
y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in (0,1) \\
y(t,0) = 0, \quad y(t,1) = u(t), \quad t \in (0,T) \\
y_{tt}(t,x) = y_{xx}(t,x), \quad (t,x) \in (0,T) \times (0,1) \\
y(T,x) = 0, \quad y_t(T,x) = 0, \quad x \in (0,1).
\end{cases}
\end{align*}
\]
Solution of Problem (EC)

- M. Gugat, G. Leugering, G. Sklyar, *Lp optimal boundary control for the wave equation*, SICON 2005

Problem **EC** has a solution $u$ that is uniquely determined.

The optimal control $u^*$ is 2 periodic.

$$u^*(t) = \begin{cases} 
1 & t \in (0, 1) \\
-\int_0^1 y_1(s) \, ds + r + y_0(1-t) & t \in (1, 2)
\end{cases}$$

with $r = \int_0^1 \int_0^t y_1(s) \, ds \, dt$.

To make life simple, for numerical purposes, it is sometimes useful to replace the exact end conditions by a penalty term:

$$\min 1 \gamma \| u \|_{L^2(0,T)} + \| y(T, \cdot) \|_{L^2(0,1)} + \| Y \|_{L^2(0,1)}$$

For the error we get

$$\| u_\gamma - u^* \|_{L^2(0,T)} / \| u^* \|_{L^2(0,T)} = \frac{1}{2} \kappa_{\gamma + 1}.$$
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Problem EC has a solution $u$ that is uniquely determined.

The optimal control $u_*$ is 2 periodic.

$$u_*(t) = \begin{cases} 
\frac{1}{T} \left(- \int_0^{1-t} y_1(s) \, ds + r + y_0(1-t) \right), & t \in (0, 1) \\
\frac{1}{T} \left(- \int_0^{t-1} y_1(s) \, ds + r - y_0(t-1) \right), & t \in (1, 2)
\end{cases}$$

with $r = \int_0^1 \int_0^t y_1(s) \, ds \, dt$. 

To make life simple, for numerical purposes, it is sometimes useful to replace the exact end conditions by a penalty term:

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$$\|u_\gamma - u_*\|_{L^2(0,T)} \leq \frac{1}{2} \kappa \gamma + 1.$$
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Problem EC has a solution $u$ that is uniquely determined.

The optimal control $u_\ast$ is 2 periodic.

$$
 u_\ast(t) = \begin{cases} 
 \frac{1}{T} \left( - \int_0^{1-t} y_1(s) \, ds + r + y_0(1-t) \right), & t \in (0,1) \\
 \frac{1}{T} \left( - \int_0^{t-1} y_1(s) \, ds + r - y_0(t-1) \right), & t \in (1,2)
\end{cases}
$$

with $r = \int_0^1 \int_0^t y_1(s) \, ds \, dt$.

To make life simple, for numerical purposes, it is sometimes useful to replace the exact end conditions by a **penalty term**:

$$
\min \frac{1}{\gamma} \| u \|_{L^2(0,T)}^2 + \| y(T, \cdot) \|_{L^2(0,1)}^2 + \| Y \|_{L^2(0,1)}^2, \quad Y'(x) = y_t(T, x).
$$
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- For the error we get

\[
 \frac{\|u_{\gamma} - u_*\|_{L^2(0,T)}}{\|u_*\|_{L^2(0,T)}} = \frac{1}{2k\gamma + 1}.
\]

M. Gugat: Penalty Techniques for State Constrained Optimal Control Problems with the Wave Equation, SICON 2009
Solution of Problem (EC)

- **M. Gugat, G. Leugering, G. Sklyar, Lp optimal boundary control for the wave equation, SICON 2005**

Problem **EC** has a solution \( u \) that is uniquely determined.

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\[
\begin{align*}
    u^*_1(t) &= \begin{cases} 
        \frac{1}{T} \left(-\int_0^{1-t} y_1(s) \, ds + r + y_0(1-t)\right), & t \in (0,1) \\
        \frac{1}{T} \left(-\int_t^{t-1} y_1(s) \, ds + r - y_0(t-1)\right), & t \in (1,2)
    \end{cases}
\end{align*}
\]

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- To make life simple, for numerical purposes, it is sometimes useful to replace the exact end conditions by a **penalty term**:

\[
\min \frac{1}{\gamma} \| u \|^2_{L^2(0,T)} + \| y(T,\cdot) \|^2_{L^2(0,1)} + \| Y \|^2_{L^2(0,1)}, \quad Y'(x) = y_t(T, x).
\]

- For the error we get \[
\frac{\| u^*_\gamma - u^* \|^2_{L^2(0,T)}}{\| u^* \|^2_{L^2(0,T)}} = \frac{1}{2k\gamma+1}.
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*M. Gugat: Penalty Techniques for State Constrained Optimal Control Problems with the Wave Equation, SICON 2009*

- This problem has a solution also for \( T \to 0+ \)!
Example

- Let $y_0(x) = x$, $y_1(x) = 0$. 

We get $r = \int_1^0 \int_t^0 y_1(s) \, ds \, dt = 0$ and the optimal control is the 2-periodic extension of $u^* (t) = \{ 1 \}$ for $t \in (0, 1)$ and $u^* (t) = \{ 1 \}$ for $t \in (1, 2)$. Hence $u^* (t) = 1$, $t \in (0, 2)$. 

Thus if $T > 2$, we have a jump at time $t = 2$: $u(2 -) = -1 \neq 1 = u(2 +)$. 

Hence also for continuous data, the optimal state for Dirichlet control is in general discontinuous. Continuity is an additional constraint, see M. Gugat; Optimal boundary control of a string to rest in finite time with continuous state, ZAMM, 86 (2006) pp. 134-150.
Example

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  u^*_t(t) = \begin{cases} 
  \frac{1}{T} (1 - t), & t \in (0,1) \\
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Example

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    \frac{1}{T} (1 - t), & t \in (1, 2) 
  \end{cases} \]
- Hence $u_*(t) = \frac{1}{T} (1 - t), t \in (0, 2)$.
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- Hence \( u^*_t(t) = \frac{1}{T} (1 - t), \ t \in (0, 2). \)

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- Hence also for continuous data, the optimal state for Dirichlet control is in general discontinuous. Continuity is an additional constraint, see M. Gugat;

Example

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- Hence $u_*(t) = \frac{1}{T} (1 - t), t \in (0, 2)$.

Thus if $T > 2$, we have a **jump** at time $t = 2$: $u(2-) = -\frac{1}{T} \neq \frac{1}{T} = u(2+)$. 

Hence also for continuous data, the optimal state for **Dirichlet** control is in general **discontinuous**. **Continuity** is an **additional constraint**, see M. Gugat;


- To do this, we need $y_0 \in H^1(0, 1)$, $y_1 \in L^2(0, 1)$. 

Continuous states

- The following optimal control problem admits only continuous states:

\[
\begin{align*}
&\text{minimize } \|(u_0', u_1')\|_{2, (0, T)} \text{ subject to} \\
&u_0, u_1 \in H^1[0, T] \\
y(0, x) = y_0(x), \ y_t(0, x) = y_1(x), \ x \in (0, 1) \\
\begin{array}{|c|}
\hline
y(t, 0) = u_0(t), \ y(t, 1) = u_1(t), \ t \in [0, T] \\
\hline
\end{array} \\
y_{tt}(t, x) = y_{xx}(t, x), \ (t, x) \in (0, T) \times (0, 1) \\
y(T, x) = 0, \ y_t(T, x) = 0, \ x \in (0, 1) \\
y_0(0) = u_0(0), \ y_0(1) = u_1(0), \ 0 = u_0(T), \ 0 = u_1(T).
\end{align*}
\]

In the last line you see $C^0$–compatibility conditions.
Continuous states

Let $T = 2$, $y_0(x) = -1$ and $y_1(x) = 0$.

Optimal controls: $u_0(t) = u_1(t) = -1 + t/2$. 

![3D plot showing the space and time intervals with a function graph]
Continuous states

- With Neumann control, Continuity is not an additional constraint!
Continuous states

- With Neumann control, Continuity is not an additional constraint!
- We will come to this later!
  Let us first look at the \( L^\infty \)-case:
  Do we get bang-bang controls?
$L^\infty$-case: Weakness of the bang-bang principle

- $y_0 \in L^\infty(0, 1)$, $y_1 \in W^{-1,\infty}(0, 1)$.

\[
\begin{align*}
\text{(DEC}_\infty \text{)} \quad \min \ & \frac{1}{2} \|u\|_{L^\infty(0,T)}^2 \\
\text{subject to} \quad & y(0, x) = \sin(x\pi), \ y_t(0, x) = 0, \ x \in (0, 1) \\
& y(t, 0) = 0, \ y(t, 1) = u(t), \ t \in (0, T) \\
& y_{tt}(t, x) = y_{xx}(t, x), \ (t, x) \in (0, T) \times (0, 1). \\
& y(T, x) = 0, \ y_t(T, x) = 0, \ x \in (0, 1).
\end{align*}
\]
$L^\infty$-case: Weakness of the bang-bang principle

- $y_0 \in L^\infty(0, 1)$, $y_1 \in W^{-1,\infty}(0, 1)$.

\[
\begin{aligned}
(\text{DEC}^\infty) \quad &\min \frac{1}{2} \|u\|^2_{L^\infty(0,T)} \text{ subject to } \\
&y(0, x) = \sin(x\pi), \; y_t(0, x) = 0, \; x \in (0, 1) \\
&y(t, 0) = 0, \; y(t, 1) = u(t), \; t \in (0, T) \\
&y_{tt}(t, x) = y_{xx}(t, x), \; (t, x) \in (0, T) \times (0, 1).
\end{aligned}
\]

- For $T = 2$ an optimal control is

$$u(t) = \frac{1}{2} \sin(t\pi).$$

All admissible controls have the form $u(t) + \text{const}$, so there is no admissible bang-bang control.
$L^\infty$-case: Weakness of the bang-bang principle

- $y_0 \in L^\infty(0, 1)$, $y_1 \in W^{-1,\infty}(0, 1)$.

\[
\min \frac{1}{2} \|u\|_{L^\infty(0, T)}^2 \text{ subject to }
\begin{align*}
y(0, x) &= \sin(x\pi), \quad y_t(0, x) = 0, \quad x \in (0, 1) \\
y(t, 0) &= 0, \quad y(t, 1) = u(t), \quad t \in (0, T) \\
y_{tt}(t, x) &= y_{xx}(t, x), \quad (t, x) \in (0, T) \times (0, 1). \\
y(T, x) &= 0, \quad y_t(T, x) = 0, \quad x \in (0, 1).
\end{align*}
\]

- For $T = 2$ an optimal control is

$$u(t) = \frac{1}{2} \sin(t\pi).$$

All admissible controls have the form $u(t) + \text{const}$, so there is no admissible bang-bang control.

- Let $T = 2k$. States that can be reached by bang-bang-off controls:

$$y(x, T) \in y_0(x) + \|u\|_{\infty,(0, T)}\{-2k, -2k + 1, \ldots, 2k - 1, \ldots, 2k\}.$$
Now: Neumann boundary control

\[ y_x(t, 1) = u(t) \]
Let $y_0 \in H^1(0, 1), y_1 \in L^2(0, 1)$. 
The Problem of optimal exact control: Neumann

- Let \( y_0 \in H^1(0,1) \), \( y_1 \in L^2(0,1) \).

\[
\begin{align*}
\text{(EC)} & \quad \begin{cases}
\text{minimize } \|u\|_{L^2(0,T)}^2 \\
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& y(T,x) = 0, \quad y_t(T,x) = 0, \quad x \in (0,1).
\end{align*}
\end{cases}
\end{align*}
\]
Méthode des caractéristiques - la clé du problème

Peinture de Maurice Quentin de La Tour.


\[ y(t, x) = \alpha(x + t) + \beta(x - t) \]

From the initial conditions for \( t \in (0, 1) \):

\[ \alpha(t) = \frac{1}{2} \left( y_0(t) + \int_0^t y_1(s) \, ds \right) + C, \quad \beta(t) = \frac{1}{2} \left( y_0(t) - \int_0^t y_1(s) \, ds \right) - C. \]
The optimal Neumann control

**Theorem** [Gugat 2013] Let \( T = K + 1 \) be even.
Theorem [Gugat 2013] Let $T = K + 1$ be even. Then the optimal control is 4–periodic, with

$$u(t) = \begin{cases} \frac{2}{T} \beta'(1 - t) = \frac{1}{T} (y'_0(1 - t) - y_1(1 - t)), & t \in (0, 1) \\ \frac{2}{T} \alpha'(t - 1) = \frac{1}{T} (y'_0(t - 1) + y_1(t - 1)), & t \in (1, 2). \end{cases}$$
The optimal Neumann control

- **Theorem** [Gugat 2013] Let $T = K + 1$ be even.
- Then the optimal control is 4–periodic, with

$$u(t) = \begin{cases} \frac{2}{T} \beta'(1 - t) = \frac{1}{T} (y_0'(1 - t) - y_1(1 - t)), & t \in (0, 1) \\ \frac{2}{T} \alpha'(t - 1) = \frac{1}{T} (y_0'(t - 1) + y_1(t - 1)), & t \in (1, 2). \end{cases}$$

- For $k \in \{0, 1, ..., (K - 1)/2\}$, $t \in (0, 2)$ we have:

$$u(t + 2k) = (-1)^k u(t).$$
Example: Optimal Neumann Control

Let $y_0(x) = 4 \sin(\frac{\pi}{2}x)$, $y_1(x) = 0$. Then $\alpha(x) = \beta(x) = 2 \sin(\frac{\pi}{2}x)$.

We obtain the optimal control

$$u(t) = \begin{cases} 
\frac{2}{T} \pi \cos(\frac{\pi}{2}(1 - t)), & t \in (0, 1); \\
\frac{2}{T} \pi \cos(\frac{\pi}{2}(t - 1)), & t \in (1, 2).
\end{cases}$$

By continuation we get

$$u(t) = \frac{2}{T} \pi \cos\left(\frac{\pi}{2}(t - 1)\right).$$
Example: Minimal Control Time $T = 2$:

Optimal state for the minimal control time $T = 2$:

State $y(t, x)$ and $y_x(t, x)$ with optimal Neumann boundary control, $T = 2$. The state is continuous.
Example: Control time $T = 10$

Optimal state for the control time $T = 10$:

State $y(t, x)$ and $y_x(t, x)$ with optimal Neumann boundary control, $T = 10$. 
Example: Control time \( T = 20 \)

Optimal state for the control time \( T = 20 \):

State \( y(t,x) \) and \( y_x(t,x) \) with optimal \textsc{Neumann} boundary control, \( T = 20 \).
How is the state steered to rest exactly?

- How does the exact control steer the state to rest?

For $T = 2^n$, we have $n$ time intervals of equal length. We have a conservative system, so if the control is switched off, the energy is conserved. Moreover we have a linear system so superposition is possible. Thus it is optimal, to take in each time interval the $n$th part of the energy out of the system. So the system decays like $1, \frac{1}{n-1}, \frac{1}{n-2}, \ldots, \frac{1}{2}, \frac{1}{1}, 0$.

Exponential decay is completely different: Like $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ it never reaches zero!
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So the system decays like $1, \frac{1}{n}, \frac{1}{2n}, \frac{1}{3n}, \ldots, \frac{1}{n}, 0$.

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The open loop control depends on the initial state \((y_0, y_1)\). In general, this state is \textbf{not} known. What happens, if the true initial state is different from \((y_0, y_1)\)?
The open loop control depends on the initial state \((y_0, y_1)\).
In general, this state is **not** known.
What happens, if the true initial state is a different from \((y_0, y_1)\)?

**Example:** \(\tilde{y}_0(x) = 2x\), \(y_1(x) = 0\).
Let $f$ be a real number. This is our feedback parameter.
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Introduce a feedback law (closed loop control) at $x = 1$:

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On the optimization of stabilizing time-varying feedback controls.
Exponential Stability of the System

We consider the **Energy**

\[
E(t) = \frac{1}{2} \int_0^1 (y_x(t, x))^2 + (y_t(t, x))^2 \, dx.
\]
Exponential Stability of the System

- We consider the **Energy**

\[ E(t) = \frac{1}{2} \int_0^1 (y_x(t, x))^2 + (y_t(t, x))^2 \, dx. \]

- For all \( f > 0 \) System **STAB** is *exponentially stable*, that is there exist \( C_1, \mu \in (0, \infty) \) such that

\[ E(t) \leq C_1 E(0) \exp(-\mu t), \ (t \in [0, \infty)). \]
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- For \( f = 1 \) **STAB** satisfies \( y(2, x) = y_t(2, x) = 0 \), for all initial states! (*Komornik, Cox and Zuazua*)
Example: Feedback

- Feedback switched off $f = 0$ (Conservation of energy):

$$y(t, x) \text{ with } f = 0, \text{ Zero control}$$
Example: Feedback

- Feedback switched off \( f = 0 \) (Conservation of energy):

\[
y(t, x) \text{ with } f = 0, \text{ Zero control}
\]

- Feedback with \( f = 1 \):

\[
\text{State } y(t, x) \text{ with feedback for } y_0 = 4 \sin(\pi x/2), \ y_1 = 0
\]
Example: Combination $y_x = -y_t + u$

- **Example** State for the control time $T = 10$ with $f = 1$ and the optimal control from (EC) for $y_0 = 4\sin\left(\frac{\pi}{2}x\right)$, $y_1(x) = 0$ with $\tilde{y}_0(x) = 2x$, $y_1(x) = 0$.

$$y_x(t, 1) = -y_t(t, 1) + u(t)$$

state $y(t, x)$ with **Neumann**-boundary control $y_x = -y_t + u_0$, $T = 10$
Example: Combination $y_x = -y_t + u$

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state $y(t, x)$ with **Neumann**-boundary control $y_x = -y_t + u_0$, $T = 10$

- Can we do better?
Optimized Feedback

To guarantee stability of the system also if an optimal control is used, we look at *optimized Feedback*. Let a feedback parameter $f \geq 0$ be given.

\[
\begin{align*}
\minimize_{u \in L_2(0, T)} & \quad \|y_{x}(t, 1)\|_2, \\
\text{subject to} & \quad y(0, x) = y_0(x), \\
& \quad y_t(0, x) = y_1(x), \\
& \quad x \in (0, 1), \\
& \quad y(t, 0) = 0, \\
& \quad y_{xx}(t, 1) = -fy_t(t, 1) + u(t), \\
& \quad t \in (0, T), \\
& \quad y_{tt}(t, x) = y_{xx}(t, x), \\
& \quad (t, x) \in (0, T) \times (0, 1), \\
& \quad y(T, x) = 0, \\
& \quad y_t(T, x) = 0,
\end{align*}
\]

For $f = 0$ we get again (EC). Here the optimal control depends on $y_0$, $y_1$ and $f$. Due to the objective function, the optimal value is independent of $f$. After time $T$ the control $u$ is switched off: $u(t) = 0$ for $t > T$. This yields exponential stability of the system.
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\[
\begin{aligned}
\text{minimize}_{u \in L^2(0, T)} & \| y_x(t, 1) \|_{L^2(0, T)}^2 \\
\text{subject to} & \\
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y(t, 0) = 0, & \ y_x(t, 1) = -fy_t(t, 1) + u(t), \ t \in (0, T) \\
y_{tt}(t, x) = y_{xx}(t, x), & \ (t, x) \in (0, T) \times (0, 1) \\
y(T, x) = 0, & \ y_t(T, x) = 0, \ x \in (0, 1). 
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y(t,0) = 0, \ y_x(t,1) = -fy_t(t,1) + u(t), \ t \in (0,T) \\
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\[
\begin{align*}
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\text{(OF)} \left\{ \begin{array}{rl}
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\end{array} \right.
\]

- For \( f = 0 \) we get again **(EC)**.
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The optimal control

**Theorem** [Gugat 2013] Let \( T = K + 1 \) be even.
The optimal control

- **Theorem** [Gugat 2013] Let $T = K + 1$ be even.
- Then the optimal control for $k \in \{0, 1, \ldots, (K - 1)/2\}$, $t \in (0, 2)$ is:

$$u(t + 2k) = \begin{cases} 
\frac{(-1)^k}{T} [1 - f (T - (2k + 1))] 2\beta'(1 - t), & t \in (0, 1) \\
\frac{(-1)^k}{T} [1 - f (T - (2k + 1))] 2\alpha'(t - 1), & t \in (1, 2) 
\end{cases}$$

For the minimal control time $T = 2$ we get

$$u(t) = \begin{cases} 
[1 - f] \beta'(1 - t), & t \in (0, 1) \\
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In particular for $f = 1$ we get $u(t) = 0$.

In this case the feedback law already yields the optimal control!
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In particular for $f = 1$ we get $u(t) = 0$.

**In this case the feedback law already yields the optimal control!**
Example: Minimal Control Time \( T = 2 \)

- State \( y \) for \( f = 0 \) and the optimal control from (EC) for \( y_0 = 4 \sin \left( \frac{\pi}{2} x \right) \), \( y_1(x) = 0 \) with \( \dot{y}_0(x) = 2x \), \( y_1(x) = 0 \)
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- State $y$ with $f = 1$ and the optimal control $u = 0$ from (OF)
Example: Minimal Control Time $T = 2$

- state $y$ with $f = \frac{1}{2}$ and the optimal control from (OF)
Example: Minimal Control Time $T = 2$

- state $y$ with $f = \frac{1}{2}$ and the optimal control from (OF)

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- State $y$ with $f = 2$ and the optimal control from (OF)

- With initial state $y_0 = 4 \sin(\frac{\pi}{2}x)$, $y_1(x) = 0$ the picture is independent of $f$!
The optimal control for $f = 1$ and general $T$

Let $T = K + 1$ be even. For $k \in \{0, 1, \ldots, (K - 1)/2\}$, $t \in (0, 2)$ we have the optimal control

$$u(t + 2k) = \begin{cases} \displaystyle (-1)^k \frac{2}{T} [1 - f (T - (2k + 1))] \beta'(1 - t), & t \in (0, 1) \\ \displaystyle (-1)^k \frac{2}{T} [1 - f (T - (2k + 1))] \alpha'(t - 1), & t \in (1, 2). \end{cases}$$

Hence for $f = 1$ the optimal control satisfies $u(t) \big|_{[T - 2, T]} = 0$.
The optimal control for \( f = 1 \) and general \( T \)

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\end{cases}
\]

- For \( 2k = T - 2 \) this implies

\[
u(t + T - 2) = \begin{cases} 
(-1)^k \frac{2}{T} [1 - f] \beta'(1 - t), & t \in (0, 1) \\
(-1)^k \frac{2}{T} [1 - f] \alpha'(t - 1), & t \in (1, 2).
\end{cases}
\]
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  \end{cases}$$

- Hence for $f = 1$ the optimal control satisfies
  
  $$u(t)|_{[T-2, T]} = 0.$$  

With $f = 1$ and $u$ with all initial states at time $T$ the zero state is reached exactly!
Example: Control Time $T = 20$

- The optimal control from (OF) for $T = 20$ and $f = 1$
Example: Control Time $T = 20$

- The optimal control from $\text{(OF)}$ for $T = 20$ and $f = 1$

- The generated state with this control and initial state $\tilde{y}_0(x) = 2x$, $y_1(x) = 0$
Example: Control Time $T = 20$

- State $y$ with $\tilde{y}_0(x) = 2x$, $y_1(x) = 0$, $f = \frac{1}{2}$
Example: Control Time $T = 20$

- State $y$ with $\tilde{y}_0(x) = 2x$, $y_1(x) = 0$, $f = \frac{1}{2}$

- State $y$ with $\tilde{y}_0(x) = 2x$, $y_1(x) = 0$, $f = 0$ (Feedback control switched off)
Korteweg-de Vries

*Cerpa and Coron 2013:*
Feedback stabilization with **exponential stability** with a suitably chosen kernel $k$ for initial state with a sufficiently small $L^2$-norm:

\[
\begin{align*}
  y(0, x) &= y_0(x) \in L^2(0, 1) \\
  y_t + y_x + y_{xxx} + yy_x &= 0 \\
  y(t, 1) &= 0 \\
  y_x(t, 1) &= 0 \\
  y(t, 0) &= \int_0^1 k(0, z)y(t, z) \, dz
\end{align*}
\]

Method: Backstepping.
For the integral feedback, the information in $y(t, z)$, $z \in (0, 1)$ is used.
Time-varying Feedback Control: Korteweg-de Vries

The System is locally exactly controllable to zero.

*L. Rosier: Control of the surface of a fluid by a wavemaker, ESAIM:COCV 10 (2004)*


Optimized Feedback stabilization (with respect to $y_0$)

\[
\begin{aligned}
\inf_u \int_0^T y(t,0)^2 \, dt \quad &\text{subject to} \\
y(0,x) = y_0(x) \in L^2(0,1) \quad &\text{small} \\
y_t + y_x + y_{xxx} + y y_x = 0 \\
y(t,1) = 0 \\
y_x(t,1) = 0 \\
y(t,0) = \int_0^1 k(0,z)y(t,z) \, dz + u(t) \\
y(T,x) = 0.
\end{aligned}
\]

If the initial state $y_0$ is known exactly (which is never the case), this gives exact control to zero. Otherwise exponential stability (with $u(t) = 0$ for $t \geq T$).
Optimized Feedback Control: Korteweg-de Vries

- **Step 1:** From *(Glass, Guerrero)*: Determine an exact control \( v \) that is \( \varepsilon \)-optimal/feasible for

\[
\begin{aligned}
\inf_v \int_0^T (v(t))^2 \, dt \quad &\text{subject to} \\
y(0, x) = y_0(x) \in L^2(0, 1) \text{ small} \\
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y(t, 1) = 0 \\
y_x(t, 1) = 0 \\
y(t, 0) = v(t) \\
y(T, x) = 0.
\end{aligned}
\]

- **Step 2:** Set

\[
u(t) = v(t) - \int_0^1 k(0, z)y_{y_0, v}(t, z) \, dz
\]

where \( v(t) = 0 \) for \( t > T \).

Then by *Cerpa, Coron* the system with control

\[
y(t, 0) = \int_0^1 k(0, z)y(t, z) \, dz + u(t)
\]

is exponentially stable and if \( y(0, \cdot) = y_0 \), it is steered to zero at time \( T \).
Stabilization of semilinear wave equations
Semilinear wave equation

For initial data $y_0 \in L^\infty(0, 1), \ y_1 \in W^{-1, \infty}(0, 1)$ consider a system with the nonlinear wave equation (includes telegraph equation, waterhammer eqn.)

$$y_{tt}(t, x) - 2g_y(x, y(t, x)) \ y_t(t, x) = y_{xx}(t, x) \quad (1)$$

where

$$|g_y(x, y)| \leq w \quad (2)$$

with the boundary conditions

$$y(t, 0) = 0, \quad y_x(t, 1) = -y_t(t, 1), \quad t \in (0, T).$$
Semilinear wave equation

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- For $w < 1/20$, $\|y(t, \cdot)\|_{L^\infty(0,1)}$ decays exponentially with rate

$$\mu = |\ln(20w)|.$$
Semilinear wave equation

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Thus the decay rate becomes arbitrarily large for $w \to 0$. 

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$$y_{tt}(t, x) - 2g_y(x, y(t, x))y_t(t, x) = y_{xx}(t, x)$$  \hspace{1cm} (1)

where

$$|g_y(x, y)| \leq w$$  \hspace{1cm} (2)

with the boundary conditions

$$y(t, 0) = 0, \quad y_x(t, 1) = -y_t(t, 1), \quad t \in (0, T).$$

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$$\mu = |\ln(20w)|.$$

Thus the decay rate becomes arbitrarily large for $w \to 0$.

- Consider now stability of ISS type (see Mazenc, Prieur, MCRF 1, 2011).
Semilinear wave equation: ISS stability

For initial data $y_0 \in L^\infty(0, 1), y_1 \in W^{-1,\infty}(0, 1)$ consider a perturbed system

$$y_{tt}(t, x) - 2g_y(x, y(t, x)) y_t(t, x) = y_{xx}(t, x) + D(t, x)$$

with continuous uniformly bounded $D$ and $(|g_y(x, y)| \leq w)$ with the boundary feedback $y(t, 0) = 0, y_x(t, 1) = -y_t(t, 1)$.
Semilinear wave equation: ISS stability

- For initial data $y_0 \in L^\infty(0,1), y_1 \in W^{-1,\infty}(0,1)$ consider a **perturbed** system

\[
y_{tt}(t,x) - 2g_y(x, y(t,x)) y_t(t,x) = y_{xx}(t,x) + D(t,x) \tag{3}
\]

with continuous uniformly bounded $D$ and $(|g_y(x, y)| \leq w)$ with the boundary feedback $y(t, 0) = 0$, $y_x(t, 1) = -y_t(t, 1)$

**Related:** For the linear wave equation $g = 0$ in *Gugat, Tucsnak, Sigalotti: Robustness analysis for the boundary control of the string equation*, 2007) the influence of the position coefficient $b$ in the feedback

\[
y_x(t, 1) = -fy_t(t, 1) - by(t, 1)
\]
on the robustness is studied:

In some cases with $b > 0$, the system is more robust with respect to $D$ than for $b = 0$. 
Semilinear wave equation: ISS stability ($L^\infty$)

Let $\delta$ solve the linear closed loop system $\delta_{tt} = \delta_{xx} + D$, $\delta(0, x) = \delta_t(0, x) = 0$, $\delta(t, 0) = 0$, $\delta_x(t, 1) = -\delta_t(t, 1)$.
Semilinear wave equation: ISS stability \((L^\infty)\)

- Let \(\delta\) solve the linear closed loop system \(\delta_{tt} = \delta_{xx} + D\), \(\delta(0, x) = \delta_t(0, x) = 0\), \(\delta(t, 0) = 0\), \(\delta_x(t, 1) = -\delta_t(t, 1)\).

- Due to the feedback law, the solution \(\delta\) has limited memory with respect to \(D\): \(\delta(t, x)\) only depends on the data \(D(s, x)|_{s \in (t-4, t)}\)!
Semilinear wave equation: ISS stability \((L^\infty)\)

- Let \(\delta\) solve the linear closed loop system \(\delta_{tt} = \delta_{xx} + D\), 
  \[ \delta(0, x) = \delta_t(0, x) = 0, \quad \delta(t, 0) = 0, \quad \delta_x(t, 1) = -\delta_t(t, 1). \]

- Due to the feedback law, the solution \(\delta\) has limited memory with respect to \(D\): \(\delta(t, x)\) only depends on the data \(D(s, x)|_{s \in (t-4, t)}\)!

  This implies in particular, that

  \[
  \text{ess sup}_t \|\delta(t, \cdot)\|_{L^\infty(0,1)}
  \]

  remains bounded if \(D\) is uniformly bounded.
Semilinear wave equation: ISS stability ($L^\infty$)

- Let $\delta$ solve the linear closed loop system $\delta_{tt} = \delta_{xx} + D$,
  $\delta(0, x) = \delta_t(0, x) = 0$, $\delta(t, 0) = 0$, $\delta_x(t, 1) = -\delta_t(t, 1)$.
- Due to the feedback law, the solution $\delta$ has limited memory with respect to $D$: $\delta(t, x)$ only depends on the data $D(s, x)|_{s \in (t-4, t)}$.
  This implies in particular, that

$$\text{ess sup}_t \| \delta(t, \cdot) \|_{L^\infty(0,1)}$$

remains bounded if $D$ is uniformly bounded.
- We get the **robustness estimate** (for $k \in \{1, 2, 3, \ldots\}$)

$$\text{ess sup}_{s \in [2k, 2k+2]} \| y(s, \cdot) \|_{L^\infty(0,1)}$$

$$\leq (20w)^k \text{ess sup}_{s \in [0,2]} \| y(s, \cdot) \|_{L^\infty(0,1)} + \frac{1 - (20w)^k}{1 - 20w} \text{ess sup}_{t \in [0,2k+2]} \| \delta(t, \cdot) \|_{L^\infty(0,1)}.$$
Stabilization of quasilinear wave equations
Quasilinear wave equation

In a paper with Leugering, Wang, Tamasoiu, we have studied the pde

\[ \ddot{\tilde{u}} + 2\tilde{u}\dot{\tilde{u}}_{x} - (a^2 - \tilde{u}^2)\tilde{u}_{xx} = \tilde{F}(\tilde{u}, \tilde{u}_{x}, \tilde{u}_{t}). \]  (4)

with Neumann boundary control.
Quasilinear wave equation

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  \] (4)

  with Neumann boundary control.

- To stabilize the system governed by the quasilinear wave equation (4) locally around a stationary state \(\bar{u}(x)\), we use boundary feedback given by
  \[
  x = 0 : \tilde{u}_x = \bar{u}_x(0) + k\tilde{u}_t, \\
  x = L : \tilde{u} = \bar{u}(L),
  \]

  with a feedback parameter \( k \in (0, \infty) \).
Quasilinear wave equation

- In a paper with Leugering, Wang, Tamasoiu, we have studied the pde

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\ddot{u} + 2\dot{u}\dot{u}_x - (a^2 - \ddot{u})u_{xx} = \tilde{F}(\ddot{u}, \dot{u}_x, \dot{u}_t).
\] (4)

with Neumann boundary control.

- To stabilize the system governed by the quasilinear wave equation (4) locally around a stationary state \(\bar{u}(x)\), we use boundary feedback given by

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\begin{align*}
  x = 0 & : \ddot{u}_x = \bar{u}_x(0) + k\ddot{u}_t, \\
  x = L & : \ddot{u} = \bar{u}(L),
\end{align*}
\]

with a feedback parameter \(k \in (0, \infty)\).

- For certain \(k > 0\), \(C^2\) solutions \(u = \ddot{u} - \bar{u}\) of the system decay exponentially:

\[
\|(u(t, \cdot), u_t(t, \cdot))\|_{H^2(0,L) \times H^1(0,L)} \leq \eta_1 \|(u(0, \cdot), u_t(0, \cdot))\|_{H^2(0,L) \times H^1(0,L)} \exp(-\bar{\mu}t)
\]
Quasilinear wave equation

- The analysis is based upon the Lyapunov function:

\[
E(t) = \int_0^L h_1(x) \left[ \left( (a^2 - \tilde{u}^2)u_x^2 + u_t^2 \right) + \left( (a^2 - \tilde{u}^2)u_{xx}^2 + u_{tx}^2 \right) \right] \\
-2h_2(x) \left[ \left( \tilde{u} u_x^2 + u_t u_x \right) + \left( \tilde{u} u_{xx}^2 + u_{tx} u_{xx} \right) \right] \, dx
\]

with the exponential weights \( h_1(x) = ke^{-\mu_1 x}, \ h_2(x) = e^{-\mu_2 x}. \)
Quasilinear wave equation

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\]

with the exponential weights \( h_1(x) = ke^{-\mu_1 x}, \ h_2(x) = e^{-\mu_2 x}. \)

- If \( \max_{(t,x)} |u(t,x)| \) is sufficiently small, the numbers \( k, \mu_1, \mu_2 \) can be chosen such that

\[
\| u_x \|^2_{H^1(0,L)} + \| u_t \|^2_{H^1(0,L)} \leq C_0 \, E(t).
\]
Conclusion

- Problems of optimal exact control provide optimal controls that should be combined with a feedback law to enhance stability.
Conclusion

- Problems of optimal exact control provide optimal controls that should be combined with a feedback law to enhance stability.
- In engineering practice, we often have nonlinear dynamics on networks:

![Network Image]

There are lots of open questions!
Merci!

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