Backstepping methods for boundary stabilization of 1-D hyperbolic balance laws

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Groupe de travail : Contrôle, September 18, 2015, Paris, France

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Outline

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Introduction

Our hyperbolic balance laws is

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \quad (t,x) \in [0,T] \times [0,L], \tag{1.1}$$

where,

- $u = (u_1, \ldots, u_n)^T$ is a vector function of (t,x);
- $A(u)$ has n real eigenvalues $\lambda_i(u)$ $(i = 1, \cdots, n)$ and a complete set of left (resp. right) eigenvectors $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{1i}(u), \cdots, l_{ni}(u))^T, (i = 1, \cdots, n)$):

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \text{ (resp. } A(u)r_i(u) = \lambda_i(u)r_i(u). \text{)} \tag{1.2}$$

- $F(u) = (f_1(u), \cdots, f_n(u))^T$ is a given vector function of u with $F(0) = 0$.

Remark

In general, we call the following systems

$$\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = F(u) \tag{1.3}$$

to be hyperbolic balance laws, where the flux $g := (g_1, \cdots, g_n)$ is a vector function of u. Obviously, system (1.3) can be written in the quasilinear form as (1.1) with the Jacobian matrix

$$A(u) := D(g(u)). \tag{1.4}$$

Introduction

Many physical models are governed by linear and quasilinear hyperbolic balance laws, for example:

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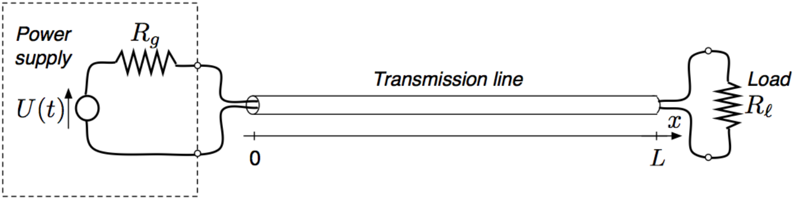
- The telegrapher equations (Heaviside, O. (1892))

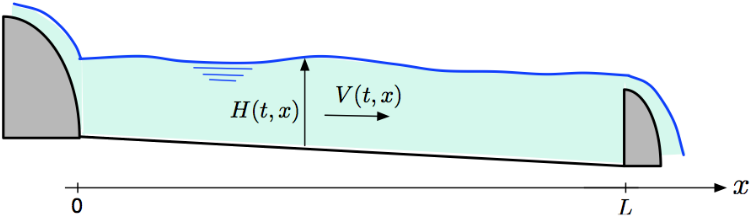
$$\partial_t \begin{pmatrix} I \\ V \end{pmatrix} + \partial_x \begin{pmatrix} -L_e^{-1} V \\ -C_e^{-1} I \end{pmatrix} = - \begin{pmatrix} R_e L_e^{-1} I \\ G_e C_e^{-1} V \end{pmatrix}.$$



Introduction

- The Saint-Venant equations (Barré de Saint-Venant (1871))

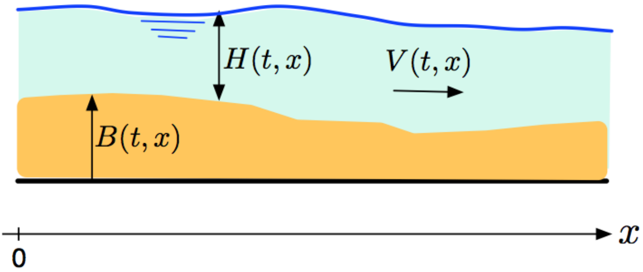
$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(HV) = 0,$$
$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}\left(\frac{V^2}{2} + gH\right) = gS_b - CV^2H^{-1}.$$



Introduction

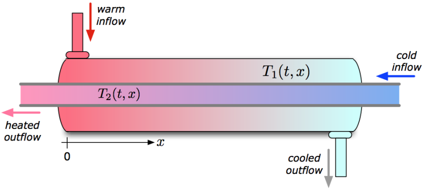
- The Saint-Venant-Exner equations (Hudson-Sweby (2003))

$$
\begin{aligned}
&\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} + H\frac{\partial V}{\partial x} = 0,\\
&\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial H}{\partial x} + g\frac{\partial B}{\partial x} = gS_b - C\frac{V^2}{H},\\
&\frac{\partial B}{\partial t} + aV^2\frac{\partial V}{\partial x} = 0.
\end{aligned}
$$



- Heat exchangers (G.Bastin-J.-M.Coron (2016))

$$
\begin{aligned}
&\partial_t H_1 + V_1 \partial_x H_1 + \frac{c^2}{g}\partial_x V_1 = 0\\
&\partial_t V_1 + \partial_x (g H_1 + \frac{V_1^2}{2}) + \frac{C}{2d} V_1 |V_1| = 0\\
&\partial_t T_1 + \partial_x (V_1 T_1) - k_1 (T_1 - T_2) - k_0 (T_1 - T_e) = 0\\
&\partial_t H_2 + V_2 \partial_x H_2 + \frac{c^2}{g}\partial_x V_2 = 0,\\
&\partial_t V_2 + \partial_x (g H_2 + \frac{V_2^2}{2}) + \frac{C}{2d} V_2 |V_2| = 0\\
&\partial_t T_2 + \partial_x (V_2 T_2) + k_2 (T_1 - T_2) = 0
\end{aligned}
$$



Introduction

All these models can be rewritten as inhomogeneous hyperbolic systems:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = Bu \text{ (Telegrapher equations)}$$

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u) \text{ (Saint-Venant(-Exner), Heat exchangers equations)}$$

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Remarks

- u is a vector;

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Remarks

- u is a vector;
- Balance Laws : $B \not\equiv 0$, $F(u) \not\equiv 0$, otherwise conservation laws.

Assumption

- Suppose that there is no zero eigenvalues for the matrix A.

Introduction

Problem (Boundary Stabilization)

How to find a boundary feedback law such that the solution $u = u(t,x)$ of the hyperbolic systems with any given initial data satisfies

$$u(t,\cdot) \to 0, \quad \text{as } t \to +\infty? \tag{1.5}$$

Remarks

- Exponential stability: $\exists\ C, \lambda > 0$, such that

$$\|u(t,\cdot)\|_X \leq Ce^{-\lambda t}\|u(0,\cdot)\|_X, \quad \forall t > 0 \tag{1.6}$$

- Finite-time stability: $\exists t_F$, such that

$$u(t,\cdot) \equiv 0, \quad \forall t \geq t_F. \tag{1.7}$$

Introduction

Homogeneous case: (i.e. $B \equiv 0$, $F(u) \equiv 0$)

(1) **Characteristic method (T.T. Li, T.H. Qin (1983, 1985, 1994))**

- Quasilinear hyperbolic systems

$$u_t + A(u)u_x = 0$$

with the boundary conditions

$$\begin{pmatrix} u_-(t,1) \\ u_+(t,0) \end{pmatrix} = \mathcal{F}\begin{pmatrix} u_-(t,0) \\ u_+(t,1) \end{pmatrix} \tag{1.8}$$

- Framework of solution: C^1 norm,
- Local exponential stability (i.e $\|u(0,\cdot)\|_{C^1}$ is suitably small);
- Boundary is "dissipative":

$$\rho_\infty(\mathcal{F}'(0)) < 1 \tag{1.9}$$

$$\mathcal{F}(0) = 0 \quad \text{and} \quad \rho_\infty(\mathcal{F}'(0)) := \mathrm{Inf}\{\|\Delta\mathcal{F}'(0)\Delta^{-1}\|_\infty ; \Delta \in \mathcal{D}_{n,+}\},$$

where $\mathcal{D}_{n,+}$ denotes the set of $n \times n$ real diagonal matrices with strictly positive diagonal elements.

Homogeneous case: (i.e. $B \equiv 0$, $F(u) \equiv 0$)

(2) **Control Lyapunov Functions method (G.Bastin, J.-M.Coron, B. d'Andréa-Novel (1999, 2007, 2008, 2014))**

- Quasilinear hyperbolic systems,

$$u_t + A(u)u_x = 0 \tag{1.10}$$

with the boundary conditions

$$\begin{pmatrix} u_-(t,1) \\ u_+(t,0) \end{pmatrix} = \mathcal{F}\begin{pmatrix} u_-(t,0) \\ u_+(t,1) \end{pmatrix} \tag{1.11}$$

- Local exponential stability.
- Boundary is "dissipative":

$$\rho_\infty(\mathcal{F}'(0)) < 1 \text{ for } C^1 \text{ norm;} \tag{1.12}$$

$$\rho_2(\mathcal{F}'(0)) < 1 \text{ for } H^2 \text{ norm.} \tag{1.13}$$

Complements for hyperbolic balance laws

- Characteristic method and Control Lyapunov Functions method : hyperbolic balance laws,

$$u_t + A(u)u_x = F(u) \tag{1.14}$$

can be exponentially stabilized by boundary feedback provided $\|\nabla F(0)\|$ is small enough (see T.T. Li (1994), J.-M. Coron-G. Bastin-B. d'Andréa-Novel (2008) and C. Prieur $et.al.$ (2008)).

Complements for hyperbolic balance laws

What happens if $\|\nabla F(0)\|$ is not small?

- Characteristic method: hyperbolic balance laws

$$u_t + A(u)u_x = F(u) \tag{1.15}$$

  exponentially decays to zero if both boundary conditions and $F(u)$ are "dissipative" in some sense (see C.M. Liu and Y.Z. Li (2015)).
- Control Lyapunov Functions method : A sufficient and necessary condition is given for the existence of basic quadratic strict control Lyapunov function for 2×2 linear hyperbolic balance laws. (see G.Bastin-J.M.Coron (2011))

Introduction

All the above works are based on the **static boundary output feedback** (i.e a feedback of the state values at the boundaries only).
However, **static boundary output feedback** can not treat all inhomogeneous case.

Counter example (G.Bastin-J.-M.Coron, 2016)

If

$$L \geq \frac{\pi}{c}. \tag{1.16}$$

there is no $k \in \mathbb{R}$ such that the equilibrium $(0,0)^T \in L^2(0,L)^2$ is exponentially stable for the closed loop system

$$\begin{aligned} &\partial_t S_1 + \partial_x S_1 + cS_2 = 0, \\ &\partial_t S_2 - \partial_x S_2 + cS_1 = 0, \quad t \in [0,+\infty),\ x \in [0,1], \\ &S_1(t,0) = kS_2(t,0), \quad S_2(t,L) = S_1(t,L). \end{aligned} \tag{1.17}$$

Introduction

Inhomogeneous case: (i.e. $B \neq 0$ and $F(u) \not\equiv 0$)

- **Backstepping Method (J.-M.Coron-R. Vazquez-M. Krstic-G.Bastin (2011, 2013)**
  - 2×2 linear hyperbolic balance laws;

$$u_t + Au_x = Bu \tag{1.18}$$

  - $B \in \mathcal{M}_{2,2}$ and $A = \text{diag}(-\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 > 0$;
  - Full-state feedback

$$u_-(t, L) = \int_0^L k(L, \xi) u(t, \xi) d\xi \tag{1.19}$$

  - Finite-time stability.

Remark

The backstepping method can be extended to deal with the boundary stabilization problem of inhomogeneous quasilinear 2×2 hyperbolic systems (See J.-M.Coron-R. Vazquez-M. Krstic-G.Bastin (2013) and R.Vazquez-J.-M.Coron-M. Krstic-G.Bastin (2011))

Inhomogeneous case: ($n \times n$ hyperbolic balance laws)

- **Backstepping Method (F. Di Meglio-R.Vazquez-M.Krstic, 2013)**
  - 1 of the PDEs is controlled at its boundary and $n-1$ other PDEs, which convect in the opposite direction, are not controlled and all have arbitrary interconnections, e.g.

$$u_t + Au_x = Bu \tag{1.20}$$

where $A = \text{diag}(-\lambda_1, -\lambda_2, \cdots, -\lambda_{n-1}, \lambda_n)$ with

$$\lambda_i > 0, \; i = 1, \cdots, n.$$

and the only boundary feedback control is acting on $u_n(t, 0)$.

Unfortunately, the method presented in J.-M. Coron $et.al.$ (2013) and R. Vazquez $et.al.$ (2011,2013) and Di Meglio $et.al.$ (2011, 2013) can not be directly extended to $n \times n$ systems, especially when several states convecting in the same direction are controlled.

Open Question (J.-M.Coron-R. Vazquez-M. Krstic-G.Bastin, SICON, 2013)

Can we stabilize the general inhomogeneous hyperbolic systems by multi-boundary feedback controls?

Outline

System Description

We consider the following general linear hyperbolic system

$$u_t(t,x) + \Lambda^+ u_x(t,x) = \Sigma^{++} u(t,x) + \Sigma^{+-} v(t,x) \quad (2.1)$$

$$v_t(t,x) - \Lambda^- v_x(t,x) = \Sigma^{-+} u(t,x) + \Sigma^{--} v(t,x) \quad (2.2)$$

where $u = \begin{pmatrix} u_1 \cdots u_n \end{pmatrix}^T, v = \begin{pmatrix} v_1 \cdots v_m \end{pmatrix}^T$. and

$$\Lambda^+ = diag(\lambda_1 \cdots \lambda_n) \quad \Lambda^- = diag(\mu_1 \cdots \mu_m) \quad (2.3)$$

with

$$-\mu_1 < \cdots < -\mu_m < 0 < \lambda_1 \leq \cdots \leq \lambda_n \quad (2.4)$$

System Description

$\Sigma^{\pm\pm}$ are matrices and without loss of generality, we assume that we assume that

$$\forall j = 1, ..., m \qquad \sigma_{jj}^{--} = 0, \tag{2.5}$$

Remark

One can do some coordinate transformation of v in order to guarantee (2.5).

System Description

The boundary conditions are as follows

$$u(t,0) = Q_0 v(t,0), \qquad v(t,1) = R_1 u(t,1) + U(t) \tag{2.6}$$

where Q_0 and R_1 are constant matrices. $U(t) = (U_1, \cdots, U_m)^T$ are boundary controls.

Goal

Our objective is to design a feedback control law for $U(t)$ in order to ensure that the closed-loop system vanishes in finite time.

Methods: Backstepping

What is Backstepping method?

- Mapping the original system to a target system which has “good” property (e.g. finite-time stable or exponential stable) by using a invertible backstepping transformation.

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Difficulty of this method

- How to choose a good transformation?
- How to choose a suitable target system?

Linear Cases

Review of 2×2 (i.e. $\Lambda \in \mathcal{M}_{2,2}$) case (JMC $et.al.$ (2013) and R. Vazquez $et.al.$ (2011))

- Transformation:

$$u(t,x) = \gamma(t,x) - \int_0^x K(x,\xi)\gamma(t,\xi)d\xi, \tag{2.7}$$

- Target system

$$\gamma_t(t,x) + \Lambda\gamma_x(t,x) = 0. \tag{2.8}$$

- The K-kernel is a matrix function of C^2 on the domain

$$\mathcal{T} = \{(x,\xi) | 0 \le \xi \le x \le 1\}. \tag{2.9}$$

Linear Cases

Unfortunately, we can not deal with the general $n \times n$ cases by using the above transformation and target system. If so, an overdetermined problem appears to the K-kernel.

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Question

- Is it possible to change only the transformation or the target system to achieve our purpose?

Linear Cases

Unfortunately, we can not deal with the general $n \times n$ cases by using the above transformation and target system. If so, an overdetermined problem appears to the K-kernel.

Question

- Is it possible to change only the transformation or the target system to achieve our purpose?

Answer: Yes!

New target system

$$\begin{aligned}\alpha_t(t,x) + \Lambda^+\alpha_x(t,x) = \Sigma^{++}\alpha(t,x) + \Sigma^{+-}\beta(t,x) \\ + \int_0^x C^+(x,\xi)\alpha(\xi)d\xi + \int_0^x C^-(x,\xi)\beta(\xi)d\xi\end{aligned} \tag{2.10}$$

$$\beta_t(t,x) - \Lambda^-\beta_x(t,x) = G(x)\beta(0) \tag{2.11}$$

with the following boundary conditions

$$\alpha(t,0) = Q_0\beta(t,0), \qquad \beta(t,1) = 0 \tag{2.12}$$

where C^+ and C^- are L^∞ matrix functions on the domain

$$\mathcal{T} = \{0 \le \xi \le x \le 1\}, \tag{2.13}$$

New target system

while $G \in L^\infty(0,1)$ is a lower triangular matrix with the following structures

$$G(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}. \tag{2.14}$$

New target system

Finite-time stabilization (LH, F. Di Meglio, R. Vazquez and M.Krstic (2015 a))

The zero equilibrium of the target system is reached in finite time $t = t_F$, where

$$t_F := \frac{1}{\lambda_1} + \sum_{j=1}^{m} \frac{1}{\mu_j}. \tag{2.15}$$

Sketch of the proof

It is easy to see that β vanishes in finite time t_1 with

$$t_1 = \sum_{r=1}^{m} \frac{1}{|\lambda_r(s)|} ds. \tag{2.16}$$

From the time $t = t_1$ on, we find α becomes the solution of the following system

$$\alpha_t(t,x) + \Lambda^+ \alpha_x(t,x) = \Sigma^{++}\alpha(t,x) + \int_0^x C^+(x,\xi)\alpha(\xi)d\xi \tag{2.17}$$

with the boundary conditions

$$\alpha(t,0) = 0. \tag{2.18}$$

Changing the status of t and x, and Equations (2.17) can be rewritten as

$$\alpha_x(t,x) + (\Lambda^+)^{-1}\alpha_t(t,x) = (\Lambda^+)^{-1}\Sigma^{++}\alpha(t,x) + \int_0^x (\Lambda^+)^{-1} C^+(x,\xi)\alpha(\xi)d\xi \tag{2.19}$$

with the initial condition (2.18).

Sketch of the proof

Then by the uniqueness of the system (2.18),(2.19), and noting the order of the transport speeds of the α–system (see (2.4)), this yields that α identically vanishes for

$$t \geq \frac{1}{\lambda_1} + \sum_{j=1}^{m} \frac{1}{\mu_j} \tag{2.20}$$

Open Question

How to reduce the finite-time-control time t_F?

Backstepping transformation

We consider the following backstepping (Volterra) transformation

$$\alpha(t,x) = u(t,x) \tag{2.21}$$

$$\beta(t,x) = v(t,x) - \int_0^x \left[K(x,\xi)u(\xi) + L(x,\xi)v(\xi)\right] d\xi \tag{2.22}$$

where the kernels to be determined K and L are defined on the triangular domain $\mathcal{T}$.

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where the kernels to be determined K and L are defined on the triangular domain $\mathcal{T}$.

Important: $\alpha(t,0) = u(t,0), \beta(t,0) = v(t,0)$

Backstepping transformation

The original system (2.1) is mapped into the target system (2.6) if K and L satisfy the following equations

for $1 \leq i \leq m$, $1 \leq j \leq n$

$$\mu_i \partial_x K_{ij}(x,\xi) - \lambda_j \partial_\xi K_{ij}(x,\xi) = \sum_{k=1}^{n} \sigma_{kj}^{++} K_{ik}(x,\xi) + \sum_{p=1}^{m} \sigma_{pj}^{-+} L_{ip}(x,\xi) \quad (2.23)$$

for $1 \leq i \leq m$, $1 \leq j \leq m$

$$\mu_i \partial_x L_{ij}(x,\xi) + \mu_j \partial_\xi L_{ij}(x,\xi) = \sum_{p=1}^{m} \sigma_{pj}^{--} L_{ip}(x,\xi) + \sum_{k=1}^{n} \sigma_{kj}^{+-} K_{ik}(x,\xi) \quad (2.24)$$

along with the following set of boundary conditions

$$K_{ij}(x,x) = -\frac{\sigma_{ij}^{-+}}{\mu_i + \lambda_j} \triangleq k_{ij} \qquad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq n \quad (2.25)$$

$$L_{ij}(x,x) = -\frac{\sigma_{ij}^{--}}{\mu_i - \mu_j} \triangleq l_{ij} \qquad \text{for } 1 \leq i, j \leq m, \quad i \neq j \quad (2.26)$$

$$\mu_j L_{ij}(x,0) = \sum^{n} \lambda_k K_{ik}(x,0) q_{k,j} \qquad \text{for } 1 \leq i \leq j \leq m. \quad (2.27)$$

Backstepping transformation

To ensure well-posedness of the kernel equations, we add the following artificial boundary conditions for $L_{ij}(i > j)$

$$L_{ij}(1, \xi) = l_{ij}, \text{ for } 1 \leq j < i \leq m. \tag{2.28}$$

Remark

We can select that

$$L_{ij}(1, \xi) = 0, \text{ for } 1 \leq j < i \leq m. \tag{2.29}$$

The existence of K and L

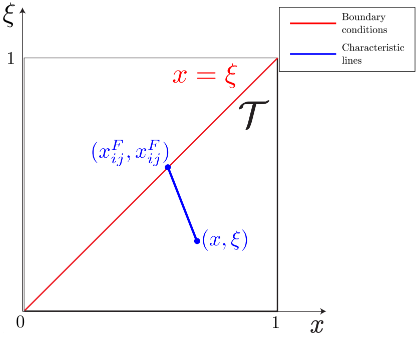
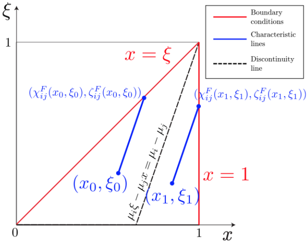
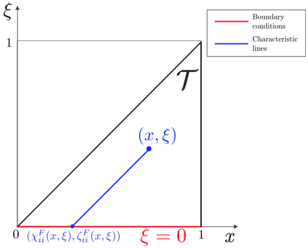


Figure 1 : Characteristic lines of the K kernels

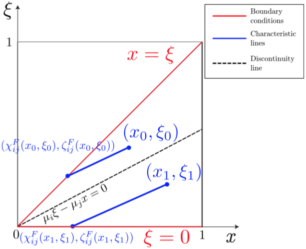
The existence of K and L



(a) Characteristic lines of the kernels L_{ij} for $i > j$



(b) Characteristic lines of the kernels L_{ii}



(c) Characteristic lines of the kernels L_{ij} for $i < j$

The existence of K and L

Successive approximation method (LH, F. DiMeglio, R. Vazquez and M. Krstic (2015 a)

$$K(x,\xi) = \sum_{q=0}^{+\infty} \Delta\mathbf{K}^q(x,\xi) \tag{2.30}$$

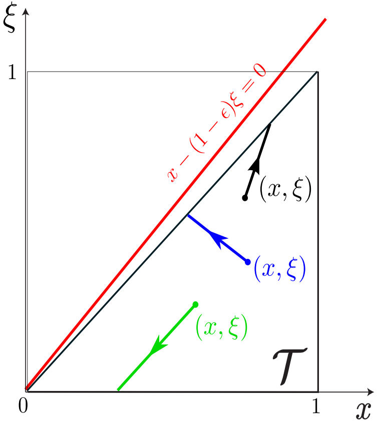
$$L(x,\xi) = \sum_{q=0}^{+\infty} \Delta\mathbf{L}^q(x,\xi) \tag{2.31}$$

where

$$\Delta\mathbf{K}^q(x,\xi), \Delta\mathbf{L}^q(x,\xi) \leq C\frac{M^q(x-(1-\varepsilon)\xi)^q}{q!}. \tag{2.32}$$

in which $C, M > 0$.

Improved successive approximation method (LH, F. DiMeglio, R. Vazquez and M. Krstic (2015 a)



Key point:

The proof is based on the fact that, starting from any point (x,ξ), all the characteristic lines "get closer" to the line defined by $x-(1-\varepsilon)\xi=0$.

Since $\beta(t,1)=0$ and

$$\begin{pmatrix} \alpha(t,x) \\ \beta(t,x) \end{pmatrix} = \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} - \int_0^x \begin{pmatrix} 0 & 0 \\ K(x,\xi) & L(x,\xi) \end{pmatrix} \begin{pmatrix} u(t,\xi) \\ v(t,\xi) \end{pmatrix} d\xi. \tag{2.33}$$

Our feedback laws finally is

$$U(t) = \int_0^1 \left[K(1,\xi)u(\xi) + L(1,\xi)v(\xi) \right] d\xi - R_1 u(t,1). \tag{2.34}$$

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$$U(t) = \int_0^1 [K(1,\xi)u(\xi) + L(1,\xi)v(\xi)]\, d\xi - R_1 u(t,1). \tag{2.34}$$

Theorem: Finite-time stabilization (LH, F. DiMeglio, R. Vazquez and M. Krstic (2015 a))

By $U(t)$, the zero equilibrium of the original system is reached in finite time $t = t_F$.

Proof. Good! (2.34) is always invertible, i.e. there exists a matrix function $\mathcal{R} \in (L^\infty(\mathcal{T}))^{(n+m)\times(n+m)}$, such that

$$\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = \begin{pmatrix} \alpha(t,x) \\ \beta(t,x) \end{pmatrix} - \int_0^x \mathcal{R}(x,\xi) \begin{pmatrix} \alpha(t,\xi) \\ \beta(t,\xi) \end{pmatrix} d\xi. \tag{2.35}$$

Since $(\alpha,\beta)^T$ goes to zero in finite time $t = t_F$, therefore $(u,v)^T$ shares this property.

Remarks

[1] We can deal with the observer problem by using also the backstepping approach.

[2] Our method is still valid for the case if the coefficients $(\Lambda^{\pm},\ \Sigma^{\pm,\pm})$ are depending on x.

[3] Counter Example: G.Bastin-JMC-2016.

$$S_1(t,0) = \int_0^L K(0,\xi)S_1(t,\xi) + L(0,\xi)S_2(t,\xi)d\xi \tag{2.36}$$

Remarks

[5] Our kernel is probably not continuous, which is different with previous works.

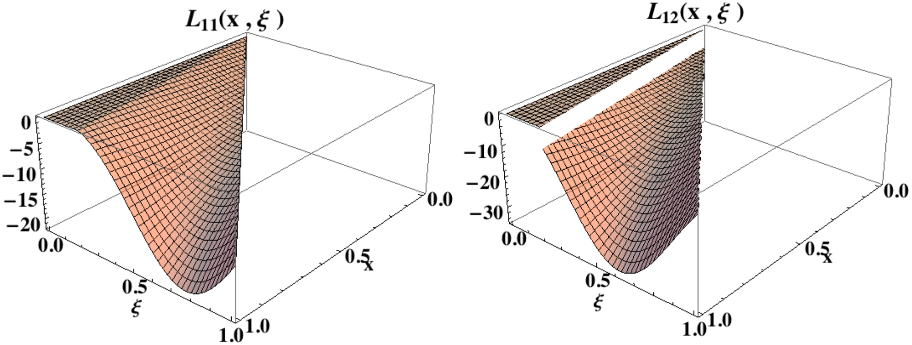

Figure 3 : Kernels $L_{11}(x,\xi)$ and $L_{12}(x,\xi)$ ($n=0$, $m=2$).

Outline

System Description

The system considered is

$$\frac{\partial u}{\partial t} + A(x,u)\frac{\partial u}{\partial x} = F(x,u) \quad (t,x) \in [0,T] \times [0,L], \tag{3.1}$$

System Description

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where

- $u = (u_1, \ldots, u_n)^T$ is a vector function of (t,x);
- $A(x,u) := (a_{ij}(x,u))_{n\times n}$ is of class C^2, $A(x,0)$ is a diagonal matrix with distinct and nonzero eigenvalues $A(x,0) = \mathrm{diag}(\Lambda_1(x), \cdots, \Lambda_n(x))$, which are ordered as follows:

$$\Lambda_1(x) < \Lambda_2(x) < \cdots < \Lambda_m(x) < 0 < \Lambda_{m+1}(x) < \cdots < \Lambda_n(x), \forall x \in [0,1]. \tag{3.2}$$

System Description

- $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued function with C^2 components $f_i(x,u)(i = 1, \cdots, n)$ with respect to u and

$$F(x,0) \equiv 0. \tag{3.3}$$

- Denote

$$\frac{\partial F}{\partial u}(x,0) := (f_{ij}(x))_{n\times n}, \tag{3.4}$$

we assume that $f_{ij} \in C^2([0,1])$ and

$$f_{ii}(x) \equiv 0. \tag{3.5}$$

Remark

One can make some coordinate transformations in order to guarantee that (3.5) is valid (see JMC (2014) and L. Hu & F. DiMeglio (2015)).

System Description

The boundary conditions are given as follows:

$$x = 0 : u_s = G_s(u_1, \cdots, u_m), \quad s = m+1, \cdots, n, \tag{3.6}$$
$$x = 1 : u_r = h_r(t), \quad r = 1, \cdots, m, \tag{3.7}$$

where G_s are C^2 functions, and we assume that they vanish at the origin, i.e.

$$G_s(0, \cdots, 0) \equiv 0, s = m+1, \cdots, n, \tag{3.8}$$

while $H = (h_1, \cdots, h_m)^T$ are boundary controls.

Remark

- Local well-posedness: see JMC (2008) and Tatsien Li (1994) [Remark 1.3 on page 171] etc.

Main Results

Theorem (LH, R. Vazquez, F. Di Meglio and M. Krstic (2015 b))

For every $\lambda > 0$, there exist $\delta > 0$, $c > 0$ and a continuous linear feedback control H, such that 0 is the exponential stable point of $u = u(t,x)$, i.e.

$$\|u(t,\cdot)\|_{H^2} \leq ce^{-\lambda t}\|u(0,\cdot)\|_{H^2}, \tag{3.9}$$

provided that $\|u(0,\cdot)\|_{H^2} \leq \delta$.

Remark

- For simplicity, we skip of C^1 compatibility conditions at the points $(t,x) = (0,0)$ and $(0,1)$;

Sketch of the proof

[1] Linearized system is

$$\begin{cases} u_t + \Lambda(x) u_x = \Sigma(x) u \\ x = 0 : u_s(t,0) = \sum_{j=1}^m Q_{sj} u_j(t,0) \\ x = L : u_j(t,1) = h_j(t). \end{cases} \tag{3.10}$$

where $\Lambda(x) = A(x,0)$ and $\Sigma(x) = \frac{\partial F}{\partial u}(x,0)$, then we can find

$$h_j(t) = \int_0^1 \sum_{l=1}^n K_{jl}(1,\xi) u_l(t,\xi) d\xi, \quad (j = 1, \cdots, m), \tag{3.11}$$

which can stabilize the linearized system in finite-time.

Sketch of the Proof

[2] We also use the volterra transformation

$$\gamma(t,x) = u(t,x) - \int_0^x K(x,\xi)u(t,\xi)dt. \tag{3.12}$$

Lemma:Regularity of the direct kernel (LH, R. Vazquez, F. DiMeglio and M. Krstic (2015 b))

Let $N \in \mathbb{N}^+$. Under the assumption that $\sigma_{ij} \in C^N[0,1],\ \lambda_i \in C^N[0,1](i,j = 1,\cdots,n)$, there exists a unique piecewise $C^N(\mathcal{T})$ solution to K kernel. Moreover, then $K(\cdot,\cdot) \in C^{N-1}(0,1)$, $K(\cdot,0) \in C^{N-1}(0,1)$ with bounded C^{N-1} norm.

Remark

- The H^2 norm of γ is equivalent to the H^2 norm of u.

Then the nonlinear target system is

$$\begin{aligned}&\gamma_t(t,x)+\Lambda(x)\gamma_x(t,x)-G(x)\gamma(t,0)\\=&F_3[\gamma,\gamma_x]+F_4[\gamma],\end{aligned}\tag{3.13}$$

The boundary conditions are

$$x=0:\gamma_+(t,0)=Q\gamma_-(t,0)+G_{NL}(\gamma_-(t,0))\tag{3.14}$$

and

$$x=1:\gamma_-(t,1)=0.\tag{3.15}$$

[3] Luckily! The usual Lyapunov function (see JMC-R.Vazquez-M.Krstic -G.Bastin, SICON 2013) can be also used to exponentially stabilize this γ system.

Control Lyapunov Functions

- Estimate of $\|\gamma\|_{L^2}$
  Define

$$V_1(t) = \int_0^1 e^{-\delta x}\gamma_+(t,x)^T \left(\Lambda_+(x)\right)^{-1}\gamma_+(t,x)dx - \int_0^1 e^{\delta x}\gamma_-(t,x)^T B\left(\Lambda_-(x)\right)^{-1}\gamma_-(t,x)dx.$$

  We have

Proposition 1

For any given $\lambda_1 > 0$, there exists $\delta_1 > 0$ and $K_2 > 0$, such that

$$\dot{V}_1 \leq -\lambda_1 V_1 + K_2\big(V_1^{\frac{3}{2}} + \|\gamma_x\|_\infty V_1\big), \tag{3.16}$$

provided $\|\gamma\|_\infty \leq \delta_1$.

- Estimate of $\|\gamma_t\|_{L^2}$
  Define $\zeta = \gamma_t$ and

$$V_2(t) = \int_0^1 \zeta^T(t,x) R[\gamma] \zeta(t,x) dx, \tag{3.17}$$

where $R[\gamma]$ is a positive matrix. We have

Proposition 2

For any given $\lambda_2 > 0$, there exists $\delta_2 > 0$ and $K_7 > 0$, such that

$$\dot{V}_2 \leq -\lambda_2 V_2 + K_7 \big(\|\zeta\|_\infty + \|\gamma\|_\infty\big) V_2, \tag{3.18}$$

provided that $\|\gamma\|_\infty \leq \delta_2$.

- Estimate of $\|\gamma_{tt}\|_{L^2}$
  Define $\theta = \gamma_{tt}$ and

$$V_3(t) = \int_0^1 \theta^T(t,x)R[\gamma]\theta(t,x)dx, \quad (3.19)$$

We have

Proposition 3

For any given $\lambda_3 > 0$, there exists $\delta_3 > 0$ and positive constants K_{10}, K_{11}, K_{12}, K_{13} and K_{14}, such that

$$\dot{V}_3 \leq -\lambda_3 V_3 + K_{10}\|\gamma\|_\infty V_3 + K_{11}V_3V_2^{\frac{1}{2}} + K_{12}V_2V_3^{\frac{1}{2}} + K_{13}V_3^{\frac{3}{2}} + K_{14}\|\zeta\|_\infty^3, \quad (3.20)$$

provided that $\|\gamma\|_\infty + \|\zeta\|_\infty \leq \delta_3$.

Proof of main result

Denote $W = V_1 + V_2 + V_3$, by Proposition 1–3, one can show that for any given $\lambda > 0$, there exists $\delta > 0$ and $K_{15} > 0$, such that

$$\dot{W} \leq -\lambda W + K_{15} W^{\frac{3}{2}}, \tag{3.21}$$

provided that $\|\gamma\|_\infty + \|\zeta\|_\infty \leq \delta$.

Outline

Perspectives

Non-local hyperbolic system is considered as

$$\begin{cases} u_t = u_x + \int_0^L g(x,y)u(t,y)dy \\ u(t,L) = U(t) \end{cases} \tag{4.1}$$

- u is a scalar $U(t)$ is the boundary feedback.
- (4.1) may involve the traffic laws, some KdV-like equation and PDE-ODE interconnected system (see M. Krstic, A.Smyshlyaev (2008), F. Bribiesca and M. Krstic (2015)).

Known results

- M. Krstic, A.Smyshlyaev (2008): $g(x,y)=0,\ x\le y$.
  Map (4.1) into

$$\begin{cases} w_t = w_x \\ w(t,L) = 0. \end{cases} \tag{4.2}$$

  by using Volterra transformation

$$u(t,x) = w(t,x) - \int_0^x k(x,y)w(t,y)dy \tag{4.3}$$

- F. Bribiesca and M. Krstic (2015): For general g Volterra transformation fails. Map (4.1) into (4.2) by using Fredholm transformation

$$u(t,x) = w(t,x) - \int_0^L \tilde{k}(x,y)w(t,y)dy \tag{4.4}$$

provided $\|g\|$ is small.

Remark

The smallness of $\|g\|$ is used to guarantee the existence of $\tilde{k}$ and the invertibility of the the Fredholm transformation (4.4).

Main results (J.-M. Coron, LH and G. Olive (2015))

Suppose $g \in H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+)$, where

$$\mathcal{T}_- = \{(x,y) \in (0,L) \times (0,L) | x > y\},$$
$$\mathcal{T}_+ = \{(x,y) \in (0,L) \times (0,L) | x < y\},$$

Then (4.1) is finite-time stabilizable in time L if and only if (4.1) is exactly controllable at time L.

Sketch of proof

We also use the Fredholm transformation

$$u(t,x) = w(t,x) - \int_0^L h(x,y)w(t,y)dy \tag{4.5}$$

to map the system (4.2) into (4.1). The kernel h satisfies

$$\begin{cases} h_x(x,y) + h_y(x,y) + \int_0^L G(x,\tau)h(\tau,y)d\tau - G(x,y) = 0 \\ h(x,0) = h(x,L) = 0 \end{cases} \tag{4.6}$$

- We use the controllability of the original system (4.1) to prove the existence of h;
- The invertibility of h can be guaranteed if the original system (4.1) is controllability.

Remarks

- We can also treat the equation of the more general form

$$\begin{cases} u_t(t,x) = u_x(t,x) + \alpha(x)u(t,x) + \beta(x)u(t,0) + \int_0^L g(x,y)u(t,y)dy \\ u(t,L) = \int_0^L \gamma(x)u(t,x)dx + U(t) \\ u(0,x) = u^0(x). \end{cases}$$

where $\alpha,\ \beta,\ \gamma : (0,L) \to \mathbb{C}$ are regular enough.

Remarks

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$$\begin{cases} u_t(t,x) = u_x(t,x) + \alpha(x)u(t,x) + \beta(x)u(t,0) + \int_0^L g(x,y)u(t,y)dy \\ u(t,L) = \int_0^L \gamma(x)u(t,x)dx + U(t) \\ u(0,x) = u^0(x). \end{cases}$$

  where $\alpha,\ \beta,\ \gamma : (0,L) \to \mathbb{C}$ are regular enough.
- We expect that (4.1) is always controllable. However, this is NOT true.
  - If $g(x,y) = 0, x \leq y$, (4.1) is controllable. (M. Krstic, A.Smyshlyaev (2008))
  - If $\|g\|$ is small enough, (4.1) is controllable. (F. Bribiesca and M. Krstic (2015))
  - If $g(x,y) = g(x)$, (4.1) is controllable if and only if (J.M.Coron, LH and G.Olive (2015))

$$\text{ker}(\lambda - A^*) \cap \text{ker}B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{4.7}$$

  Question: What about the general case $g(x,y)$?

Pas un jour sans contrôle

Thank you for your attention !