The Hautus test for infinite dimensional systems

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Plan of the talk

• Background on exact observability

• The Hautus test

• High and low frequency decomposition

• From the wave to the Schrödinger equation

• A problem of fluid structure interaction
Background on exact observability
Let $X$ and $Y$ be Hilbert spaces, $A : \mathcal{D}(A) \to X$ et $C \in \mathcal{L}(\mathcal{D}(A), Y)$.

(1) \[ \dot{w}(t) =Aw(t), \; y(t) = Cw(t). \]

Assume that $A$ generates a $C^0$ semigroup, denoted $\mathbb{T}$, in $X$.

**Definition 1.** $C \in \mathcal{L}(\mathcal{D}(A), Y)$ is an **admissible observation operator for** $\mathbb{T}$ if there exist $\tau > 0$, $k_\tau > 0$ such that

\[ k_\tau^2 \int_0^\tau \|C\mathbb{T}_t z_o\|_Y^2 dt \leq \|z_0\|_X^2 \quad \forall \; z_0 \in \mathcal{D}(A). \]

**Definition 2.** Let $\tau > 0$ and let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an admissible observation operator for $\mathbb{T}$. The pair $(A, C)$ is **exactly observable in time $\tau$** if there exists $K_\tau > 0$ such that

\[ K_\tau^2 \int_0^\tau \|C\mathbb{T}_t z_0\|_Y^2 dt \geq \|z_0\|_X^2 \quad \forall \; z_0 \in \mathcal{D}(A). \]

(1)
The duality observability-controllability

Proposition 1 (Dolecki and Russell, 1973). The pair \((A, C)\) is exactly observable in time \(\tau\) iff the pair \((A^*, C^*)\) is exactly controllable in time \(\tau\) (this means that for every \(z_0 \in X\) there exists \(u \in L^2([0, \tau]; U)\) s.t.

\[
\dot{z}(t) = A^*z(t) + C^*u(t), \quad z(0) = 0, \quad z(\tau) = z_0.
\]

Moreover, the control cost for \((A^*, C^*)\) coincides with the observation cost of \((A, C)\).
The wave equation with Neumann boundary observation

Theorem 1. (Bardos, Lebeau, Rauch, 1992).
Assume that $\Gamma \subset \partial \Omega$ and that $\tau > 0$ and consider the system

$$
\begin{align*}
\dot{w} - \Delta w &= 0 \quad (x \in \Omega, \ t \geq 0), \\
w &= 0 \quad (x \in \partial \Omega, \ t \geq 0).
\end{align*}
$$

Then the following conditions are equivalent:

1. There exists $K_{\tau, \Gamma} > 0$ s.t.

$$
K_{\tau, \Gamma}^2 \int_0^\tau \int_{\Gamma} \left| \frac{\partial w}{\partial \nu} \right|^2 \, d\Gamma \geq \int_\Omega \left( |\nabla w(0)|^2 + \dot{w}(0)^2 \right) \, dx d\tau,
$$

for every solution $w$.

2. $\Gamma$ satisfies the geometric optics condition (also called Bardos, Lebeau, Rauch condition).
The Hautus test
The finite dimensional case

Let $U, X$ and $Y$ be finite-dimensional inner product spaces with $\dim X = n$. Let $A, B, C$ be linear operators such that

$$A : X \rightarrow X, \quad B : U \rightarrow X, \quad C : X \rightarrow Y.$$

**Theorem 2. (Hautus)** The following assertions are equivalent:

1. The pair $(A, C')$ is observable;

2. $\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \forall \lambda \in \sigma(A)$;

3. $C\phi \neq 0$ for every eigenvector $\phi$ of $A$. 
References


The Russell-Weiss necessary condition

Let $X$ and $Y$ be Hilbert spaces.
Let $T$ be an exponentially stable semigroup on $X$, with generator $A$.
Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $T$.
Denote $C_\infty = \{ s \in \mathbb{C} \mid \text{Re}\, s < 0 \}$.

Theorem 3. (Russell and Weiss, 1994).
If $(A, C)$ is exactly observable in infinite time, then there is an $m > 0$ such that for every $s \in C_\infty$ and every $z \in \mathcal{D}(A)$,

$$\frac{1}{|\text{Re}\, s|^2} \|(sI - A)z\|^2 + \frac{1}{|\text{Re}\, s|} \|Cz\|^2 \geq m \cdot \|z\|^2.$$
Sketch of the proof (I)

Let $\Psi : X \to L^2_{\text{loc}}([0, \infty); Y)$ be defined by

$$(\Psi z_0)(t) = C T_t z_0 \quad \forall \, z_0 \in \mathcal{D}(A), \, t \geq 0.$$ 

We know that $\Psi \in \mathcal{L}(X, L^2([0, \infty), Y)$ and that there exists $\delta > 0$ such that

$$\|\Psi z_0\|_{L^2([0, \infty), Y)} \geq \delta \|z_0\| \quad \forall \, z_0 \in \mathcal{D}(A).$$

It suffices to show that there exists $\mu > 0$ s.t. for all $s \in \mathbb{C}_-$ and $z \in \mathcal{D}(A),$

$$\frac{1}{|\text{Re} \, s|^2} \|(sI - A)z\|^2 + \frac{1}{|\text{Re} \, s|} \|Cz\|^2 \geq \mu \cdot \|\Psi z\|_{L^2}^2.$$
Sketch of the proof (II)

Denote $q = (A - sI)z$, and $\xi : [0, \infty) \to X$, $\xi(t) = T_t z$. Then

$$\dot{\xi}(t) = T_t Az = T_t (sz + q) = s\xi(t) + T_t q,$$

whence

$$\begin{align*}
(\Psi z)(t) &= C\xi(t) = e^{st} Cz + \int_0^t e^{s(t-\sigma)} C T_\sigma q d\sigma = e^{st} Cz + (e_s * \Psi q)(t),
\end{align*}$$

where $e_s(t) = e^{st}$. We obtain that

$$\begin{align*}
\|\Psi z\|_{L^2} &\leq \|e_s\|_{L^2} \cdot \|Cz\| + \|e_s\|_{L^1} \cdot \||\Psi q||_{L^2} \\
&\leq \frac{1}{\sqrt{2|\text{Re} s|}} \|Cz\| \\
+ \frac{1}{|\text{Re} s|} \|\Psi\| \cdot \||q|| &\leq \left(\frac{1}{2} + \|\Psi\|^2\right) \left[\frac{1}{|\text{Re} s|^2} \|q\|^2 + \frac{1}{|\text{Re} s|} \|Cz\|^2\right].
\end{align*}$$
The skew-adjoint case

Let $\mathbb{T}$ be a group of isometries on $X$, with generator $A$.
Let $C \in \mathcal{L}(X, Y)$ be an admissible observation operator for $\mathbb{T}$.

Theorem 4. (Miller (2005))
The pair $(A, C)$ is exactly observable iff there exists $M, m > 0$ s.t.

$$M^2 \|(i\omega I - A)z_0\|^2 + m^2 \|Cz_0\|^2 \geq \|z_0\|^2 \quad \forall \omega \in \mathbb{R}, \ z_0 \in \mathcal{D}(A).$$

If the above estimate holds then
$(A, C)$ is exactly observable in any time $\tau > M\pi$. 
Proof of the necessity

If \((A, C)\) is exactly observable then \(A - I\) generates an exponentially stable semigroup and \((A - I, C)\) is exactly observable. Applying Theorem 3 with \(\text{Re} s = -1\), we obtain that there exists \(m_0 > 0\) such that

\[
\| (i\omega I - A) z_0 \|^2 + \| C z_0 \|^2 \geq m_0 \cdot \| z_0 \|^2,
\]

for all \(z_0 \in \mathcal{D}(A)\) and for all \(\omega \in \mathbb{R}\).

Remark. The proof of the sufficiency is slightly harder.
High and low frequency decomposition
First decomposition result

Proposition 2. (Bardos, Lebeau, Rauch (1992), Tucsnak, Weiss (2000))
Assume that there exists an orthonormal basis $(\phi_k)_{k \in \mathbb{N}}$ formed of
eigenvectors of $A$ and the corresponding eigenvalues $\lambda_k$ satisfy
$\lim |\lambda_k| = \infty$. Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation
operator for $T$. For some $J\alpha > 0$ denote

$$E_\alpha = \text{span} \{ \phi_k \mid |\mu_k| \leq \alpha \}$$

and let $A_\alpha$ be the part of $A$ in $E_\alpha$. Let $C_\alpha$ be the restriction
of $C_\alpha$ to $\mathcal{D}(A_\alpha)$. Assume that $(A_\alpha, C_\alpha)$ is exactly observable in time
$\tau_0 > 0$ and that $C\phi \neq 0$ for every eigenvector $\phi$ of $A$. Then
$(A, C)$ is exactly observable in any time $\tau > \tau_0$. 
Application to the Hautus test (I)

Notation:

- $A : D(A) \to X$ with compact resolvents et $A = -A^*$.
- $(\phi_k)_{k \in \mathbb{N}}$ an orthonormal basis of eigenvectors of $A$
- $i\mu_k$ is the eigenvalue associated to $\phi_k$.
- For $\alpha > 0$ we denote $E_\alpha = \text{span} \{ \phi_k \mid |\mu_k| < \alpha \}^\perp$.
- $C \in \mathcal{L}(D(A), Y)$ is an admissible observation operator.
Application to the Hautus test (II)

Proposition 3. Assume that

1. There exist $M, m, \alpha > 0$ s.t. for every $\omega \in \mathbb{R}$ with $|\omega| > \alpha$, we have

$$M^2 \|(i\omega I - A)z_0\|^2 + m^2 \|Cz_0\|^2 \geq \|z_0\|^2 \quad \forall z_0 \in E_\alpha \cap D(A),$$

2. $C\phi \neq 0$ for every eigenvector $\phi$ of $A$.

Then $(A, C)$ is exactly observable in any time $\tau > M\pi$.

Proof. Just combine Theorem 4 and Proposition 3. □
From the wave to the Schrödinger equation
Notation

- $A_0 : \mathcal{D}(A_0) \to H$ with compact resolvents and $A_0 = A_0^* > 0$.
- $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$ with the norm $\|w\|_{\frac{1}{2}} = \|A_0^{\frac{1}{2}}w\|$
- $X = H_{\frac{1}{2}} \times H$, $A : \mathcal{D}(A) \to X$, $A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}$
- $C_1 \in \mathcal{L}(H_1, Y)$, $C \in \mathcal{L}(X_1, Y)$, $C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$. 
Proposition 4 (Miller (2005), Tucsnak and Weiss(2009)).
If \((A, C)\) is exactly observable then \((iA_0, C_1)\), with the state space \(H_{\frac{1}{2}}\), is exactly observable in any time \(\tau > 0\).

Proof. By Theorem 4 there exist \(M, m > 0\) s.t.

\[
M^2 \|(i\sqrt{\omega}I - A)\tilde{z}\|^2 + m^2 \|C\tilde{z}\|^2 \geq \|\tilde{z}\|^2 \quad \forall \omega > 0, \tilde{z} \in \mathcal{D}(A)
\]

If we choose \(\tilde{z} = \begin{bmatrix} \frac{1}{2} & 1 & iA_0^2 \end{bmatrix} \), with \(z \in \mathcal{D}(A_0^{\frac{3}{2}})\), we obtain

\[
\frac{M^2}{\omega} \|(\omega I - A_0)z\|_{\frac{1}{2}}^2 + \frac{m^2}{2} \|C_1z\|^2 \geq \|z\|_{\frac{1}{2}}^2 \quad \forall \omega > 0, z \in \mathcal{D}(A_0^{\frac{3}{2}}).
\]

We conclude using Proposition 3.
The Schrödinger equation with Neumann boundary observation

**Theorem.** Assume that \( \Gamma \subset \partial \Omega \) satisfies the Bardos-Lebeau-Rauch condition. Then for every \( \tau > 0 \), there exists \( K_{\tau, \Gamma} > 0 \) s.t.

\[
K_{\tau, \Gamma}^2 \int_0^\tau \int_\Gamma \left| \frac{\partial w}{\partial \nu} \right|^2 d\Gamma \geq \int_\Omega |\nabla w(0)|^2 dx dt,
\]

for every solution \( w \).

\[
\begin{cases}
\dot{w} + i\Delta w = 0 \quad (x \in \Omega, \ t \geq 0), \\
w = 0 \quad (x \in \partial \Omega, \ t \geq 0).
\end{cases}
\]

**Remark.** The above result has been obtained by micro-local analysis by Lebeau in 1992. If \( \Omega \) is a rectangle then \( \Gamma \) may be arbitrarily small (Tenenbaum and Tucsnak (2009)).
A problem of fluid structure interaction
Statement of the problem

\[ (CP) \]

\[
\begin{align*}
\ddot{w} - \Delta w &= u\chi_\Omega \\
w &= 0 \\
\frac{\partial w}{\partial n} &= \dot{s} \cdot n \\
w(0) &= f \quad \text{and} \quad \dot{w}(0) = g \\
\ddot{s} + s &= -\int_{\Gamma} \dot{w} n \\
s(0) &= l \quad \text{and} \quad \dot{s}(0) = k
\end{align*}
\]

in \( Q \), \( \Sigma_e \), \( \Sigma \), \( \Omega \), \( (0, T) \), \( \mathbb{R}^2 \).
The controllability result (I)

Proposition 5. (Tucsnak and Vanninathan (2009))
Assume that there exists $\varepsilon > 0$ such that $\mathcal{O} \supset N_\varepsilon \cap \Omega$, where

$$N_\varepsilon = \{ x \in \mathbb{R}^2 \mid d(x, \partial \Omega) < \varepsilon \}.$$  

Then the system (CP) is exactly controllable.

Proof.
First step (see Raymond an Vaninnathan (2005)):
Write (CP) as $\dot{z} = Az + Bu$ with $A = -A^*$.  
Second step:
Show that $B^* \phi \neq 0$ for every eigenvector $\phi$ of $A$.
This is done by Holmgren’s uniqueness theorem.
The controllability result (II)

Third step:
Assume that there exist real sequences \((M_p)_{p \geq 1}, (\omega_p)_{p \geq 1}\)
and the sequence \((\phi_p)_{p \geq 1}\) in \(\mathcal{D}(A)\) such that

\[
\lim_{p \to \infty} M_p = \infty, \quad |\omega_p| > M_p, \quad \|\phi_p\|_X = 1 \quad (p \geq 1).
\]

\[
\phi_p \in E_{M_p} \cap \mathcal{D}(A) \quad (p \geq 1), \quad \lim_{p \to \infty} (i\omega_p \dot{\phi}_p - A\phi_p) = 0 \quad \text{in} \quad X,
\]

\[
B^* \phi_p \to 0 \quad \text{in} \quad U.
\]

After some multiplier calculations we obtain that the problem
\[
\begin{aligned}
\ddot{w} - \Delta w &= u \chi_\Omega \\
w &= 0 \\
\frac{\partial w}{\partial n} &= 0 \\
w(0) &= f \quad \text{and} \quad \dot{w}(0) = g
\end{aligned}
\]

is not exactly controllable, which is absurd.