Backstepping methods for boundary stabilization of 1-D hyperbolic balance laws

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Groupe de travail : Contrôle, September 18, 2015, Paris, France

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Outline

1. Introduction
2. Boundary stabilization of linear hyperbolic balance laws
3. Boundary stabilization of quasilinear hyperbolic balance laws
4. Perspectives-Stabilization of nonlocal hyperbolic system
Outline

1. **Introduction**

2. **Boundary stabilization of linear hyperbolic balance laws**

3. **Boundary stabilization of quasilinear hyperbolic balance laws**

4. **Perspectives-Stabilization of nonlocal hyperbolic system**
Our hyperbolic balance laws is

\[
\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \quad (t, x) \in [0, T] \times [0, L],
\]

where,

\begin{itemize}
\item $u = (u_1, \ldots, u_n)^T$ is a vector function of $(t, x)$;
\item $A(u)$ has $n$ real eigenvalues $\lambda_i(u)$ ($i = 1, \cdots, n$) and a complete set of left (resp. right) eigenvectors $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{1i}(u), \cdots, r_{ni}(u))^T$, $(i = 1, \cdots, n)$):
\end{itemize}

\[
l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \quad (1.2)
\]

\begin{itemize}
\item $F(u) = (f_1(u), \cdots, f_n(u))^T$ is a given vector function of $u$ with $F(0) = 0$.
\end{itemize}
In general, we call the following systems

\[
\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = F(u)
\]  

(1.3)

to be hyperbolic balance laws, where the flux \( g := (g_1, \cdots, g_n) \) is a vector function of \( u \). Obviously, system (1.3) can be written in the quasilinear form as (1.1) with the Jacobian matrix

\[
A(u) := D(g(u)).
\]  

(1.4)
Many physical models are governed by linear and quasilinear hyperbolic balance laws, for example:

\[
\frac{\partial}{\partial t} (I_0 V) + \frac{\partial}{\partial x} (-L - 1/e V - C - 1/e I) = -\left( R_L e L - 1/e I G_e C - 1/e V \right).
\]
Many physical models are governed by linear and quasilinear hyperbolic balance laws, for example:

- The telegrapher equations (Heaviside, O. (1892))

\[
\partial_t \begin{pmatrix} I \\ V \end{pmatrix} + \partial_x \begin{pmatrix} -L^{-1}eV \\ -C^{-1}eI \end{pmatrix} = - \begin{pmatrix} ReL^{-1}eI \\ GeC^{-1}eV \end{pmatrix}.
\]
The Saint-Venant equations (Barré de Saint-Venant (1871))

\[
\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(HV) = 0,
\]

\[
\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}\left(\frac{V^2}{2} + gH\right) = gS_b - CV^2H^{-1}.
\]
The Saint-Venant-Exner equations (Hudson-Sweby (2003))

\[
\begin{align*}
\frac{\partial H}{\partial t} + V \frac{\partial H}{\partial x} + H \frac{\partial V}{\partial x} &= 0, \\
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial H}{\partial x} + g \frac{\partial B}{\partial x} &= g S_b - C \frac{V^2}{H}, \\
\frac{\partial B}{\partial t} + a V^2 \frac{\partial V}{\partial x} &= 0.
\end{align*}
\]
Heat exchangers (G. Bastin-J.-M. Coron (2016))

\[
\begin{align*}
\partial_t H_1 + V_1 \partial_x H_1 + \frac{c^2}{g} \partial_x V_1 &= 0 \\
\partial_t V_1 + \partial_x (g H_1 + \frac{V_1^2}{2}) + \frac{C}{2d} V_1 |V_1| &= 0 \\
\partial_t T_1 + \partial_x (V_1 T_1) - k_1 (T_1 - T_2) - k_0 (T_1 - T_e) &= 0 \\
\partial_t H_2 + V_2 \partial_x H_2 + \frac{c^2}{g} \partial_x V_2 &= 0, \\
\partial_t V_2 + \partial_x (g H_2 + \frac{V_2^2}{2}) + \frac{C}{2d} V_2 |V_2| &= 0 \\
\partial_t T_2 + \partial_x (V_2 T_2) + k_2 (T_1 - T_2) &= 0
\end{align*}
\]
All these models can be rewritten as inhomogeneous hyperbolic systems:

\[
\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = Bu \quad \text{(Telegrapher equations)}
\]

\[
\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = F(u) \quad \text{(Saint-Venant(-Exner), Heat exchangers equations)}
\]
All these models can be rewritten as inhomogeneous hyperbolic systems:

\[
\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = Bu \text{ (Telegrapher equations)}
\]

\[
\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = F(u) \text{ (Saint-Venant(-Exner), Heat exchangers equations)}
\]

**Remarks**

- \( u \) is a vector;
All these models can be rewritten as inhomogeneous hyperbolic systems:

\[
\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = Bu \quad (\text{Telegrapher equations})
\]

\[
\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = F(u) \quad (\text{Saint-Venant(-Exner), Heat exchangers equations})
\]

**Remarks**

- \(u\) is a vector;
- Balance Laws: \(B \neq 0, \ F(u) \neq 0\), otherwise conservation laws.

**Assumption**

- Suppose that there is no zero eigenvalues for the matrix \(A\).
Problem (Boundary Stabilization)

How to find a boundary feedback law such that the solution \( u = u(t, x) \) of the hyperbolic systems with any given initial data satisfies

\[
u(t, \cdot) \to 0, \quad \text{as } t \to +\infty? \tag{1.5}\]

Remarks

- Exponential stability: \( \exists C, \lambda > 0, \) such that

\[
\|u(t, \cdot)\|_X \leq Ce^{-\lambda t}\|u(0, \cdot)\|_X, \quad \forall t > 0 \tag{1.6}
\]

- Finite-time stability: \( \exists t_F, \) such that

\[
u(t, \cdot) \equiv 0, \quad \forall t \geq t_F. \tag{1.7}\]
Homogeneous case: (i.e. $B \equiv 0$, $F(u) \equiv 0$)


- Quasilinear hyperbolic systems
  \[
  u_t + A(u)u_x = 0
  \]
  with the boundary conditions
  \[
  \left( \begin{array}{c}
  u_-(t, 1) \\
  u_+(t, 0)
  \end{array} \right) = \mathcal{F} \left( \begin{array}{c}
  u_-(t, 0) \\
  u_+(t, 1)
  \end{array} \right)
  \tag{1.8}
  \]

- Framework of solution: $C^1$ norm,
- Local exponential stability (i.e. $\|u(0, \cdot)\|_{C^1}$ is suitably small);
- Boundary is "dissipative":
  \[
  \rho_\infty(\mathcal{F}'(0)) < 1
  \tag{1.9}
  \]

\[
\mathcal{F}(0) = 0 \quad \text{and} \quad \rho_\infty(\mathcal{F}'(0)) := \inf\{\|\Delta \mathcal{F}'(0)\Delta^{-1}\|_\infty; \Delta \in \mathcal{D}_{n,+}\},
\]

where $\mathcal{D}_{n,+}$ denotes the set of $n \times n$ real diagonal matrices with strictly positive diagonal elements.
Homogeneous case: \( (\text{i.e. } B \equiv 0, \ F(u) \equiv 0) \)


- Quasilinear hyperbolic systems,

\[
 u_t + A(u)u_x = 0 
\] (1.10)

with the boundary conditions

\[
 \begin{pmatrix}
     u_-(t, 1) \\
     u_+(t, 0)
\end{pmatrix}
 = F \begin{pmatrix}
     u_-(t, 0) \\
     u_+(t, 1)
\end{pmatrix} 
\] (1.11)

- Local exponential stability.
- Boundary is ”dissipative”:

\[
 \rho_{\infty}(F'(0)) < 1 \text{ for } C^1 \text{ norm}; 
\] (1.12)

\[
 \rho_2(F'(0)) < 1 \text{ for } H^2 \text{ norm}. \] (1.13)
Complements for hyperbolic balance laws

- Characteristic method and Control Lyapunov Functions method: hyperbolic balance laws,

\[ u_t + A(u)u_x = F(u) \]  \hspace{1cm} (1.14)

can be exponentially stabilized by boundary feedback provided \( \|\nabla F(0)\| \) is small enough (see T.T. Li (1994), J.-M. Coron-G. Bastin-B. d’Andréa-Novel (2008) and C. Prieur et.al. (2008)).
Complements for hyperbolic balance laws

What happens if $\|\nabla F(0)\|$ is not small?

- Characteristic method: hyperbolic balance laws

$$u_t + A(u)u_x = F(u) \quad (1.15)$$

exponentially decays to zero if both boundary conditions and $F(u)$ are "dissipative" in some sense (see C.M. Liu and Y.Z. Li (2015)).

- Control Lyapunov Functions method: A sufficient and necessary condition is given for the existence of basic quadratic strict control Lyapunov function for $2 \times 2$ linear hyperbolic balance laws. (see G.Bastin-J.M.Coron (2011))
All the above works are based on the static boundary output feedback (i.e. a feedback of the state values at the boundaries only). However, static boundary output feedback can not treat all inhomogeneous case.

Counter example (G.Bastin-J.-M.Coron, 2016)

If

\[ L \geq \frac{\pi}{c}. \]  \hspace{2cm} (1.16)

there is no \( k \in \mathbb{R} \) such that the equilibrium \((0, 0)^T \in L^2(0, L)^2\) is exponentially stable for the closed loop system

\[
\begin{align*}
\partial_t S_1 + \partial_x S_1 + cS_2 &= 0, \\
\partial_t S_2 - \partial_x S_2 + cS_1 &= 0, \quad t \in [0, +\infty), \ x \in [0, 1], \\
S_1(t, 0) &= kS_2(t, 0), \quad S_2(t, L) = S_1(t, L). 
\end{align*}
\]  \hspace{2cm} (1.17)
Introduction

Inhomogeneous case: (i.e. \( B \neq 0 \) and \( F(u) \neq 0 \))

  - \( 2 \times 2 \) linear hyperbolic balance laws;
  \[
  u_t + Au_x = Bu
  \] (1.18)
  - \( B \in \mathcal{M}_{2,2} \) and \( A = \text{diag}(-\lambda_1, \lambda_2) \) with \( \lambda_1, \lambda_2 > 0 \);
  - Full-state feedback
  \[
  u_-(t, L) = \int_0^L k(L, \xi) u(t, \xi) d\xi
  \] (1.19)
  - Finite-time stability.

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Stabilization of hyperbolic balance laws

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Remark

The backstepping method can be extended to deal with the boundary stabilization problem of inhomogeneous quasilinear $2 \times 2$ hyperbolic systems (See J.-M. Coron-R. Vazquez-M. Krstic-G. Bastin (2013) and R. Vazquez-J.-M. Coron-M. Krstic-G. Bastin (2011))
Inhomogeneous case: ( $n \times n$ hyperbolic balance laws)

- **Backstepping Method** (F. Di Meglio-R. Vazquez-M. Krstic, 2013)
  - 1 of the PDEs is controlled at its boundary and $n - 1$ other PDEs, which convect in the opposite direction, are not controlled and all have arbitrary interconnections, e.g.

  \[
  u_t + Au_x = Bu
  \]

  (1.20)

  where $A = \text{diag}(-\lambda_1, -\lambda_2, \cdots, -\lambda_{n-1}, \lambda_n)$ with

  \[
  \lambda_i > 0, \quad i = 1, \cdots, n.
  \]

  and the only boundary feedback control is acting on $u_n(t, 0)$. 


Unfortunately, the method presented in J.-M. Coron *et al.* (2013) and R. Vazquez *et al.* (2011, 2013) and Di Meglio *et al.* (2011, 2013) cannot be directly extended to $n \times n$ systems, especially when several states convecting in the same direction are controlled.


Can we stabilize the general inhomogeneous hyperbolic systems by multi-boundary feedback controls?
Outline

1 Introduction

2 Boundary stabilization of linear hyperbolic balance laws

3 Boundary stabilization of quasilinear hyperbolic balance laws

4 Perspectives-Stabilization of nonlocal hyperbolic system
We consider the following general linear hyperbolic system

\begin{align}
    u_t(t, x) + \Lambda^+ u_x(t, x) &= \Sigma^{++} u(t, x) + \Sigma^{+-} v(t, x) \tag{2.1} \\
    v_t(t, x) - \Lambda^- v_x(t, x) &= \Sigma^{-+} u(t, x) + \Sigma^{--} v(t, x) \tag{2.2}
\end{align}

where \( u = (u_1 \cdots u_n)^T \), \( v = (v_1 \cdots v_m)^T \). and

\[ \Lambda^+ = \text{diag}(\lambda_1 \cdots \lambda_n) \quad \Lambda^- = \text{diag}(\mu_1 \cdots \mu_m) \tag{2.3} \]

with

\[-\mu_1 < \cdots < -\mu_m < 0 < \lambda_1 \leq \cdots \leq \lambda_n \tag{2.4}\]
\( \Sigma^{\pm\pm} \) are matrices and without loss of generality, we assume that
\[
\forall j = 1, \ldots, m \quad \sigma_{jj} = 0, \quad (2.5)
\]

**Remark**

One can do some coordinate transformation of \( \nu \) in order to guarantee (2.5).
The boundary conditions are as follows

$$u(t, 0) = Q_0 v(t, 0), \quad v(t, 1) = R_1 u(t, 1) + U(t) \quad (2.6)$$

where $Q_0$ and $R_1$ are constant matrices. $U(t) = (U_1, \cdots, U_m)^T$ are boundary controls.

**Goal**

Our objective is to design a feedback control law for $U(t)$ in order to ensure that the closed-loop system vanishes in finite time.
What is Backstepping method?

- Mapping the original system to a target system which has “good” property (e.g. finite-time stable or exponential stable) by using a invertible backstepping transformation.
Methods: Backstepping

What is Backstepping method?
- Mapping the original system to a target system which has “good” property (e.g. finite-time stable or exponential stable) by using a invertible backstepping transformation.

Difficulty of this method
- How to choose a good transformation?
- How to choose a suitable target system?
Review of $2 \times 2$ (i.e. $\Lambda \in \mathcal{M}_{2,2}$) case (JMC et al. (2013) and R. Vazquez et al. (2011))

- **Transformation:**

\[ u(t, x) = \gamma(t, x) - \int_0^x K(x, \xi)\gamma(t, \xi)d\xi, \]  
(2.7)

- **Target system**

\[ \gamma_t(t, x) + \Lambda \gamma_x(t, x) = 0. \]  
(2.8)

- **The K-kernel is a matrix function of $C^2$ on the domain**

\[ \mathcal{T} = \{(x, \xi) | 0 \leq \xi \leq x \leq 1\}. \]  
(2.9)
Unfortunately, we can not deal with the general $n \times n$ cases by using the above transformation and target system. If so, an overdetermined problem appears to the K-kernel.

Question: Is it possible to change only the transformation or the target system to achieve our purpose?
Answer: Yes!
Unfortunately, we can not deal with the general $n \times n$ cases by using the above transformation and target system. If so, an overdetermined problem appears to the K-kernel.

**Question**

Is it possible to change only the transformation or the target system to achieve our purpose?
Unfortunately, we can not deal with the general $n \times n$ cases by using the above transformation and target system. If so, an overdetermined problem appears to the K-kernel.

**Question**

Is it possible to change only the transformation or the target system to achieve our purpose?

**Answer:** Yes!
New target system

\[ \alpha_t(t, x) + \Lambda^+ \alpha_x(t, x) = \Sigma^{++} \alpha(t, x) + \Sigma^{+-} \beta(t, x) \]
\[ + \int_0^x C^+(x, \xi) \alpha(\xi) d\xi + \int_0^x C^-(x, \xi) \beta(\xi) d\xi \] (2.10)

\[ \beta_t(t, x) - \Lambda^- \beta_x(t, x) = G(x) \beta(0) \] (2.11)

with the following boundary conditions

\[ \alpha(t, 0) = Q_0 \beta(t, 0), \quad \beta(t, 1) = 0 \] (2.12)

where \( C^+ \) and \( C^- \) are \( L^\infty \) matrix functions on the domain

\[ \mathcal{T} = \{0 \leq \xi \leq x \leq 1\} \] (2.13)
New target system

while $G \in L^\infty(0, 1)$ is a lower triangular matrix with the following structures

$$G(x) = \begin{pmatrix}
0 & \cdots & \cdots & 0 \\
g_{2,1}(x) & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0
\end{pmatrix}. \quad (2.14)$$
The zero equilibrium of the target system is reached in finite time $t = t_F$, where

$$t_F := \frac{1}{\lambda_1} + \sum_{j=1}^{m} \frac{1}{\mu_j}.$$ 

(2.15)
Sketch of the proof

It is easy to see that $\beta$ vanishes in finite time $t_1$ with

$$t_1 = \sum_{r=1}^{m} \frac{1}{|\lambda_r(s)|} ds. \quad (2.16)$$

From the time $t = t_1$ on, we find $\alpha$ becomes the solution of the following system

$$\alpha_t(t, x) + \Lambda^+ \alpha_x(t, x) = \Sigma^{++} \alpha(t, x) + \int_0^x C^+(x, \xi) \alpha(\xi) d\xi \quad (2.17)$$

with the boundary conditions

$$\alpha(t, 0) = 0. \quad (2.18)$$

Changing the status of $t$ and $x$, and Equations (2.17) can be rewritten as

$$\alpha_x(t, x) + (\Lambda^+)^{-1} \alpha_t(t, x) = (\Lambda^+)^{-1} \Sigma^{++} \alpha(t, x) + \int_0^x (\Lambda^+)^{-1} C^+(x, \xi) \alpha(\xi) d\xi \quad (2.19)$$

with the initial condition (2.18).
Then by the uniqueness of the system (2.18),(2.19), and noting the order of the transport speeds of the $\alpha$–system (see (2.4)), this yields that $\alpha$ identically vanishes for

$$
t \geq \frac{1}{\lambda_1} + \sum_{j=1}^{m} \frac{1}{\mu_j}
$$

(2.20)

Open Question

How to reduce the finite-time-control time $t_F$?
We consider the following backstepping (Volterra) transformation

\[
\alpha(t, x) = u(t, x) \quad (2.21) 
\]
\[
\beta(t, x) = v(t, x) - \int_0^x [K(x, \xi)u(\xi) + L(x, \xi)v(\xi)] d\xi \quad (2.22) 
\]

where the kernels to be determined \( K \) and \( L \) are defined on the triangular domain \( \mathcal{T} \).
Backstepping transformation

We consider the following backstepping (Volterra) transformation

\[ \alpha(t, x) = u(t, x) \]  
\[ \beta(t, x) = v(t, x) - \int_{0}^{x} [K(x, \xi)u(\xi) + L(x, \xi)v(\xi)] d\xi \]

where the kernels to be determined \( K \) and \( L \) are defined on the triangular domain \( \mathcal{T} \).

Important: \( \alpha(t, 0) = u(t, 0), \beta(t, 0) = v(t, 0) \)
The original system (2.1) is mapped into the target system (2.6) if $K$ and $L$ satisfy the following equations

for $1 \leq i \leq m$, $1 \leq j \leq n$

$$\mu_i \partial_x K_{ij}(x, \xi) - \lambda_j \partial_\xi K_{ij}(x, \xi) = \sum_{k=1}^{n} \sigma_{kj}^{++} K_{ik}(x, \xi) + \sum_{p=1}^{m} \sigma_{pj}^{-+} L_{ip}(x, \xi)$$  \hspace{1cm} (2.23)

for $1 \leq i \leq m$, $1 \leq j \leq m$

$$\mu_i \partial_x L_{ij}(x, \xi) + \mu_j \partial_\xi L_{ij}(x, \xi) = \sum_{p=1}^{m} \sigma_{pj}^{--} L_{ip}(x, \xi) + \sum_{k=1}^{n} \sigma_{kj}^{+-} K_{ik}(x, \xi)$$  \hspace{1cm} (2.24)

along with the following set of boundary conditions

$$K_{ij}(x, x) = - \frac{\sigma_{ij}^{-+}}{\mu_i + \lambda_j} \triangleq k_{ij}$$  \hspace{1cm} for $1 \leq i \leq m$, $1 \leq j \leq n$  \hspace{1cm} (2.25)

$$L_{ij}(x, x) = - \frac{\sigma_{ij}^{--}}{\mu_i - \mu_j} \triangleq l_{ij}$$  \hspace{1cm} for $1 \leq i, j \leq m$, $i \neq j$  \hspace{1cm} (2.26)

$$\mu_j L_{ij}(x, 0) = \sum_{k=1}^{n} \lambda_k K_{ik}(x, 0) q_{k,j}$$  \hspace{1cm} for $1 \leq i \leq j \leq m$.  \hspace{1cm} (2.27)
To ensure well-posedness of the kernel equations, we add the following artificial boundary conditions for $L_{ij}(i > j)$

$$L_{ij}(1, \xi) = l_{ij}, \text{ for } 1 \leq j < i \leq m.$$  \hfill (2.28)

**Remark**

We can select that

$$L_{ij}(1, \xi) = 0, \text{ for } 1 \leq j < i \leq m.$$  \hfill (2.29)
The existence of $K$ and $L$

**Figure 1**: Characteristic lines of the $K$ kernels
The existence of $K$ and $L$

(a) Characteristic lines of the kernels $L_{ij}$ for $i > j$

(b) Characteristic lines of the kernels $L_{ii}$

(c) Characteristic lines of the kernels $L_{ij}$ for $i < j$
The existence of $K$ and $L$

Successive approximation method (LH, F. DiMeglio, R. Vazquez and M. Krstic (2015 a))

\[
K(x, \xi) = \sum_{q=0}^{+\infty} \Delta K^q(x, \xi) \tag{2.30}
\]

\[
L(x, \xi) = \sum_{q=0}^{+\infty} \Delta L^q(x, \xi) \tag{2.31}
\]

where \[
\Delta K^q(x, \xi), \Delta L^q(x, \xi) \leq C \frac{M^q(x - (1 - \varepsilon)\xi)^q}{q!}.
\] (2.32)

in which $C, M > 0$. 
Improved successive approximation method (LH, F. DiMeglio, R. Vazquez and M. Krstic (2015 a))

Key point:
The proof is based on the fact that, starting from any point \((x, \xi)\), all the characteristic lines "get closer" to the line defined by \(x - (1 - \varepsilon)\xi = 0\).
Since $\beta(t, 1) = 0$ and
\[
\begin{pmatrix}
\alpha(t, x) \\
\beta(t, x)
\end{pmatrix} = \begin{pmatrix}
u(t, x) \\
v(t, x)
\end{pmatrix} - \int_0^x \begin{pmatrix}
0 & 0 \\
K(x, \xi) & L(x, \xi)
\end{pmatrix} \begin{pmatrix}
u(t, \xi) \\
v(t, \xi)
\end{pmatrix} d\xi.
\] (2.33)

Our feedback laws finally is
\[
U(t) = \int_0^1 \left[ K(1, \xi)u(\xi) + L(1, \xi)v(\xi) \right] d\xi - R_1 u(t, 1).
\] (2.34)
Since $\beta(t, 1) = 0$ and

$$
\begin{pmatrix}
\alpha(t, x) \\
\beta(t, x)
\end{pmatrix} = 
\begin{pmatrix}
u(t, x) \\
v(t, x)
\end{pmatrix} - \int_{0}^{x}
\begin{pmatrix}
0 & 0 \\
K(x, \xi) & L(x, \xi)
\end{pmatrix}
\begin{pmatrix}
u(t, \xi) \\
v(t, \xi)
\end{pmatrix}
\, d\xi.
$$

(2.33)

Our feedback laws finally is

$$
U(t) = \int_{0}^{1} [K(1, \xi)u(\xi) + L(1, \xi)v(\xi)] \, d\xi - R_1 u(t, 1).
$$

(2.34)

**Theorem:** Finite-time stabilization (LH, F. DiMeglio, R. Vazquez and M. Krstic (2015 a))

By $U(t)$, the zero equilibrium of the original system is reached in finite time $t = t_F$.

**Proof.** Good! (2.34) is always invertible, i.e. there exists a matrix function $R \in (L^\infty(\mathcal{T}))(n+m) \times (n+m)$, such that

$$
\begin{pmatrix}
u(t, x) \\
v(t, x)
\end{pmatrix} = 
\begin{pmatrix}
\alpha(t, x) \\
\beta(t, x)
\end{pmatrix} - \int_{0}^{x} R(x, \xi)
\begin{pmatrix}
\alpha(t, \xi) \\
\beta(t, \xi)
\end{pmatrix}
\, d\xi.
$$

(2.35)

Since $(\alpha, \beta)^T$ goes to zero in finite time $t = t_F$, therefore $(u, v)^T$ shares this property.
[1] We can deal with the observer problem by using also the backstepping approach.

[2] Our method is still valid for the case if the coefficients \((\Lambda^\pm, \Sigma^\pm)\) are depending on \(x\).


\[
S_1(t, 0) = \int_0^L K(0, \xi) S_1(t, \xi) + L(0, \xi) S_2(t, \xi) d\xi \tag{2.36}
\]
Remarks

[5] Our kernel is probably not continuous, which is different with previous works.

Figure 3: Kernels $L_{11}(x, \xi)$ and $L_{12}(x, \xi)$ ($n = 0$, $m = 2$).
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3. Boundary stabilization of quasilinear hyperbolic balance laws
4. Perspectives-Stabilization of nonlocal hyperbolic system
The system considered is

$$\frac{\partial u}{\partial t} + A(x, u) \frac{\partial u}{\partial x} = F(x, u) \quad (t, x) \in [0, T] \times [0, L],$$  \hspace{1cm} (3.1)
The system considered is

\[
\frac{\partial u}{\partial t} + A(x, u) \frac{\partial u}{\partial x} = F(x, u) \quad (t, x) \in [0, T] \times [0, L],
\]

(3.1)

where

- \( u = (u_1, \ldots, u_n)^T \) is a vector function of \((t, x)\);
- \( A(x, u) := (a_{ij}(x, u))_{n \times n} \) is of class \( C^2 \), \( A(x, 0) \) is a diagonal matrix with distinct and nonzero eigenvalues \( A(x, 0) = \text{diag}(\Lambda_1(x), \ldots, \Lambda_n(x)) \), which are ordered as follows:

\[
\Lambda_1(x) < \Lambda_2(x) < \cdots < \Lambda_m(x) < 0 < \Lambda_{m+1}(x) < \cdots < \Lambda_n(x), \quad \forall x \in [0, 1].
\]

(3.2)
$F : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector valued function with $C^2$ components $f_i(x, u)(i = 1, \cdots, n)$ with respect to $u$ and

$$F(x, 0) \equiv 0. \quad (3.3)$$

Denote

$$\frac{\partial F}{\partial u}(x, 0) := (f_{ij}(x))_{n \times n}, \quad (3.4)$$

we assume that $f_{ij} \in C^2([0, 1])$ and

$$f_{ii}(x) \equiv 0. \quad (3.5)$$

Remark

One can make some coordinate transformations in order to guarantee that (3.5) is valid (see JMC (2014) and L. Hu & F. DiMeglio (2015)).
The boundary conditions are given as follows:

\begin{align*}
x = 0 : & \quad u_s = G_s(u_1, \cdots, u_m), \quad s = m + 1, \cdots, n, \quad (3.6) \\
x = 1 : & \quad u_r = h_r(t), \quad r = 1, \cdots, m, \quad (3.7)
\end{align*}

where $G_s$ are $C^2$ functions, and we assume that they vanish at the origin, i.e.

\[ G_s(0, \cdots, 0) \equiv 0, \quad s = m + 1, \cdots, n, \quad (3.8) \]

while \( H = (h_1, \cdots, h_m)^T \) are boundary controls.

**Remark**

- Local well-posedness: see JMC (2008) and Tatsien Li (1994) [Remark 1.3 on page 171] etc.
Main Results

Theorem (LH, R. Vazquez, F. Di Meglio and M. Krstic (2015 b))

For every $\lambda > 0$, there exist $\delta > 0$, $c > 0$ and a continuous linear feedback control $H$, such that 0 is the exponential stable point of $u = u(t, x)$, i.e.

$$\|u(t, \cdot)\|_{H^2} \leq ce^{-\lambda t}\|u(0, \cdot)\|_{H^2},$$

(3.9)

provided that $\|u(0, \cdot)\|_{H^2} \leq \delta$.

Remark

- For simplicity, we skip of $C^1$ compatibility conditions at the points $(t, x) = (0, 0)$ and $(0, 1)$;
Sketch of the proof

[1] Linearized system is

\[
\begin{cases}
    u_t + \Lambda(x)u_x = \Sigma(x)u \\
    x = 0 : u_s(t, 0) = \sum_{j=1}^{m} Q_{sj} u_j(t, 0) \\
    x = L : u_j(t, 1) = h_j(t).
\end{cases}
\]  

(3.10)

where \( \Lambda(x) = A(x, 0) \) and \( \Sigma(x) = \frac{\partial F}{\partial u}(x, 0) \), then we can find

\[
h_j(t) = \int_{0}^{1} \sum_{l=1}^{n} K_{jl}(1, \xi) u_l(t, \xi) d\xi, \quad (j = 1, \cdots, m),
\]  

(3.11)

which can stabilize the linearized system in finite-time.
We also use the Volterra transformation
\[ \gamma(t, x) = u(t, x) - \int_0^x K(x, \xi) u(t, \xi) \, dt. \] (3.12)

**Lemma: Regularity of the direct kernel (LH, R. Vazquez, F. DiMeglio and M. Krstic (2015 b))**

Let \( N \in \mathbb{N}^+ \). Under the assumption that \( \sigma_{ij} \in C^N[0, 1] \), \( \lambda_i \in C^N[0, 1] \) \((i, j = 1, \cdots, n)\), there exists a unique piecewise \( C^N(T) \) solution to \( K \) kernel. Moreover, then \( K(\cdot, \cdot) \in C^{N-1}(0, 1) \), \( K(\cdot, 0) \in C^{N-1}(0, 1) \) with bounded \( C^{N-1} \) norm.

**Remark**
- The \( H^2 \) norm of \( \gamma \) is equivalent to the \( H^2 \) norm of \( u \).
Then the nonlinear target system is

$$
\gamma_t(t,x) + \Lambda(x)\gamma_x(t,x) - G(x)\gamma(t,0) = F_3[\gamma, \gamma_x] + F_4[\gamma],
$$

(3.13)

The boundary conditions are

$$
x = 0 : \gamma_+(t,0) = Q\gamma_-(t,0) + G_{NL}(\gamma_-(t,0))
$$

(3.14)

and

$$
x = 1 : \gamma_-(t,1) = 0.
$$

(3.15)

[3] Luckily! The usual Lyapunov function (see JMC-R. Vazquez-M. Krstic -G. Bastin, SICON 2013) can be also used to exponentially stabilize this $\gamma$ system.
Estimate of $\|\gamma\|_{L^2}$

Define

$$V_1(t) = \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} \gamma_+(t, x) dx - \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \gamma_-(t, x) dx.$$ 

We have

**Proposition 1**

For any given $\lambda_1 > 0$, there exists $\delta_1 > 0$ and $K_2 > 0$, such that

$$\dot{V}_1 \leq -\lambda_1 V_1 + K_2 (V_1^{\frac{3}{2}} + \|\gamma_x\|_\infty V_1),$$

provided $\|\gamma\|_\infty \leq \delta_1$. 

$(3.16)$
Control Lyapunov Functions

- Estimate of $\|\gamma_t\|_{L^2}$
  - Define $\zeta = \gamma_t$ and

  $$V_2(t) = \int_0^1 \zeta^T(t, x) R[\gamma] \zeta(t, x) \, dx,$$

  where $R[\gamma]$ is a positive matrix. We have

**Proposition 2**

For any given $\lambda_2 > 0$, there exists $\delta_2 > 0$ and $K_7 > 0$, such that

$$\dot{V}_2 \leq -\lambda_2 V_2 + K_7 (\|\zeta\|_\infty + \|\gamma\|_\infty) V_2,$$

provided that $\|\gamma\|_\infty \leq \delta_2$. 
Estimate of $\|\gamma_{tt}\|_{L^2}$
Define $\theta = \gamma_{tt}$ and

$$V_3(t) = \int_0^1 \theta^T(t, x) R[\gamma] \theta(t, x) dx,$$  \hspace{1cm} (3.19)

We have

**Proposition 3**

For any given $\lambda_3 > 0$, there exists $\delta_3 > 0$ and positive constants $K_{10}$, $K_{11}$, $K_{12}$, $K_{13}$ and $K_{14}$, such that

$$\dot{V}_3 \leq -\lambda_3 V_3 + K_{10} \|\gamma\|_\infty V_3 + K_{11} V_3 V_2^{1/2} + K_{12} V_2 V_3^{1/2} + K_{13} V_3^{3/2} + K_{14} \|\zeta\|_\infty^3,$$ \hspace{1cm} (3.20)

provided that $\|\gamma\|_\infty + \|\zeta\|_\infty \leq \delta_3$. 
Denote $W = V_1 + V_2 + V_3$, by Proposition 1–3, one can show that for any given $\lambda > 0$, there exists $\delta > 0$ and $K_{15} > 0$, such that

$$\dot{W} \leq -\lambda W + K_{15}W^{3/2},$$

(3.21)

provided that $\|\gamma\|_{\infty} + \|\zeta\|_{\infty} \leq \delta$.
Non-local hyperbolic system is considered as

\[
\begin{aligned}
&u_t = u_x + \int_0^L g(x, y)u(t, y)\,dy \\
&u(t, L) = U(t)
\end{aligned}
\]  

(4.1)

- \( u \) is a scalar \( U(t) \) is the boundary feedback.
- (4.1) may involve the traffic laws, some KdV-like equation and PDE-ODE interconnected system (see M. Krstic, A. Smyshlyaev (2008), F. Bribiesca and M. Krstic (2015)).
Known results

- M. Krstic, A. Smyshlyaev (2008): \( g(x, y) = 0, \ x \leq y \). Map (4.1) into

\[
\begin{cases}
   w_t = w_x \\
   w(t, L) = 0.
\end{cases}
\]  

(4.2)

by using Volterra transformation

\[
u(t, x) = w(t, x) - \int_0^x k(x, y)w(t, y)dy\]  

(4.3)
Known results

- F. Bribiesca and M. Krstic (2015): For general $g$ Volterra transformation fails. Map (4.1) into (4.2) by using Fredholm transformation

$$u(t, x) = w(t, x) - \int_{0}^{L} \tilde{k}(x, y)w(t, y)dy$$ \hspace{1cm} (4.4)

provided $\|g\|$ is small.

Remark

The smallness of $\|g\|$ is used to guarantee the existence of $\tilde{k}$ and the invertibility of the Fredholm transformation (4.4).
Main results (J.-M. Coron, LH and G. Olive (2015))

Suppose $g \in H^1(T_-) \cap H^1(T_+)$, where

$T_- = \{(x, y) \in (0, L) \times (0, L) | x > y\}$,

$T_+ = \{(x, y) \in (0, L) \times (0, L) | x < y\}$,

Then (4.1) is finite-time stabilizable in time $L$ if and only if (4.1) is exactly controllable at time $L$. 

We also use the Fredholm transformation

\[ u(t, x) = w(t, x) - \int_0^L h(x, y)w(t, y)dy \]  \hspace{1cm} (4.5)

to map the system (4.2) into (4.1). The kernel \( h \) satisfies

\[
\begin{cases}
  h_x(x, y) + h_y(x, y) + \int_0^L G(x, \tau)h(\tau, y)d\tau - G(x, y) = 0 \\
  h(x, 0) = h(x, L) = 0
\end{cases}
\]  \hspace{1cm} (4.6)

- We use the controllability of the original system (4.1) to prove the existence of \( h \);
- The invertibility of \( h \) can be guaranteed if the original system (4.1) is controllability.
We can also treat the equation of the more general form

\[
\begin{cases}
    u_t(t, x) = u_x(t, x) + \alpha(x)u(t, x) + \beta(x)u(t, 0) + \int_0^L g(x, y)u(t, y)dy \\
    u(t, L) = \int_0^L \gamma(x)u(t, x)dx + U(t) \\
    u(0, x) = u^0(x).
\end{cases}
\]

where \( \alpha, \beta, \gamma : (0, L) \rightarrow \mathbb{C} \) are regular enough.
We can also treat the equation of the more general form

\[
\begin{aligned}
&u_t(t, x) = u_x(t, x) + \alpha(x)u(t, x) + \beta(x)u(t, 0) + \int_0^L g(x, y)u(t, y)dy \\
u(t, L) = \int_0^L \gamma(x)u(t, x)dx + U(t) \\
u(0, x) = u^0(x).
\end{aligned}
\]

where \(\alpha, \beta, \gamma : (0, L) \to \mathbb{C}\) are regular enough.

We expect that (4.1) is always controllable. However, this is NOT true.

- If \(g(x, y) = 0, x \leq y\), (4.1) is controllable. (M. Krstic, A.Smyshlyaev (2008))
- If \(\|g\|\) is small enough, (4.1) is controllable. (F. Bribiesca and M. Krstic (2015))
- If \(g(x, y) = g(x)\), (4.1) is controllable if and only if (J.M.Coron, LH and G.Olive (2015))

\[
\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}.
\]  

(4.7)

Question: What about the general case \(g(x, y)\)?
Pas un jour sans contrôle

Thank you for your attention!